Contents

8 Perturbation theory and gravitational waves

8.1 Linearised metric and gauge transformations

8.1.1 Metric, connection and Riemann tensor

When discussing the Newtonian limit in chapter 4, we assumed that the perturbation of the metric is diagonal and consists of only one function. Let us now consider general linear perturbations around Minkowski space, also dropping the small velocity approximation we used.

We assume that the spacetime is perturbatively near Minkowski space, i.e. gravitational fields are weak. Precisely speaking, this means that there exists a coordinate system where the metric can be written as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ , \tag{8.1}$$

where $|h_{\alpha\beta}| \ll 1$. We take the derivatives of $h_{\alpha\beta}$ to be of the same order of smallness as $h_{\alpha\beta}$. (This is not the case in all applications of perturbation theory in GR.) The metric $g_{\alpha\beta}$ is a tensor, but $\eta_{\alpha\beta}$ is not a tensor, so $h_{\alpha\beta}$ is not a tensor either. However, if we work only to linear order in $h_{\alpha\beta}$, it behaves as a tensor in Minkowski space, so we can treat it like a field in flat spacetime. This is illustrated in figure 1. For example, the indices of $h_{\alpha\beta}$ are lowered with $\eta_{\alpha\beta}$ and raised with $\eta^{\alpha\beta}$. As $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha{}_\beta$ we have, to first order,

$$g^{\alpha\beta} \simeq \eta^{\alpha\beta} - h^{\alpha\beta} \ , \tag{8.2}$$

where $h^{\alpha\beta} \equiv \eta^{\alpha\gamma}\eta^{\beta\delta}h_{\gamma\delta}$.

The perturbation $h_{\alpha\beta}$ has 10 components. It is useful to decompose it in terms of quantities that behave in a specific way under spatial rotations. (As Minkowski space has no preferred time slices, the choice of time coordinate and spatial slicing

(a) The real, perturbed, spacetime.
The metric is $g_{\alpha\beta}$.
Neither $\eta_{\alpha\beta}$ nor $h_{\alpha\beta}$ are tensors.

(b) The fictitious background spacetime.
The metric is $\eta_{\alpha\beta}$. It is a tensor.
The field $h_{\alpha\beta}$ is a tensor.

Figure 1

is arbitrary, though we restrict to Cartesian coordinates, i.e. we slice the spacetime with Euclidean spatial sections.) Under spatial rotations, h_{00} is a scalar, h_{0i} is a vector, h_{ij} is a symmetric rank 2 tensor, and the spatial trace $\delta^{ij}h_{ij}$ is a scalar. We write

$$\begin{aligned} \mathrm{d}s^2 &= -(1-h_{00})\mathrm{d}t^2 + 2h_{0i}\mathrm{d}t\mathrm{d}x^i + (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j \\ &= -(1+2\phi)\mathrm{d}t^2 + 2w_i\mathrm{d}t\mathrm{d}x^i + [(1-2\psi)\delta_{ij} + 2S_{ij}]\mathrm{d}x^i\mathrm{d}x^j \ , \end{aligned} \tag{8.3}$$

where $S_{ij} = S_{(ij)}$ is called the **strain**. It is traceless, $\delta^{ij}S_{ij} = 0$. The indices of w_i and S_{ij} are raised and lowered with the Euclidean metric δ_{ij}. The new perturbation functions in terms of the components $h_{\alpha\beta}$ read

$$\phi \equiv -\frac{1}{2}h_{00} \tag{8.4}$$

$$w_i \equiv h_{0i} = h_{i0} \tag{8.5}$$

$$\psi \equiv -\frac{1}{6}\delta^{ij}h_{ij} \tag{8.6}$$

$$S_{ij} \equiv \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}h_{kl}\right) \ . \tag{8.7}$$

The inverse of (8.4)–(8.7) can be written as

$$h_{00} = -2\phi \tag{8.8}$$

$$h_{0i} = w_i \tag{8.9}$$

$$h_{ij} = -2\delta_{ij}\psi + 2S_{ij} \tag{8.10}$$

$$h = 2\phi - 6\psi \ , \tag{8.11}$$

where $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$. The expressions (8.4)–(8.7) do not give a full decomposition into **irreducible representations** of the rotation group.[1] The perturbation h_{0i} has a scalar and an irreducible vector part, and S_{ij} has scalar, irreducible vector and irreducible rank 2 tensor parts. To get the full decomposition, we write

$$\begin{aligned} w_i &= B_i + B_{,i} \\ S_{ij} &= C_{ij} + C_{(i,j)} + C_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2 C \ , \end{aligned} \tag{8.12}$$

[1] Irreducible representations are those that contain no smaller subrepresentations that are closed under the group action.

where $\partial_i B^i = \partial_i C^i = 0$, $\partial_i C^{ij} = 0$, $C_{ij} = C_{(ij)}$, $C^i{}_i = 0$, and indices are again raised and lowered with the Euclidean metric δ_{ij}. This extends the well-known **Helmholtz decomposition** of a vector in three-dimensional Euclidean space. Here B_i and C_i are irreducible vectors and C_{ij} is an irreducible rank 2 tensor.[2]

Let us count the number of degrees of freedom. In the decomposition (8.4)–(8.7), ϕ and ψ are 2 functions, w_i contains 3 functions, and the symmetric traceless 3×3 matrix S_{ij} contains $6-1=5$ functions. That is 10 functions in total. From the irreducible representation (8.12) we see that w_i contains one scalar and one irreducible vector, and S_{ij} contains one scalar, one irreducible vector and one irreducible rank 2 tensor. An irreducible (three-)vector has 2 degrees of freedom (3 components minus 1 for being divergence-free), as does an irreducible symmetric rank 2 tensor (the 5 independent components of S_{ij} minus 1 for one scalar and minus 2 for one irreducible vector). So in total the metric has 4 scalar degrees of freedom, 4 irreducible vector degrees of freedom and 2 irreducible tensor degrees of freedom. However, due to coordinate invariance, in perturbation theory called **gauge invariance**, not all of them are physical.

8.1.2 Gauge transformations

The split into background and perturbations is not uniquely defined. Consider unperturbed Minkowski space in Cartesian coordinates. Now do a small coordinate transformation, and linearise in the small parameter of the transformation (for example consider the merry-go-round coordinates introduced in chapter 1, with a small angular velocity). The new metric has the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, but $h_{\alpha\beta}$ is a coordinate artifact, and does not correspond to physical degrees of freedom. A general coordinate transformation $x^\alpha \to x'^\alpha(x)$ contains 4 arbitrary functions, so only $10-4=6$ of the 10 functions in the metric are independent. In the context of perturbation theory, this freedom corresponds to **gauge transformations**. They are small coordinate transformations such that the background is left invariant and the change is absorbed in the perturbations. A choice of division into the background and perturbations means choosing a map from the real spacetime into a Minkowski spacetime. Gauge transformations are changes in this mapping. Concretely, we have

$$x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha(x) \ , \tag{8.13}$$

where ξ^α is of the same order of smallness as the metric perturbations. In (6.29), we showed that the metric transforms as

$$\begin{aligned} g_{\alpha\beta} \to g'_{\alpha\beta} &= (M^{-1})^\gamma{}_\alpha (M^{-1})^\delta{}_\beta g_{\gamma\delta} \\ &\simeq g_{\alpha\beta} - 2\nabla_{(\beta}\xi_{\alpha)} \\ &\simeq \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \\ &\equiv \eta_{\alpha\beta} + h'_{\alpha\beta} \ , \end{aligned} \tag{8.14}$$

[2] Physicists are sloppy with language. The word tensor refers to an invariant object on the manifold. However, it is also used to specifically mean a spatial rank 2 tensor. In the context of perturbation theory, it most often means an irreducible rank 2 tensor, or its components. The meaning is usually clear from the context.

where $\xi_\alpha \equiv \eta_{\alpha\beta}\xi^\beta$. So the perturbation changes as

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \ . \tag{8.15}$$

Under the gauge transformation (8.15), the decomposed fields (8.4)–(8.7) transform as

$$\phi \ \to \ \phi' = \phi + \dot{\xi}_0 \tag{8.16}$$
$$w_i \ \to \ w'_i = w_i - \dot{\xi}_i - \partial_i \xi_0 \tag{8.17}$$
$$\psi \ \to \ \psi' = \psi + \frac{1}{3}\partial_i \xi^i \tag{8.18}$$
$$S_{ij} \ \to \ S'_{ij} = S_{ij} - \frac{1}{2}(\xi_{i,j} + \xi_{j,i}) + \frac{1}{3}\delta_{ij}\partial_k \xi^k \ . \tag{8.19}$$

We can divide the gauge transformation three-vector into irreducible parts as $\xi_i = \partial_i \xi + \tilde{\xi}_i$, where $\partial_i \tilde{\xi}^i = 0$. In terms of the irreducible representation (i.e. using the decomposition (8.12)), the gauge transformations (8.16)–(8.19) read

$$\phi' = \phi + \dot{\xi}_0 \tag{8.20}$$
$$B' = B - \dot{\xi} - \xi_0 \tag{8.21}$$
$$B'_i = B_i - \dot{\tilde{\xi}}_i \tag{8.22}$$
$$\psi' = \psi + \frac{1}{3}\nabla^2 \xi \tag{8.23}$$
$$C' = C - \xi \tag{8.24}$$
$$C'_i = C_i - \tilde{\xi}_i \tag{8.25}$$
$$C'_{ij} = C_{ij} \ . \tag{8.26}$$

We have 4 **gauge degrees of freedom**: two scalars ξ and ξ_0, and one irreducible vector $\tilde{\xi}_i$. We calculated after (8.12) that the perturbations have 4 scalar, 4 vector, and 2 tensor degrees of freedom. Subtracting the number of gauge degrees of freedom, we now see that the 6 physical degrees are divided into 2 scalar, 2 vector and 2 tensor degrees of freedom. The gauge degrees of freedom are a nuisance in the sense that we have to take them into account to be sure of the physical interpretation of our solutions: we should not mistake gauge artifacts for physics. One way to do so is to consider only **gauge-invariant** quantities, i.e. combinations of the perturbations that do not change under gauge transformations. For example, $B_i - \dot{C}_i$ does not change under a gauge transformation, and C_{ij} is gauge-invariant by itself. On the other hand, we can use the gauge degrees of freedom to our advantage, because a judicious choice of ξ^α, called a **gauge choice**, can considerably simplify the equations of motion, as we will see.

8.1.3 Connection and Riemann tensor

To first order, the connection coefficients for the metric (8.3) are

$$\begin{aligned}\Gamma^\gamma_{\alpha\beta} &= \frac{1}{2}g^{\gamma\mu}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}) \\ &\simeq \frac{1}{2}\eta^{\gamma\mu}(\partial_\alpha h_{\beta\mu} + \partial_\beta h_{\mu\alpha} - \partial_\mu h_{\alpha\beta}) \ . \end{aligned}\tag{8.27}$$

In terms of the decomposition (8.3), we have

$$\Gamma^0_{00} \simeq \dot\phi \tag{8.28}$$
$$\Gamma^0_{j0} \simeq \partial_j\phi \tag{8.29}$$
$$\Gamma^0_{jk} \simeq -\frac{1}{2}(\partial_j w_k + \partial_k w_j - \dot h_{jk}) \tag{8.30}$$
$$\Gamma^i_{00} \simeq \dot w_i + \partial_i\phi \tag{8.31}$$
$$\Gamma^i_{j0} \simeq \frac{1}{2}(\partial_j w_i - \partial_i w_j + \dot h_{ij}) \tag{8.32}$$
$$\Gamma^i_{jk} \simeq \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ij} - \partial_i h_{jk}) \ , \tag{8.33}$$

where dot denotes ∂_t and we have not divided h_{ij} into the traceless part $2S_{ij}$ and the trace -6ψ.

The corresponding Riemann tensor is

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &\simeq \eta_{\alpha\mu}\partial_\gamma\Gamma^\mu_{\delta\beta} - \eta_{\alpha\mu}\partial_\delta\Gamma^\mu_{\gamma\beta} \\ &\simeq \frac{1}{2}(\partial_\gamma\partial_\beta h_{\alpha\delta} - \partial_\gamma\partial_\alpha h_{\delta\beta} - \partial_\delta\partial_\beta h_{\alpha\gamma} + \partial_\delta\partial_\alpha h_{\gamma\beta}) \ , \end{aligned} \tag{8.34}$$

The gauge transformation (8.15) leaves the Riemann tensor unchanged. This can be seen straightforwardly by substituting (8.15) into (8.34) to get

$$R_{\alpha\beta\gamma\delta} \to R'_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} \ . \tag{8.35}$$

This is analogous to how the gauge transformation $A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha\sigma$ in electromagnetism leaves the field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ unchanged.

The Riemann tensor can be written as

$$\begin{aligned} R_{0j0l} &\simeq \frac{1}{2}(\partial_j\dot h_{0l} + \partial_l\dot h_{0j} - \partial_0\dot h_{lj} - \partial_l\partial_j h_{00}) \\ &\simeq \partial_j\partial_l\phi + \frac{1}{2}(\partial_j\dot w_l + \partial_l\dot w_j) - \frac{1}{2}\ddot h_{jl} \end{aligned} \tag{8.36}$$

$$\begin{aligned} R_{0jkl} &\simeq \frac{1}{2}(\partial_k\partial_j h_{0l} - \partial_l\partial_j h_{0k} + \partial_l\dot h_{kj} - \partial_k\dot h_{lj}) \\ &\simeq \frac{1}{2}(\partial_k\partial_j w_l - \partial_l\partial_j w_k) - \frac{1}{2}(\partial_k\dot h_{lj} - \partial_l\dot h_{kj}) \end{aligned} \tag{8.37}$$

$$R_{ijkl} \simeq \frac{1}{2}(\partial_k\partial_j h_{il} - \partial_k\partial_i h_{lj} - \partial_l\partial_j h_{ik} + \partial_l\partial_i h_{kj}) \tag{8.38}$$

The Ricci tensor is

$$\begin{aligned} R_{\alpha\beta} &\simeq \eta^{\gamma\delta}R_{\gamma\alpha\delta\beta} \\ &\simeq \frac{1}{2}\left(\partial_\alpha\partial_\gamma h^\gamma{}_\beta + \partial_\beta\partial_\gamma h^\gamma{}_\alpha - \Box h_{\alpha\beta} - \partial_\alpha\partial_\beta h\right) \ , \end{aligned} \tag{8.39}$$

where $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$, $\Box \simeq \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. The Ricci scalar is

$$R \simeq \eta^{\alpha\beta}R_{\alpha\beta} \simeq \partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h \ , \tag{8.40}$$

so the Einstein tensor is

$$\begin{aligned} G_{\alpha\beta} &\simeq R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R \\ &\simeq \frac{1}{2}(\partial_\gamma\partial_\alpha h^\gamma{}_\beta + \partial_\beta\partial_\gamma h^\gamma{}_\alpha - \partial_\alpha\partial_\beta h - \Box h_{\alpha\beta} - \eta_{\alpha\beta}\partial_\gamma\partial_\delta h^{\gamma\delta} + \eta_{\alpha\beta}\Box h) \ . \end{aligned} \tag{8.41}$$

In terms of the decomposition (8.3), the Einstein tensor reads

$$G_{00} \simeq 2\nabla^2\psi + \partial_i\partial_j S^{ij} \tag{8.42}$$

$$G_{0j} \simeq -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_j\dot{\psi} + \partial_k\dot{S}^k{}_j \tag{8.43}$$

$$\begin{aligned} G_{ij} &\simeq \big(\delta_{ij}\nabla^2 - \partial_i\partial_j\big)(\phi - \psi) + 2\delta_{ij}\ddot{\psi} + \delta_{ij}\partial_k\dot{w}^k \\ &\quad -\frac{1}{2}(\partial_i\dot{w}_j + \partial_j\dot{w}_i) - \Box S_{ij} + \partial_k(\partial_i S^k{}_j + \partial_j S^k{}_i) - \delta_{ij}\partial_k\partial_l S^{kl} \ , \end{aligned} \tag{8.44}$$

where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$.

8.1.4 Equation of motion for particles

Let us first consider the equation of motion for particles (the geodesic equation) and then the equation of motion for the metric (the Einstein equation). Take a particle moving on timelike geodesic with tangent vector u^α. Its four-momentum is

$$p^\alpha = mu^\alpha = (E, p^i) \ , \tag{8.45}$$

where m is mass, E is energy, and p^i is three-momentum. We can write the three-momentum as

$$p^i = m\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = E\frac{\mathrm{d}x^i}{\mathrm{d}t} = Ev^i \ , \tag{8.46}$$

where we have used the relation $E = mu^0 = m\frac{\mathrm{d}t}{\mathrm{d}\tau}$.

The geodesic equation is

$$\begin{aligned} 0 &= u^\beta\nabla_\beta u^\alpha \\ &= \frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma \\ &\propto m\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma \\ &= E\frac{\mathrm{d}p^\alpha}{\mathrm{d}t} + \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma \ , \end{aligned} \tag{8.47}$$

where we have again used $E = mu^0 = m\frac{\mathrm{d}t}{\mathrm{d}\tau}$. Moving the connection coefficients to one side and dividing both sides by E, we get

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}t} = -\Gamma^\alpha_{\beta\gamma}\frac{p^\beta p^\gamma}{E} \ . \tag{8.48}$$

The component $\alpha = 0$ gives, inputting connection coefficients from (8.28)–(8.30),

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= -\Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{E} = -\Gamma^0_{00}E - 2\Gamma^0_{0i}Ev^i - \Gamma^0_{ij}Ev^iv^j \\ &= -E\left[\dot\phi + 2v^i\partial_i\phi - \left(\partial_i w_j - \frac{1}{2}\dot h_{ij}\right)v^iv^j\right] .\end{aligned} \tag{8.49}$$

For the components $\alpha = i$ we get

$$\begin{aligned}\frac{\mathrm{d}p^i}{\mathrm{d}t} &= -\Gamma^i_{\alpha\beta}\frac{p^\alpha p^\beta}{E} = -\Gamma^i_{00}E - 2\Gamma^i_{0j}Ev^j - \Gamma^i_{jk}Ev^jv^k \\ &= -E\Big[\partial_i\phi + \dot w_i + (\partial_j w_i - \partial_i w_j + \dot h_{ij})v^j \\ &\quad + \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ij} - \partial_i h_{jk})v^jv^k\Big] .\end{aligned} \tag{8.50}$$

This form of the geodesic equation suggests identifying the connection coefficient terms as forces due to gravitational fields living in Minkowski space. In addition to the field ϕ that is the only contribution in the Newtonian case, we have the extra scalar field ψ, vector field w^i and tensor field h_{ij}. Their physical meaning is more transparent if we introduce the **gravitoelectric** and **gravitomagnetic** vector fields

$$\begin{aligned}G_i &\equiv -\partial_i\phi - \dot w_i \\ H^i &\equiv \epsilon^{ijk}\partial_j w_k ,\end{aligned} \tag{8.51}$$

or, in three-vector notation,

$$\begin{aligned}\vec G &= -\nabla\phi - \dot{\vec w} \\ \vec H &= \nabla\times\vec w .\end{aligned} \tag{8.52}$$

In terms of the fields (8.52), the generalisation (8.50) of Newton's law of gravity reads

$$\begin{aligned}\frac{\mathrm{d}p^i}{\mathrm{d}t} &= E\left[G^i + (\partial_i w_j - \partial_j w_i)v^j - \dot h_{ij}v^j - \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk})v^jv^k\right] \\ &= E(\underbrace{G^i + \epsilon^{ijk}v_jH_k}_{\vec G + \vec v\times\vec H}) - E\left[\dot h_{ij}v^j + \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk})v^jv^k\right] ,\end{aligned} \tag{8.53}$$

where we have used the relation

$$\vec v\times\vec H = \epsilon_{ijk}v^jH^k = \underbrace{\epsilon_{ijk}\epsilon^{klm}}_{\epsilon_{kij}\epsilon^{klm} = \delta^l{}_i\delta^m{}_j - \delta^m{}_i\delta^l{}_j}v^j\partial_l w_m = v^j\partial_i w_j - v^j\partial_j w_i . \tag{8.54}$$

The force law (8.53) can be compared to the electromagnetic Lorentz force, which we discussed in chapter 1:

$$\frac{\mathrm{d}\vec p}{\mathrm{d}t} = q\left(\vec E + \vec v\times\vec B\right) , \tag{8.55}$$

and the gravitoelectric and gravitomagnetic fields $\vec{G}$ and $\vec{H}$ can be compared to the electric and magnetic fields:

$$\begin{aligned}\vec{E} &= -\nabla\varphi - \dot{\vec{A}} \\ \vec{B} &= \nabla \times \vec{A}\ ,\end{aligned} \tag{8.56}$$

where we have written $A^\alpha = (\varphi, A^i)$.

We see that h_{00} and h_{0i} affect the motion of a particle in the same way as the electromagnetic scalar and vector potential, respectively. Gravity couples to the energy E instead of the electric charge q. The perturbation h_{00} alone gives the Newtonian force, with the difference that it couples to E and not m. The time derivative of h_{ij} contributes at the same linear order in v^i as h_{0i}, and its spatial derivatives contribute at quadratic order in v^i.

In order to quantify the different terms, we need the magnitude of the different perturbations for a realistic source, so we have to solve the Einstein equation.

8.1.5 Equation of motion for the metric

In the components of the Einstein tensor (8.42)-(8.44), various combinations of the metric perturbations and their derivatives appear. We can simplify these components considerably by choosing a convenient gauge, such as the **transverse gauge**, defined by the **gauge conditions**

$$\begin{aligned}\partial_i w^i &= 0 \\ \partial_i S^{ij} &= 0\ .\end{aligned} \tag{8.57}$$

Let us first show that we can simultaneously impose these conditions. We have available four functions ξ^α, and (8.57) has four conditions, so seems possible that we can impose them, but we still have to show explicitly that there exists a vector ξ^α such that (8.57) holds. The conditions are easiest to handle in terms of irreducible variables defined in (8.12).

The first condition reads

$$\begin{aligned}0 &= \partial_i B^i + \nabla^2 B \\ &= \nabla^2 B\ ,\end{aligned} \tag{8.58}$$

given that $\partial_i B^i = 0$ by definition. The only solution to the equation $\nabla^2 B = 0$ that is non-singular everywhere and vanishes at spatial infinity is $B = 0$.

In terms of the irreducible variables, the second gauge condition in (8.57) reads

$$\begin{aligned}0 &= \partial^i C_{ij} + \frac{1}{2}\partial_j\partial^i C_i + \frac{1}{2}\nabla^2 C_j + \frac{2}{3}\partial_j\nabla^2 C \\ &= \frac{1}{2}\nabla^2 C_j + \frac{2}{3}\partial_j\nabla^2 C\ ,\end{aligned} \tag{8.59}$$

where we have taken into account that by definition $\partial^i C_{ij} = 0$, $\partial^i C_i = 0$. Applying ∂^j to (8.59) and using $\partial^j C_j = 0$ gives $\nabla^2\nabla^2 C = 0$. Again, the only solution that is non-singular everywhere and vanishes at spatial infinity is $C = 0$. Using this in (8.59) then gives $\nabla^2 C_j = 0$, which gives $C_j = 0$.

So we have to check whether it is possible to choose $\xi_\alpha = (\xi_0, \partial_i \xi + \tilde{\xi}_i)$ so that $B = C = 0$ and $C_j = 0$. Starting from arbitrary coordinates where these functions are not necessarily zero, under a gauge transformation they change according to (8.21), (8.24) and (8.25) as

$$\begin{aligned} B \to B' &= B - \dot{\xi} - \xi_0 \\ C \to C' &= C - \xi \\ C_i \to C'_i &= C_i - \tilde{\xi}_i \ . \end{aligned} \tag{8.60}$$

Requiring that in the new coordinates $B' = C' = 0$ and $C'_i = 0$ gives the unique solution $\xi = C$, $\xi_0 = B - \dot{C}$ and $\tilde{\xi}_i = C_i$. So it is always possible to choose the transverse gauge, and it completely fixes the gauge conditions: there are no gauge degrees of freedom left over.

In the transverse gauge, the Einstein equation $G_{\alpha\beta} = 8\pi G_{\rm N} T_{\alpha\beta}$ simplifies considerably:

$$\nabla^2 \psi \simeq 4\pi G_{\rm N} T_{00} \tag{8.61}$$

$$\nabla^2 w_j - 4\partial_j \dot{\psi} \simeq -16\pi G_{\rm N} T_{0j} \tag{8.62}$$

$$\big(\delta_{ij}\nabla^2 - \partial_i\partial_j\big)(\phi - \psi) + 2\delta_{ij}\ddot{\psi} - \frac{1}{2}(\partial_i \dot{w}_j + \partial_j \dot{w}_i) - \Box S_{ij} \simeq 8\pi G_{\rm N} T_{ij} \ . \tag{8.63}$$

We can split the ij component into the trace and the traceless part:

$$\nabla^2(\phi - \psi) + 3\ddot{\psi} \simeq 4\pi G_{\rm N} \delta^{ij} T_{ij} \tag{8.64}$$

$$\left(\frac{1}{3}\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\phi - \psi) - \partial_{(i}\dot{w}_{j)} - \Box S_{ij} \simeq 8\pi G_{\rm N} \left(T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}\right) . \tag{8.65}$$

We can solve these equations straightforwardly one step at a time. First we solve ψ from (8.61). We then input the result into (8.64) to solve for ϕ and into (8.62) to solve for w^i. Finally, we input all these results into (8.65) to solve for S_{ij}.

The solution of (8.61) is

$$\psi(t, \mathbf{x}) = -G_{\rm N} \int {\rm d}^3 x' \frac{T_{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{8.66}$$

Analogously, the solution of (8.62) is

$$\begin{aligned} w_i(t, \mathbf{x}) &= \int {\rm d}^3 x' \frac{4G_{\rm N} T_{0i}(t, \mathbf{x}') - \frac{1}{\pi}\partial_i \dot{\psi}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= G_{\rm N} \int \frac{{\rm d}^3 x'}{|\mathbf{x} - \mathbf{x}'|} \left[4T_{0i}(t, \mathbf{x}') - \frac{1}{\pi} \int {\rm d}^3 x'' \frac{\dot{T}_{00}(t, \mathbf{x}'')(x'_i - x''_i)}{|\mathbf{x}' - \mathbf{x}''|^3} \right] . \end{aligned} \tag{8.67}$$

The solution of (8.64) is likewise

$$\begin{aligned} \phi(t, \mathbf{x}) - \psi(t, \mathbf{x}) &= \int {\rm d}^3 x' \frac{-G_{\rm N}\delta^{ij} T_{ij}(t, \mathbf{x}') + \frac{3}{4\pi}\ddot{\psi}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= -G_{\rm N} \int \frac{{\rm d}^3 x'}{|\mathbf{x} - \mathbf{x}'|} \left[\delta^{ij} T_{ij}(t, \mathbf{x}') + \frac{3}{4\pi} \int {\rm d}^3 x'' \frac{\ddot{T}_{00}(t, \mathbf{x}'')}{|\mathbf{x}' - \mathbf{x}''|^3} \right] . \end{aligned} \tag{8.68}$$

We will discuss the solution for S_{ij} later. Let us first look at the Newtonian limit, now properly accounting for all of the metric perturbations.

8.1.6 Newtonian limit redux

The central assumptions of the Newtonian limit are weak fields and small velocities. The energy-momentum tensor reads, in full generality,

$$\begin{aligned} T_{\alpha\beta} &= (\rho + P)u_\alpha u_\beta + P g_{\alpha\beta} + 2q_{(\alpha}u_{\beta)} + \Pi_{\alpha\beta} \\ &\simeq (\rho + P)u_\alpha u_\beta + P\eta_{\alpha\beta} + 2q_{(\alpha}u_{\beta)} + \Pi_{\alpha\beta} \ , \end{aligned} \tag{8.69}$$

where on the second line we have taken into account that the matter quantities are taken to be the same order of smallness as the metric perturbations (which they source via a linear equation). Taking into account that u_i is also first order small and the normalisation $g_{\alpha\beta}u^\alpha u^\beta = -1$, the components of the energy-momentum tensor read

$$\begin{aligned} T_{00} &\simeq (\rho + P)u_0 u_0 - P + 2q_0 u_0 + \Pi_{00} \simeq \rho + 2q_i u^i \\ T_{0i} &\simeq (\rho + P)u_0 u_i + q_i u_0 + \Pi_{0i} \simeq -(\rho + P)u_i - q_i - \Pi_{ij}u^j \\ T_{ij} &\simeq P\delta_{ij} + 2q_{(i}u_{j)} + \Pi_{ij} \ , \end{aligned} \tag{8.70}$$

where we have used $q_\alpha u^\alpha = 0$ and $\Pi_{\alpha\beta}u^\beta = 0$ to solve for q_0 and Π_{0i}, and have kept the leading cross-terms between the velocity and the matter variables, but dropped cross-terms between matter variables and metric perturbations. If we assume that all matter contributions except the energy density are negligible, we get

$$\begin{aligned} T_{00} &\simeq \rho \\ T_{0i} &\simeq -\rho u_i \\ T_{ij} &\simeq 0 \ . \end{aligned} \tag{8.71}$$

The solution for ψ from (8.66) is now the same as in Newtonian physics, except that the source is the energy density, not the mass density,

$$\psi(t, \mathbf{x}) = -G_{\rm N} \int {\rm d}^3x' \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{8.72}$$

In this limit, the difference between ψ and ϕ is generated solely by $\ddot{\psi}$. If the density varies slowly in time, so does ψ, and $\phi \simeq \psi$.

The solution for w_i in (8.67) has two parts, one generated by ρu_i and the other by $\dot{\rho}$. If the second contribution is small, we get

$$w_i(t, \mathbf{x}) \simeq -4G_{\rm N} \int {\rm d}^3x' \frac{\rho(t, \mathbf{x}')u_i(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{8.73}$$

In the Newtonian limit, the vector perturbation w^i is suppressed by one factor of velocity compared to the the scalar perturbations ψ and ϕ. In the geodesic equation (8.53), w^i appears via a time derivative and coupled to the velocity. The contribution of w^i to the equation of motion is therefore additionally suppressed beyond the suppression of w^i over ψ and ϕ.

Let us now come back to S_{ij}. In the approximation we have adopted, its equation of motion reduces to

$$\Box S_{ij} = 0 \ . \tag{8.74}$$

This is a wave equation. Unlike all the other metric perturbations, S_{ij} does not vanish in the absence of matter, but propagates at the speed of light. Let us take a closer look at these **gravitational waves**.

8.2 Gravitational waves

8.2.1 In vacuum

Let us consider the perturbed metric in the case when the energy-momentum tensor is zero. The equations of motion then give $\phi = \psi = 0$, $w_i = 0$, and

$$0 = \Box S_{ij} = -\frac{\partial^2 S_{ij}}{\partial t^2} + \nabla^2 S_{ij} \ . \tag{8.75}$$

This is a wave equation whose solutions are waves moving at the speed of light. The general solution is a linear combination of waves with all possible null wavevectors k^α,

$$S_{ij}(t,\vec{x}) = \mathrm{Re}\left\{ \int \mathrm{d}^3k' \tilde{S}_{ij}(\vec{k}') e^{ik'_\alpha x^\alpha} \right\} \ . \tag{8.76}$$

Note that $k_\alpha x^\alpha = \vec{k}\cdot\vec{x} - kt$, where $k \equiv |\vec{k}|$. Inserting (8.76) into (8.75), we get the null condition $k'_\alpha k'^\alpha = 0$. For a monochromatic wave with wavevector $\vec{k}$, we have $\tilde{S}_{ij}(\vec{k}') = \delta^{(3)}(\vec{k}' - \vec{k}) s_{ij}(\vec{k})$, so

$$S_{ij}(t,\vec{x}) = \mathrm{Re}\left\{ s_{ij}(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - kt)} \right\} \ . \tag{8.77}$$

Because S_{ij} is symmetric, $s_{ij} = s_{ji}$. Because S_{ij} is traceless, $\delta^{ij} s_{ij} = 0$. The equation (8.75) has been derived in the transverse gauge, and the gauge condition $\partial_i S^{ij} = 0$ means $k^i s_{ij} = 0$. Let us choose the z-axis to point in the direction of $\vec{k}$. The matrix s_{ij} then reads

$$s_{ij}(\vec{k}) = \begin{pmatrix} s_{11}(\vec{k}) & s_{12}(\vec{k}) & 0 \\ s_{12}(\vec{k}) & -s_{11}(\vec{k}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{8.78}$$

From the 5 degrees of freedom of a general traceless symmetric 3×3 matrix, the transverse condition removes 3, leaving us with 2, which is the right number of physical tensor degrees of freedom, as discussed earlier. Of the 10 degrees of freedom in the metric, only 6 are physical, and only 2 propagate, the other 4 are given by constraints.

The gravitational wave is analogous to the electromagnetic wave. A photon is described by a vector field and is hence a spin 1 particle. However, because of gauge invariance, it has only 2 degrees of freedom (a general spin 1 particle has 3). A gravitational wave corresponds to a spin 2 particle[3], which in general can have 5 degrees of freedom, but in GR it has only 2, because of invariance under general coordinate transformations. In both cases, in the transverse gauge the field oscillates orthogonal to the direction of propagation. Let us now look more closely at these oscillations, i.e. at the **polarisation** of the gravitational wave.

The metric perturbation corresponding to a gravitational wave propagating in the z-direction with wavenumber k can be written as

$$h_{\alpha\beta}(t,\vec{x}) = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[k(z-t)] \ , \tag{8.79}$$

where the **polarisation modes** $h_+, h_\times$ depend on k, and the reason for the notation $+, \times$ will become clear soon.

8.2.2 Effect on matter

Let us now look at how gravitational waves affect matter. Consider massive test particles affected only by gravity. Their motion is given by the geodesic equation:

$$\begin{aligned} 0 &= u^\beta \nabla_\beta u^\alpha \\ &= \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma \ . \end{aligned} \tag{8.80}$$

Because the only non-zero metric perturbation is S_{ij}, and $S^i{}_i = 0$, $\partial_i S^{ij} = 0$, only connection coefficients with at least two spatial indices are non-zero, to first order in perturbation theory. (This can be checked explicitly from (8.28)–(8.33).) Therefore the combination $\Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma$ includes either one or two factors of $\dot{x}^i$. So if we consider particles that are initially at rest, their initial acceleration is zero, and they stay at rest.[4] By the same token, their proper time equals coordinate time, $x^0 = t$.

So the solution to the equation of motion for test particles initially at rest is simply $x^0 = t$, $x^i =$ constant. However, this does not mean they are unaffected by the gravitational wave. Their proper distance oscillates as the wave passes through, even though the coordinate distance is unaffected.[5]

Consider particles on the xy-plane ($z = 0$). For a gravitational wave with polar-

[3] This propagating perturbation of the spacetime could be called a **graviton**. However, more often the word graviton is only used to refer to a mode of the quantised gravitational field.

[4] A plane wave has infinite extension, so there is no time when it would not have have crossed our test particles. But real waves are wave packets with finite extension, and the same reasoning applies.

[5] Similarly, in the spatially homogeneous and isotropic FLRW model the proper distance between observers who stay at constant coordinate position increases or decreases as the universe expands or contracts.

isation h_+ (i.e. $h_\times = 0$), the metric is

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 + 2h_+ \cos(kt) & & \\ & & 1 - 2h_+ \cos(kt) & \\ & & & 1 \end{pmatrix} . \tag{8.81}$$

The proper distance at coordinate distance x in the x-direction is

$$\begin{aligned} L_x &= \int_0^x \mathrm{d}x \sqrt{g_{11}} \\ &\simeq \int_0^x \mathrm{d}x [1 + h_+ \cos(kt)] \\ &= x[1 + h_+ \cos(kt)] \ . \end{aligned} \tag{8.82}$$

Similarly, the proper distance in the y-direction is

$$L_y = y[1 - h_+ \cos(kt)] \ . \tag{8.83}$$

The area element on the xy-plane is unchanged, as $[1 + 2h_+ \cos(kz - kt)][1 - 2h_+ \cos(kz - kt)] \simeq 1$.

So a gravitational wave with polarisation h_+ expands and contracts the x- and y-directions concurrently in such a way that the area remains constant. This is the reason for the label $+$.

Finding the effect of a wave with polarisation $h_\times$ (i.e. in the case $h_+ = 0$) is a bit more complicated, because in the coordinate system we used, the metric is not diagonal. If we don't want to introduce more formalism (which would be overkill for our simple problem), we can first rotate the coordinate system so that the metric is diagonal, calculate the proper distance and then rotate back. Rotation of the xy-plane is the transformation

$$x^i \to x'^i = R^i{}_j x^j \ , \tag{8.84}$$

where the rotation matrix is

$$R^i{}_j(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{8.85}$$

Here the index i takes only the values 1 and 2. We could include the z-coordinate and use a 3×3 matrix, but since the action is on the xy-plane, this would only be extra notation with little benefit. As usual, the metric transforms with the inverse matrix, which in this case just has $-\theta$ in place of θ, so

$$\begin{aligned} g_{ij} \to g'_{ij} &= R^k{}_i(-\theta) R^l{}_j(-\theta) g_{kl} \\ &= R^k{}_j(-\theta) R^l{}_i(-\theta)(\delta_{kl} + h_{kl}) \\ &= \delta_{ij} + R^k{}_i(-\theta) R^l{}_j(-\theta) h_{kl} \end{aligned} \tag{8.86}$$

Let us write the transformation in matrix form, considering a general polarisation (and taking into account $R^i{}_j = (R^{-1})_j{}^i$). In matrix notation, $h \to RhR^{-1}$:

$$\begin{aligned}
&\begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \\
\to\ & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
=\ & \begin{pmatrix} (\cos^2\theta - \sin^2\theta)h_+ - 2\cos\theta\sin\theta h_\times & 2\cos\theta\sin\theta h_+ + (\cos^2\theta - \sin^2\theta)h_\times \\ 2\cos\theta\sin\theta h_+ + (\cos^2\theta - \sin^2\theta)h_\times & -(\cos^2\theta - \sin^2\theta)h_+ + 2\cos\theta\sin\theta h_\times \end{pmatrix} \\
=\ & \begin{pmatrix} \cos(2\theta)h_+ - \sin(2\theta)h_\times & \sin(2\theta)h_+ + \cos(2\theta)h_\times \\ \sin(2\theta)h_+ + \cos(2\theta)h_\times & -\cos(2\theta)h_+ + \sin(2\theta)h_\times \end{pmatrix} \\
=\ & \begin{pmatrix} -\sin(2\theta)h_\times & \cos(2\theta)h_\times \\ \cos(2\theta)h_\times & \sin(2\theta)h_\times \end{pmatrix} ,
\end{aligned} \tag{8.87}$$

where on the last line we have specialised to the situation $h_+ = 0$. We see that if $\cos(2\theta) = 0$ and $\sin(2\theta) = -1$, the $\times$ polarisation is transformed into the $+$ polarisation. This corresponds to $\theta = -\frac{\pi}{4}$.

In the transformed coordinate system, the proper length between the origin and the ring of dust particles in the x' and y' directions is

$$\begin{aligned}
L_{x'} &= x'[1 + h_\times \cos(kt)] \\
L_{y'} &= y'[1 - h_\times \cos(kt)] \ .
\end{aligned} \tag{8.88}$$

We now rotate the directions x' and y' back to the original directions with the inverse transformation, $R^i{}_j(-\theta) = R^i{}_j(\frac{\pi}{4})$

$$\begin{aligned}
x'^i &\to R^i{}_j\left(\frac{\pi}{4}\right)x'^j \\
&= \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} x' - y' \\ x' + y' \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} .
\end{aligned} \tag{8.89}$$

Combining this and (8.88), we find that the proper distance in the x-direction oscillates as

$$\begin{aligned}
L_x &= \frac{1}{\sqrt{2}}(L_{x'} - L_{y'}) \\
&= \frac{1}{\sqrt{2}}[1 + h_\times \cos(kt)]x' - \frac{1}{\sqrt{2}}[1 - h_\times \cos(kt)]y' \\
&= \frac{1}{\sqrt{2}}(x' - y') + \frac{1}{\sqrt{2}}(x' + y')h_\times \cos(kt) \\
&= x + y h_\times \cos(kt) \ .
\end{aligned} \tag{8.90}$$

Similarly, we get

$$L_y = y + x h_\times \cos(kt) \; . \tag{8.91}$$

Again, the area element is unchanged, as $\det(\delta_{ij} + h_{ij}) \simeq 1$ to first order. The $h_\times$ polarisation stretches the directions diagonally: the distance in the x-direction stretches/contracts as the y-coordinate grows, and vice versa. This is the reason for the label $\times$.

8.2.3 Generation

Let us now discuss how gravitational waves are sourced by matter. As we have to consider other components of the metric in addition to S_{ij}, it is more convenient to use a gauge other than the transverse gauge. Consider the Einstein tensor (8.41). We can simplify it by switching to a different variable, namely the **trace-reversed perturbation**

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta} \; . \tag{8.92}$$

It follows that $\bar{h} \equiv \eta^{\alpha\beta} \bar{h}_{\alpha\beta} = -h$, so

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \bar{h} \eta_{\alpha\beta} \; . \tag{8.93}$$

The relation between $\bar{h}_{\alpha\beta}$ and $h_{\alpha\beta}$ is the same as the relation between the Einstein tensor $G_{\alpha\beta}$ and the Ricci tensor $R_{\alpha\beta}$. In terms of $\bar{h}_{\alpha\beta}$, the Einstein tensor reads

$$G_{\alpha\beta} \simeq \frac{1}{2} \Big(\partial_\gamma \partial_\alpha \bar{h}^\gamma{}_\beta + \partial_\beta \partial_\gamma \bar{h}^\gamma{}_\alpha - \eta_{\alpha\beta} \partial_\gamma \partial_\delta \bar{h}^{\gamma\delta} - \Box \bar{h}_{\alpha\beta} \Big) \; , \tag{8.94}$$

i.e. we have eliminated terms proportional to $\eta_{\alpha\beta}$. So far, we have not made any gauge choice. If we impose the condition

$$\partial_\alpha \bar{h}^{\alpha\beta} = 0 \; , \tag{8.95}$$

then (8.94) simplifies to

$$G_{\alpha\beta} \simeq -\frac{1}{2} \Box \bar{h}_{\alpha\beta} \; , \tag{8.96}$$

The coordinate system where condition (8.95) holds is called the **Lorenz gauge** after the similar gauge in electromagnetism. Let us show that it is possible to impose this condition. As the Lorenz gauge condition is written in terms of the full perturbation $h_{\alpha\beta}$ rather than the components split into scalar, vector and tensor parts, it is easier to consider the gauge condition covariantly rather then in terms of the irreducible variables. Under the gauge transformation (8.15), $\bar{h}_{\alpha\beta}$ transforms as

$$\bar{h}_{\alpha\beta} \to \bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta} \xi^\gamma{}_{,\gamma} \; . \tag{8.97}$$

So, let us assume that we are in a gauge where $\partial_\alpha \bar{h}^{\alpha\beta}$ is arbitrary and show that there exists a choice of ξ^α that brings it to zero:

$$\partial_\alpha \bar{h}^{\alpha\beta} \to \partial_\alpha \bar{h}'^{\alpha\beta} = \partial_\alpha \bar{h}^{\alpha\beta} - \Box \xi^\beta = 0 \; . \tag{8.98}$$

The equation $\Box\xi^\beta = \partial_\alpha \bar{h}^{\alpha\beta}$ always has a solution, so we can choose the Lorenz gauge. In contrast to the transverse gauge condition, the Lorenz gauge condition does not fix the gauge completely. We can still shift the gauge functions as $\xi^\alpha \to \xi^\alpha + \zeta^\alpha$ as long as $\Box\zeta^\alpha = 0$.

We are now left with only one term in the Einstein tensor (8.94), and the Einstein equation reads

$$\Box\bar{h}_{\alpha\beta} = -16\pi G_{\rm N} T_{\alpha\beta} \ . \tag{8.99}$$

Note the consistency of the gauge condition (8.95) with the continuity equation $\partial_\alpha T^{\alpha\beta} = 0$. This equation can be solved as readily as the Poisson equation, with the result

$$\bar{h}^{\alpha\beta}(t, \mathbf{x}) = 4G_{\rm N} \int {\rm d}^3x' \frac{T^{\alpha\beta}(t_{\rm ret}, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{8.100}$$

where $t_{\rm ret} \equiv t - |\mathbf{x}-\mathbf{x}'|$ is the **retarded time**. We could add an arbitrary solution of the homogeneous wave equation, but we are here interested in the waves generated by matter, and so demand that the metric perturbation vanishes when the energy-momentum tensor vanishes.

In contrast to the solutions of the Poisson equation, here the integration is not over a spatial hypersurface of constant time, but over the past lightcone. The gravitational wave signal at our location is determined by the sources with a time lag corresponding to the distance times the speed of light. As far as the equations are concerned, we could equally well use the **advanced time** $t_{\rm adv} \equiv t + |\mathbf{x}-\mathbf{x}'|$ in the solution, corresponding to integrating over the future lightcone. As usual, we discard such signals that move backward in time, demanding that null geodesics can only be travelled in one direction. (As noted earlier, we do not know why this the case in GR any more than in electromagnetism.)

Let us now simplify (8.100). We first show that we need to consider only the ij components, because the others can be solved from the gauge condition (8.95). The i components of the gauge condition are

$$0 \ = \ \partial_\alpha \bar{h}^{\alpha i} = \dot{\bar{h}}^{0i} + \partial_j \bar{h}^{ji} \ , \tag{8.101}$$

from which we solve

$$\bar{h}^{0i}(t, \vec{x}) \ = \ a^i(\vec{x}) - \int^t {\rm d}t' \partial_j \bar{h}^{ji}(t', \vec{x}) \ , \tag{8.102}$$

where $a^i(\vec{x})$ is an arbitrary function of the spatial coordinates. The 0 component of the gauge condition gives

$$0 \ = \ \partial_\alpha \bar{h}^{\alpha 0} = \dot{\bar{h}}^{00} + \partial_j \bar{h}^{j0} \ , \tag{8.103}$$

so we get

$$\begin{aligned} \bar{h}^{00}(t, \vec{x}) \ &= \ a(\vec{x}) - \int^t {\rm d}t' \partial_j \bar{h}^{j0}(t', \vec{x}) \\ &= \ a(\vec{x}) - t\partial_i a^i(\vec{x}) + \int^t {\rm d}t' \partial_i \partial_j \bar{h}^{ij}(t', \vec{x}) \ , \end{aligned} \tag{8.104}$$

where $a(\vec{x})$ is an arbitrary function of the spatial coordinates.

We see that the gauge condition does not quite allow us to determine the components $\bar{h}_{0\alpha}$ from $\bar{h}_{ij}$. One possibility is that the functions a and a^i might correspond to gauge degrees of freedom – recall that we are still allowed to do gauge transformations where the gauge function ξ^α satisfies $\Box\xi^\alpha = 0$. However, with such transformations we cannot get rid of a and a^i, so in general they correspond to physical degrees of freedom. This is simple to see: the relation between a^i and ξ^α is linear, so $\Box\xi^\alpha = 0$ means we can only eliminate a^i if $\Box a^i = \nabla^2 a^i = 0$, which is only possibly if $a^i = 0$, and similarly for a. So a and a^i are physical parts of the solution. However, because a and a^i depend only on the spatial coordinates, they cannot describe waves. Because we are considering wave solutions, we can set them to zero.

Having established it's enough to consider $\bar{h}^{ij}$, let us simplify the expression for it. First, we assume that the source region where $T^{ij} \neq 0$ has compact support, and its size is small compared to its distance from the observation point $\vec{x}$. This allows us to write

$$\begin{aligned} \bar{h}^{ij}(t,\mathbf{x}) &= 4G_\mathrm{N} \int \mathrm{d}^3x' \frac{T^{ij}(t_\mathrm{ret},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &\approx \frac{4G_\mathrm{N}}{r} \int \mathrm{d}^3x' T^{ij}(t_\mathrm{ret},\mathbf{x}') \ , \end{aligned} \tag{8.105}$$

where r is the distance from the observer at $\vec{x}$ to the centre of the observation region. This approximation is excellent for all sources of gravitational waves that can be observed in the foreseeable future. They are at astrophysical or cosmological distances from us[6], and the sources have sizes comparable to the radii of compact stellar remnants or Schwarzschild radii of black holes (kilometers to billions of kilometers).

We can further simplify the expression by replacing the spatial components with time components using the continuity equation $\partial_\alpha T^{\alpha\beta} = 0$. The procedure is similar to writing $\bar{h}_{0\alpha}$ in terms of $\bar{h}_{ij}$ above, here we just go in reverse. Consider the integral

$$\int \mathrm{d}^3x (T^{ik}x^j)_{,k} = \int \mathrm{d}^3x T^{ik}{}_{,k} x^j + \int \mathrm{d}^3x T^{ij} \ . \tag{8.106}$$

The left-hand side is a total derivative, so it can be written as a boundary integral via Gauss' theorem. As the source region is compact, the boundary term vanishes, and we get

$$\begin{aligned} \int \mathrm{d}^3x T^{ij} &= -\int \mathrm{d}^3x T^{ik}{}_{,k} x^j \\ &= \int \mathrm{d}^3x T^{i0}{}_{,0} x^j \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3x T^{i0} x^j \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3x \left(T^{i0}x^j + T^{j0}x^i \right) \ , \end{aligned} \tag{8.107}$$

6 The closest source of directly detected gravitational waves so far, event GW170817, was about 100 million light years from us. In the next decade, we will be able to detect gravitational waves from sources only some thousands of light years away.

where on the second line we have applied $\partial_\alpha T^{\alpha i} = \partial_0 T^{0i} + \partial_j T^{ji} = 0$, and on the last line used the fact that since T^{ij} is symmetric, we can explicitly symmetrise the left-hand side. We can now repeat the same trick to get rid of the remaining spatial index in the energy-momentum tensor:

$$\int \mathrm{d}^3x \left(T^{0k} x^i x^j\right)_{,k} = \int \mathrm{d}^3x T^{0k}{}_{,k} x^i x^j + \int \mathrm{d}^3x \left(T^{0i} x^j + T^{0j} x^i\right) . \tag{8.108}$$

The left-hand side vanishes for the same reason as before, so the right-hand side gives

$$\begin{aligned} \int \mathrm{d}^3x \left(T^{0i} x^j + T^{0j} x^i\right) &= -\int \mathrm{d}^3x T^{0k}{}_{,k} x^i x^j \\ &= \int \mathrm{d}^3x T^{00}{}_{,0} x^i x^j , \end{aligned} \tag{8.109}$$

where on the second line we have used $\partial_\alpha T^{\alpha 0} = \partial_0 T^{00} + \partial_k T^{k0} = 0$. Inserting (8.109) into (8.107), we get

$$\int \mathrm{d}^3x T^{ij} = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \mathrm{d}^3x T^{00} x^i x^j . \tag{8.110}$$

For sources moving at non-relativistic velocities, $T^{00} \approx \rho$. Inputting this result into (8.105), we get the final expression for the metric perturbation:

$$\bar{h}^{ij}(t, \mathbf{x}) = \frac{2G_\mathrm{N}}{r} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \mathrm{d}^3x' \rho(t, \mathbf{x}') x'^i x'^j \Big|_{t=t_\mathrm{ret}} . \tag{8.111}$$

This is the **quadrupole formula** for gravitational radiation.

8.2.4 Sourced by two rotating masses

As an example, let us consider a source that consists of two masses M moving at constant angular velocity ω on opposite sides of a circle with radius a, illustrated in figure 2. This simple setup could model a physical object, like a dumbbell, or an astrophysical binary system. It gives a good first approximation for the amplitude and frequency of the gravitational waves emitted by real black hole binary systems during their inspiral phase.[7]

Let us choose coordinates so that the masses move on the xy-plane, with positions

$$\vec{x}_n(t) = (-1)^n a(\cos[\omega t], \sin[\omega t], 0) , \tag{8.112}$$

where $n = 1, 2$ labels the masses. The energy density is thus, modelling the masses as point particles (a fair approximation as long as a is much larger than their size)

$$\rho(t, \vec{x}) = M \sum_{n=1}^{2} \delta[\vec{x} - \vec{x}_n(t)] . \tag{8.113}$$

[7] The eccentricity of a black hole binary system decreases faster than its energy, so circularity is a good approximation for mature black hole binaries.

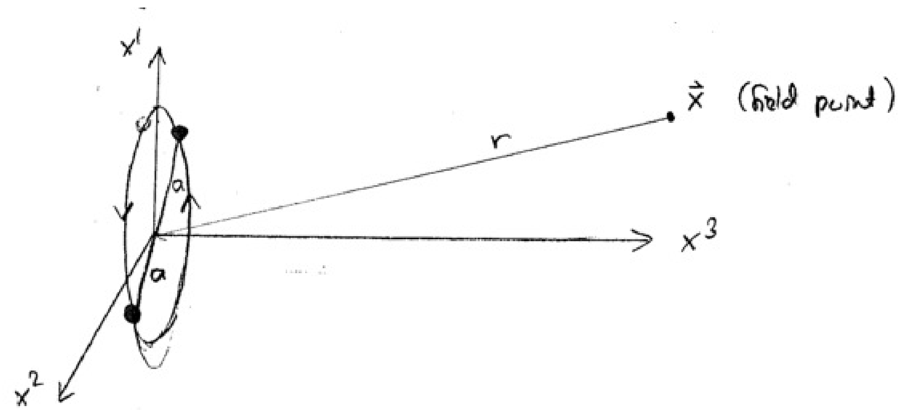

Figure 2: Binary system setup.

The quadrupole formula (8.111) gives

$$\begin{aligned}
\bar{h}^{ij}(t,\mathbf{x}) &= \frac{2G_{\rm N}M}{r}\frac{{\rm d}^2}{{\rm d}t^2}\sum_{n=1}^{2}x_n^i x_n^j\Big|_{t=t_{\rm ret}} \\
&= \frac{4G_{\rm N}M}{r}\frac{{\rm d}^2}{{\rm d}t^2}(x_1^i x_1^j)\Big|_{t=t_{\rm ret}} \\
&= \frac{4G_{\rm N}Ma^2}{r}\frac{{\rm d}^2}{{\rm d}t^2}\begin{pmatrix}\cos^2(\omega t) & \sin(\omega t)\cos(\omega t) & 0\\ \sin(\omega t)\cos(\omega t) & \sin^2(\omega t) & 0\\ 0 & 0 & 0\end{pmatrix}\Bigg|_{t=t_{\rm ret}} \\
&= \frac{2G_{\rm N}Ma^2}{r}\frac{{\rm d}^2}{{\rm d}t^2}\begin{pmatrix}1+\cos(2\omega t) & \sin(2\omega t) & 0\\ \sin(2\omega t) & 1-\cos(2\omega t) & 0\\ 0 & 0 & 0\end{pmatrix}\Bigg|_{t=t_{\rm ret}} \\
&= -\frac{8G_{\rm N}Ma^2\omega^2}{r}\begin{pmatrix}\cos(2\omega t) & \sin(2\omega t) & 0\\ \sin(2\omega t) & -\cos(2\omega t) & 0\\ 0 & 0 & 0\end{pmatrix}\Bigg|_{t=t_{\rm ret}} \\
&= -\frac{8G_{\rm N}Ma^2\omega^2}{r}\begin{pmatrix}\cos[2\omega(t-r)] & \sin[2\omega(t-r)] & 0\\ \sin[2\omega(t-r)] & -\cos[2\omega(t-r)] & 0\\ 0 & 0 & 0\end{pmatrix}\ , \qquad (8.114)
\end{aligned}$$

where $r = |\vec{x}|$.

We cannot straightforwardly apply the interpretation of gravitational waves discussed in (8.2.2) to this result, because there we used the transverse gauge ($\partial_i S^{ij} = 0$) and here we have used the Lorenz gauge ($\partial_\alpha \bar{h}^{\alpha\beta} = 0$). We have

$$\begin{aligned}
\partial_i S^{ij} &= \partial_i h^{ij} \\
&= \partial_i \bar{h}^{ij} - \frac{1}{2}\delta^{ij}\partial_i \bar{h}\ . \qquad (8.115)
\end{aligned}$$

If the observer is on the z-axis, $\partial_i \bar{h}^{ij} = 0$. It then follows from Lorenz gauge condition that $\bar{h}_{0\alpha} = 0$, as we see from (8.102) and (8.104). Because the solution (8.114) satisfies $\bar{h}^i{}_i = 0$, it follows that $\partial_i S^{ij} = 0$. So on the z-axis we can identify

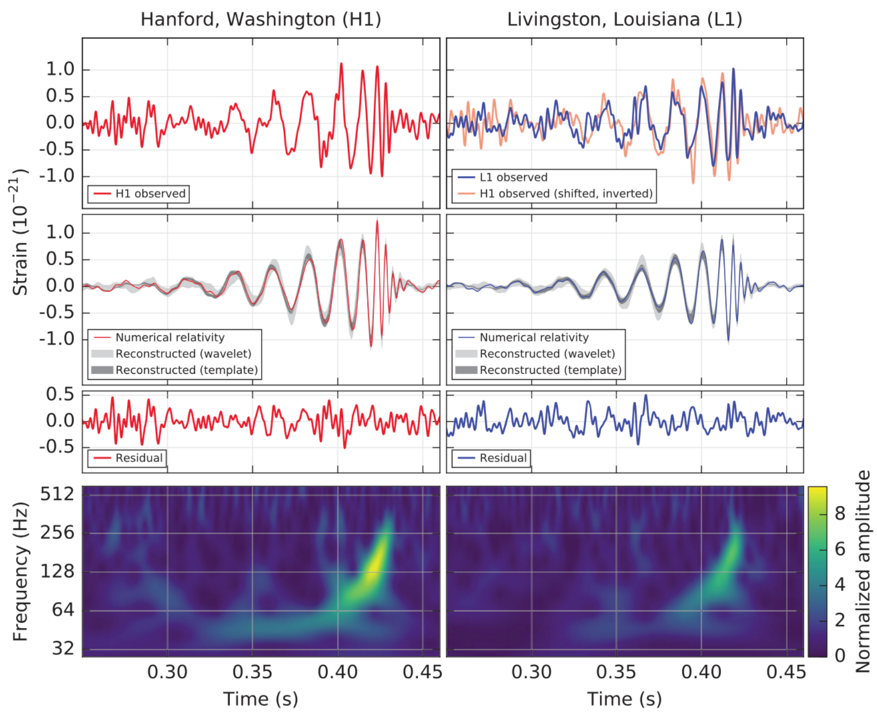

Figure 3: The first direct gravitational wave signal detected. The upper two panels show the amplitude as a function of time at the two observation sites at Hanford and Livingston, the third panel shows the residual after subtracting the model, and the bottom panels show the frequency as a function of time. (Source: https://arxiv.org/abs/1602.03837).

(8.114) as a plane wave with angular frequency 2ω, amplitude $\frac{4G_N M a^2 \omega^2}{r}$ and right-handed circular polarisation. For an observer off the z-axis, we have to do a gauge transformation from the Lorenz gauge to the transverse gauge.

Exercise. Show that the wave (8.114) is circularly polarised, i.e. particles at constant coordinate position on the xy-plane move in a circle in the clockwise direction as seen from the direction of the positive z-axis.

The first direct gravitational wave detection was made by the LIGO experiment on September 14 2015. The physical idea is simple: a light wave is split in two and sent down two 4 km long tunnels that are orthogonal to each other, reflected back at the ends, and then the phases of the waves are compared. Interferometry allows for extremely accurate measurement of changes in length (not the length itself) along the two tunnels. If gravitational wave passes through, the lengths of the two tunnels change in a different way. To make sure of the detection, there are two detectors 3 000 km apart, which also allows to measure the time lag between gravitational wave detection (varying from zero to 0.01 seconds, depending on the direction of the wave). The source of the first gravitational wave was a black hole binary system. Each black hole had mass $30M_\odot$, and the distance to the system was 1 billion light years. The signal is shown in figure 3.

Exercise. Find the frequency and the amplitude of the gravitational waves emitted by the system detected by LIGO as a function of λ (defined below). Take the black holes to be on a circular Newtonian orbit with radius $r = \lambda r_s$, where r_s is the Schwarzschild radius and $\lambda > 1$. Take the orbit to be on the xy-plane, its center to be at the origin and the observer to be on the z-axis. Approximate the black holes as pointlike and non-rotating, and neglect the expansion of the universe.

In 2017, a Nobel prize was awarded for the LIGO detection. The observation started a boom of gravitational wave studies both on the theoretical and observational side. As we mentioned in chapter, the fourth observational round of the combined LIGO-Virgo-KAGRA gravitational wave detector network is ongoing, with 10-15 candidate detections per month. The next generation experiment LISA will consist of a trio of satellites 2.5 million km apart, co-orbiting the Sun with the Earth. LISA is due to be launched in 2035 by the European Space Agency ESA in collaboration with the United States space agency NASA.

However, before direct detection, gravitational waves had already been observed indirectly from the energy loss due to their emission. Let us calculate it.

8.2.5 Energy-momentum pseudotensor

Gravitational waves are a first order perturbation of the metric. Their effect on the system emitting them is a second order effect. Second order perturbation theory can get a bit involved, but we will be able to avoid most complications, as we are mainly interested in finding the effective energy-momentum tensor of the gravitational waves. The energy it carries is then equal to the energy lost by the system. If we can deduce the amount of energy lost by the system from observing its dynamics, we can determine the energy of the emitted waves, which fixes a combination of their amplitude and frequency.

We have already remarked that we treat metric perturbations as fields living in Minkowski space. From this point of view, the gravitational wave is just a spin 2 field, which carries energy like all matter fields. However, because of coordinate invariance, it is impossible to localise the energy carried by a gravitational wave. At any point in spacetime, we can make the metric flat and its first derivative zero. To find the energy of gravitational waves, we have to average over several wavelengths. Let us first derive the local quantity that we would like to identify as the energy-momentum tensor and then average it.

Recall our perturbative calculation of Mercury's orbit and bending of light by the Sun. We first solved for the background, and then the background solution provided a source term for the first order perturbation. Here we have a similar setup, but at second order: the first order perturbation provides a source for the second order perturbation. The metric is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \eta_{\alpha\beta} + h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta} \ , \tag{8.116}$$

where $h^{(1)}_{\alpha\beta}$ is the first order perturbation we have been dealing with so far, and $h^{(2)}_{\alpha\beta}$ is the perturbation it sources; $1 \gg |h^{(1)}_{\alpha\beta}| \gg |h^{(2)}_{\alpha\beta}|$. Correspondingly, the Ricci tensor is

$$R_{\alpha\beta} \simeq \underbrace{R^{(1)}_{\alpha\beta}(h^{(1)}_{\mu\nu})}_{=0} + \underbrace{R^{(1)}_{\alpha\beta}(h^{(2)}_{\mu\nu}) + R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu})}_{=0} , \qquad (8.117)$$

where we have not written down the background Ricci tensor, as it is trivially zero. The first term $R^{(1)}_{\alpha\beta}(h^{(1)}_{\mu\nu})$ is the part of the Ricci tensor that is linear in the perturbation (given in (8.39)), evaluated with the first order perturbation $h^{(1)}_{\mu\nu}$. The second term $R^{(1)}_{\alpha\beta}(h^{(2)}_{\mu\nu})$ is the same linear Ricci tensor (8.39), but evaluated with the second order perturbation $h^{(2)}_{\mu\nu}$. Finally, $R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu})$ is the part of the Ricci tensor that is quadratic in the perturbation, evaluated with the first order perturbation $h^{(1)}_{\mu\nu}$.

We solve the Einstein equation $R_{\alpha\beta} = 0$ order by order. The first order equation is the same as before, $R^{(1)}_{\alpha\beta}(h^{(1)}_{\mu\nu}) = 0$, and gives the solutions we have discussed. The second order equation has two terms whose sum is zero, so we have

$$R^{(1)}_{\alpha\beta}(h^{(2)}_{\mu\nu}) = -R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu}) . \qquad (8.118)$$

In terms of the Einstein tensor, this reads

$$\begin{aligned} G^{(1)}_{\alpha\beta}(h^{(2)}_{\mu\nu}) &= R^{(1)}_{\alpha\beta}(h^{(2)}_{\mu\nu}) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}R^{(1)}_{\gamma\delta}(h^{(2)}_{\mu\nu}) \\ &= -R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu}) + \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}R^{(2)}_{\gamma\delta}(h^{(1)}_{\mu\nu}) \\ &\equiv 8\pi G_{\mathrm{N}} t_{\alpha\beta} , \end{aligned} \qquad (8.119)$$

where on the last line we have defined the **energy-momentum pseudotensor** $t_{\alpha\beta}$,

$$t_{\alpha\beta} = -\frac{1}{8\pi G_{\mathrm{N}}}\left[R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu}) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}R^{(2)}_{\gamma\delta}(h^{(1)}_{\mu\nu})\right] . \qquad (8.120)$$

The second order perturbation $h^{(2)}_{\mu\nu}$ is sourced by the square of the first order perturbation, and we treat the the source term as if it was an energy-momentum tensor of matter. Physically we have the small first order ripples in spacetime, which lead to smaller second order ripples due to the non-linearity of the Einstein equation. The division we have made between the genuinely second order term and the square of the first order terms is not gauge-invariant. In general, terms that are the square of a first order term in one gauge may be part of the linear second order term in another gauge. Also, recall that $t_{\alpha\beta}$ is not a tensor in the full spacetime, as the split between background and perturbations is gauge-dependent. Only the sum $g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta}$ is a tensor, not the individual parts.

Nevertheless, in a given gauge, $t_{\alpha\beta}$ functions like an energy-momentum tensor in sourcing $h^{(2)}_{\mu\nu}$, and also (after averaged over several wavelengths) in determining the energy carried by gravitational waves.

We get $R^{(2)}_{\alpha\beta}(h_{\mu\nu})$ by inserting $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ into the definition of the Ricci tensor and expanding to second order. There are quite a few terms, but the expression simplifies considerably by choosing a convenient gauge. We adopt the Lorenz gauge $\partial_\alpha \bar{h}^{\alpha\beta} = 0$, where $\bar{h}_{\alpha\beta}$ contains both the first and the second order perturbation. In vacuum, we can simultaneously impose the gauge conditions $\bar{h}^{(1)} = 0, \bar{h}^{(1)}_{\alpha 0} = 0$. (**Exercise.** Show this.) We saw that for the binary system we considered, this happens automatically in the Lorenz gauge, but it applies more generally. In these coordinates, we get the rather simple result (note that to first order, $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$)

$$\begin{aligned} R^{(2)}_{\alpha\beta}(h_{\mu\nu}) &= \frac{1}{2} h^{\mu\nu}\partial_\alpha\partial_\beta h_{\mu\nu} + \frac{1}{4}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} + \partial^\mu h^\nu{}_\alpha \partial_{[\mu} h_{\nu]\beta} \\ &\quad - h^{\mu\nu}\partial_\mu\partial_{(\alpha} h_{\beta)\nu} + \frac{1}{2}\partial_\mu (h^{\mu\nu}\partial_\nu h_{\alpha\beta}) \end{aligned} \tag{8.121}$$

Consider the continuity equation. In order to interpret $t_{\alpha\beta}$ as an effective energy-momentum tensor, $t_{\alpha\beta}$ should vanish when contracted with the covariant derivative. Let us check this.

$$\begin{aligned} 0 &= \nabla_\alpha G^{\alpha\beta} \\ &= \partial_\alpha G^{\alpha\beta} + \Gamma^\alpha_{\alpha\mu} G^{\mu\beta} + \Gamma^\beta_{\alpha\mu} G^{\alpha\mu} \\ &\simeq \partial_\alpha G^{\alpha\beta} \\ &= \partial_\alpha G^{(1)\alpha\beta}(h^{(1)}_{\mu\nu}) + \partial_\alpha G^{(1)\alpha\beta}(h^{(2)}_{\mu\nu}) + \partial_\alpha G^{(2)\alpha\beta}(h^{(1)}_{\mu\nu}) \\ &= \partial_\alpha G^{(1)\alpha\beta}(h^{(2)}_{\mu\nu}) + \partial_\alpha G^{(2)\alpha\beta}(h^{(1)}_{\mu\nu}) \ , \end{aligned} \tag{8.122}$$

where on the third line we have taken into account that as the background Einstein tensor as well as $G^{(1)\alpha\beta}(h^{(1)}_{\mu\nu})$ vanishes, there are no cross terms with the connection coefficients at second order. Applying (8.119), we have

$$\begin{aligned} \partial_\alpha t^{\alpha\beta} &= \frac{1}{8\pi G_{\rm N}}\partial_\alpha G^{(1)\alpha\beta}(h^{(2)}_{\mu\nu}) \\ &= -\frac{1}{16\pi G_{\rm N}}\partial_\alpha \Box \bar{h}^{\alpha\beta} \\ &= 0 \ , \end{aligned} \tag{8.123}$$

where on the second line we have used the result $G^{(1)}_{\alpha\beta}(h_{\mu\nu}) = -\frac{1}{2}\Box\bar{h}_{\mu\nu}$ given in (8.96), and on the third line we have applied the Lorenz gauge condition.

The only thing left is to average $t_{\alpha\beta}$ to get something that corresponds to measured energy and momentum flow. We will not go into the details of the averaging process. The important thing is that as the wave is periodic (and for real systems, contained in a tube of finite spatial width), any total derivatives can be transformed into boundary terms that vanish. Let us denote the averaging by $\langle\rangle$. Averaging

$R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu})$, we have from (8.121)

$$\begin{aligned}\langle R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu})\rangle &= \frac{1}{2}\langle h^{(1)\mu\nu}\partial_\alpha\partial_\beta h^{(1)}_{\mu\nu}\rangle + \frac{1}{4}\langle\partial_\alpha h^{(1)\mu\nu}\partial_\beta h^{(1)}_{\mu\nu}\rangle + \langle\partial^\mu h^{(1)\nu}{}_\alpha\partial_{[\mu}h^{(1)}_{\nu]\beta}\rangle \\ &\quad -\langle h^{(1)\mu\nu}\partial_\mu\partial_{(\alpha}h^{(1)}_{\beta)\nu}\rangle \\ &= -\frac{1}{2}\langle\partial_\alpha h^{(1)\mu\nu}\partial_\beta h^{(1)}_{\mu\nu}\rangle + \frac{1}{4}\langle\partial_\alpha h^{(1)\mu\nu}\partial_\beta h^{(1)}_{\mu\nu}\rangle - \langle h^{(1)\nu}{}_\alpha\partial^\mu\partial_{[\mu}h^{(1)}_{\nu]\beta}\rangle \\ &\quad +\langle\partial_\mu h^{(1)\mu\nu}\partial_{(\alpha}h^{(1)}_{\beta)\nu}\rangle \\ &= -\frac{1}{4}\langle\partial_\alpha h^{(1)\mu\nu}\partial_\beta h^{(1)}_{\mu\nu}\rangle\ , \end{aligned} \tag{8.124}$$

where we have used partial integrations, taking into account that total derivatives can be discarded, and used the equation of motion $\Box h^{(1)}_{\mu\nu} = 0$ and the gauge condition $\partial_\mu h^{(1)\mu\nu} = 0$. Let us calculate the corresponding Ricci scalar,

$$\begin{aligned}\langle R^{(2)}(h^{(1)}_{\mu\nu})\rangle &= \langle\eta^{\alpha\beta}R^{(2)}_{\alpha\beta}(h^{(1)}_{\mu\nu})\rangle \\ &= -\frac{1}{4}\langle\eta^{\alpha\beta}\partial_\alpha h^{(1)\mu\nu}\partial_\beta h^{(1)}_{\mu\nu}\rangle \\ &= \frac{1}{4}\langle\eta^{\alpha\beta}\partial_\alpha\partial_\beta h^{(1)\mu\nu}h^{(1)}_{\mu\nu}\rangle \\ &= 0\ , \end{aligned} \tag{8.125}$$

where we have again used $\Box h^{(1)}_{\mu\nu} = 0$. So we get the simple result that the energy and momentum carried by a gravitational wave are described by the pseudotensor

$$\langle t_{\alpha\beta}\rangle = \frac{1}{32\pi G_N}\langle\partial_\alpha h^{(1)\mu\nu}\partial_\beta h^{(1)}_{\mu\nu}\rangle\ . \tag{8.126}$$

As an example, consider a monochromatic wave described by the perturbation

$$h^{(1)}_{\alpha\beta} = s_{\alpha\beta}\cos(k_\gamma x^\gamma)\ . \tag{8.127}$$

Choosing coordinates such that the z-axis points in the $\vec{k}$ direction, we have $k^\alpha = (\omega, 0, 0, \omega)$, i.e. $k_\alpha = (-\omega, 0, 0, \omega)$, so

$$\begin{aligned}\langle t_{\alpha\beta}\rangle &= \frac{1}{32\pi G_N}s_{\mu\nu}s^{\mu\nu}k_\alpha k_\beta\langle\sin^2(k_\gamma x^\gamma)\rangle \\ &= \frac{\omega^2}{64\pi G_N}s_{\mu\nu}s^{\mu\nu}\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}\ , \end{aligned} \tag{8.128}$$

where on the last line we have taken into account that the average of $\sin^2 x$ over a period is $\frac{1}{2}$. This has the same form as the energy-momentum tensor of monochromatic electromagnetic radiation.

8.2.6 Energy loss

Now that we have the energy-momentum pseudotensor in hand, we can find how much energy a gravitational wave source emits. The power radiated per unit time t through a spherical surface S at distance r away from the source is the integral of $q_\alpha n^\alpha$ over the surface, where q_α is the energy flux and n^α is the radial unit vector normal to the surface. Looking at the decomposition of the energy-momentum tensor (8.69) (or equation (4.10)) we get, taking into account $u_\alpha n^\alpha = 0$, $q_\alpha n^\alpha = -T_{\alpha\beta} u^\alpha n^\beta$. We now have $\langle t_{\alpha\beta} \rangle$ in place of $T_{\alpha\beta}$, so the rate of energy passing through a spherical shell of radius r is

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= -\int_S \mathrm{d}S \langle t_{\alpha\beta} \rangle u^\alpha n^\beta \\ &= -r^2 \int \mathrm{d}\Omega \langle t_{0r} \rangle \\ &= -\frac{r^2}{32\pi G_\mathrm{N}} \int \mathrm{d}\Omega \langle \partial_0 h^{(1)\mu\nu} \partial_r h^{(1)}_{\mu\nu} \rangle \ , \end{aligned} \tag{8.129}$$

where we have input the result (8.126) for $\langle t_{\alpha\beta} \rangle$. Now we want to express the gravitational wave in terms of the source properties. The quadrupole formula (8.111) can be written as

$$h^{(1)}_{ij}(t, \mathbf{x}) = \frac{2G_\mathrm{N}}{r} \ddot{J}^{TT}_{ij}(t, \mathbf{x}) \ , \tag{8.130}$$

where J^{TT}_{ij} is the traceless transverse part of the inertial tensor I_{ij},

$$J_{ij} \equiv I_{ij} - \frac{1}{3} \delta_{ij} \delta^{kl} I_{kl} \ , \tag{8.131}$$

where

$$I_{ij} = \int \mathrm{d}^3 x' \rho(t, \mathbf{x}') x'^i x'^j \Big|_{t = t_\mathrm{ret}} \ . \tag{8.132}$$

There are three differences compared to our earlier result (8.111): we have replaced $\bar{h}_{ij}$ with h_{ij}, taken out the trace of J_{ij} and added to it the superscript TT, for traceless and transverse. These additions are due to the extra conditions we imposed when deriving $t_{\alpha\beta}$, namely $h^{(1)} = 0$ and $h^{(1)}_{0\alpha} = 0$. Because of them, $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$, and $h^{(1)}_{ij}$ is traceless and transverse, $\partial_i h^{(1)ij} = 0$.

Before looking at the transverse condition, let us simplify (8.129). We have

$$\begin{aligned} \partial_r h^{(1)}_{ij} &= -\frac{2G_\mathrm{N}}{r^2} \ddot{J}^{TT}_{ij} + \frac{2G_\mathrm{N}}{r} \partial_r \ddot{J}^{TT}_{ij} \\ &= -\frac{2G_\mathrm{N}}{r^2} \ddot{J}^{TT}_{ij} - \frac{2G_\mathrm{N}}{r} \dddot{J}^{TT}_{ij} \\ &\approx -\frac{2G_\mathrm{N}}{r} \dddot{J}^{TT}_{ij} \ , \end{aligned} \tag{8.133}$$

where on the second line we have used the fact that for any function f we have $\partial_r f(t - r) = -\partial_0 f(t - r)$, and on the last line we have assumed that $1/r$ gives a stronger suppression than an extra time derivative (for real sources, this is an excellent approximation).

Putting these results into (8.129), we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G_{\mathrm{N}}}{8\pi}\int \mathrm{d}\Omega \langle \dddot{J}^{TTij}\dddot{J}^{TT}_{ij}\rangle \ , \tag{8.134}$$

Now we should implement the traceless transverse projection and take the integral over the solid angle. The wave propagates in the radial direction, so the transverse condition $\partial_i h^{(1)ij} = 0$ reduces to $h^{(1)}_{ir} = 0$. This can be implemented with the projection operator

$$P_{\alpha\beta} = \eta_{\alpha\beta} - n_\alpha n_\beta \ , \tag{8.135}$$

We have earlier used the tensor $g_{\alpha\beta} + u_\alpha u_\beta$ to project orthogonally to the time direction u^α. The tensor $P_{\alpha\beta}$ similarly projects orthogonally to the radial direction n^α. (Note that $n^i = x^i/|\vec{x}|$.) It satisfies

$$P^\alpha{}_\gamma P^\gamma{}_\beta = P^\alpha{}_\beta \ . \tag{8.136}$$

We only need the spatial part, $P_{ij} = \delta_{ij} - n_i n_j$. The traceless transverse projection of a spatial tensor A_{ij} is

$$A^{TT}_{ij} = P^k{}_i P^l{}_j A_{kl} - \frac{1}{2} P_{ij} P^{kl} A_{kl} \ . \tag{8.137}$$

It is necessary to subtract the trace part even in the case of tensors that are explicitly traceless, because the projection could otherwise spoil the property. The contraction of two transverse traceless projected tensors is

$$\begin{aligned} A^{TT}_{ij} A^{TTij} &= \left(P^k{}_i P^l{}_j - \frac{1}{2} P_{ij} P^{kl}\right) A_{kl} \left(P^i{}_m P^j{}_n - \frac{1}{2} P_{mn} P^{ij}\right) A^{mn} \\ &= \left(P^k{}_m P^l{}_n + \frac{1}{4} P_{ij} P^{ij} P^{kl} P_{mn} - P^{kl} P_{mn}\right) A_{kl} A^{mn} \\ &= \left(P_{km} P_{ln} - \frac{1}{2} P_{kl} P_{mn}\right) A^{kl} A^{mn} \ , \end{aligned} \tag{8.138}$$

where we have on the last line taken into account $P_{ij}P^{ij} = 2$. Averaging this over all directions, we have (assuming that A_{ij} does not depend on direction, which is a good approximation far away from the source)

$$\begin{aligned} & \frac{1}{4\pi}\int \mathrm{d}\Omega A^{TT}_{ij} A^{TTij} \\ = \ & A^{kl} A^{mn} \frac{1}{4\pi} \int \mathrm{d}\Omega \left[(\delta_{km} - n_k n_m)(\delta_{ln} - n_l n_n) - \frac{1}{2}(\delta_{kl} - n_k n_l)(\delta_{mn} - n_m n_n)\right] \\ = \ & A^{kl} A^{mn} \frac{1}{4\pi} \int \mathrm{d}\Omega \left[\delta_{km}\delta_{ln} - \delta_{km} n_l n_n - \delta_{ln} n_k n_m + \frac{1}{2} n_k n_l n_m n_n\right] \ , \end{aligned} \tag{8.139}$$

where we have on the last line assumed that A is traceless.

The average of $n_i n_j$ over the unit sphere is zero unless $i = j$. (This is easy to see by parity: any component n_i has the opposite value on opposite sides of the hemisphere.) If $i = j$, all components give the same result by rotational symmetry, so $\frac{1}{4\pi}\int \mathrm{d}\Omega n_i n_j = \frac{1}{3}\delta_{ij}$, given that $\delta^{ij} n_i n_j = 1$. By similar reasoning, average over $n_k n_l n_m n_n$ gives zero unless all indices are paired off, and since all directions are equivalent, we have

$$\frac{1}{4\pi}\int \mathrm{d}\Omega n_k n_l n_m n_n = N(\delta_{kl}\delta_{mn} + \delta_{kn}\delta_{lm} + \delta_{km}\delta_{ln}) \ , \tag{8.140}$$

where N is a constant. Contracting with $\delta_{kl}\delta_{mn}$ gives 1 on the left-hand side and $15N$ on the right-hand side, so $N = \frac{1}{15}$. With these results, (8.139) gives

$$\frac{1}{4\pi}\int \mathrm{d}\Omega A^{TT}_{ij} A^{TTij} = \frac{2}{5} A_{ij} A^{ij} \ . \tag{8.141}$$

Setting $A_{ij} = \dddot{J}_{ij}$ and inputting this into (8.134), we have the final result for the power radiated by the gravitating system, expressed in terms of its inertial tensor,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G_\mathrm{N}}{5}\langle \dddot{J}^{ij}\dddot{J}_{ij}\rangle \ , \tag{8.142}$$

where $\langle\rangle$ stands for averaging over a few periods corresponding to the wavelength of the wave.

For the two-mass system discussed in section 8.2.4, we get (**Exercise.** Show this.)

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{128 G_\mathrm{N} M^2 a^4 \omega^6}{5} \\ &= \frac{128 G_\mathrm{N} M^2 a^4 \omega^6}{5c^5} \ ,\end{aligned} \tag{8.143}$$

where on the second line we have restored the speed of light for easy calculation in SI units. For example, two 1 kg balls rotating 2 m from each other with angular frequency 1 rad/s give off gravitational radiation with power 7×10^{-52} W.

Change of orbital parameters due to gravitational wave emission from the Hulse–Taylor pulsar was first observed in 1978. A Nobel prize was awarded for it in 1993. It was the first detection of gravitational waves, albeit indirect.

Exercise. Consider the binary black hole system discussed at the end of section 8.2.4.
a) Approximating that the orbit remains circular, find the decay of λ as a function of time due to gravitational wave emission.
b) What is the lifetime of the system (defined here as the time to reach $\lambda = 1$, where our approximation must break down) if the initial radius is 1) one astronomical unit or 2) $10r_\mathrm{s}$?
c) What is the total radiated energy (from the initial radius to $\lambda = 1$) in cases 1) and 2), in units of $M_\odot$?
d) At which point does the velocity of the black holes exceed 0.1? (Then our Newtonian approximation for the orbits becomes questionable.)