Due on Monday February 24 by 14.15. These are the last exercises of General relativity I. (Note that this document has two pages.)

1. **Time dilation.** Time dilation was first verified experimentally in 1971 by flying atomic clocks around the world (and the corrections are now routine for GPS satellites). Consider a flight along the equator at altitude 10 km and velocity 300 m/s. How much does the time of the clock that travelled around the world differ from the time of a clock that remained still with respect to the Earth (at sea level)

   (a) for an eastbound flight

   (b) for a westbound flight.

   Take into account that the Earth rotates. How much of the time dilation is due to gravity?

2. **Newtonian limit of the Schwarzschild solution.** In the lecture notes the relation between the Schwarzschild solution integration constant r_s and the Newtonian mass M is determined carelessly, by comparing the Schwarzschild metric written in spherical coordinates with the earlier Newtonian limit metric written in Cartesian coordinates. Do the calculation correctly, either by transforming the Newtonian limit metric (4.24) into spherical coordinates or by transforming the Schwarzschild metric into Cartesian coordinates. (In the former case, you should define the radius r so that the area of a two-sphere is $4\pi r^2$.)

3. **Shapiro time delay**.

   a) In the Schwarzschild spacetime, a radial radar signal is sent from $(r_2, \theta_0, \varphi_0)$ to $(r_1, \theta_0, \varphi_0)$. The signal is immediately reflected and travels back again. Assume $r_2 > r_1 > r_\mathrm{s}$. The signal is lightlike, so $\mathrm{d}s^2 = 0$. Find the round-trip time $\Delta\tau$ measured by an observer at $(r_2, \theta_0, \varphi_0)$.

   b) The proper distance L between the two points is given by (5.29). We might naively expect the round-trip time to be $\Delta\tilde{\tau} \equiv 2L$. The difference $\Delta\tau - \Delta\tilde{\tau}$ is called the Shapiro time delay. Show that for $r_1 \gg r_\mathrm{s}$, it is

   $$\Delta\tau - \Delta\tilde{\tau} \approx r_\mathrm{s}\left(\ln\frac{r_2}{r_1} - \frac{r_2 - r_1}{r_2}\right) .$$

   Explain the cause of the Shapiro time delay.

4. **Schwarzschild orbits.** In the Schwarzschild metric, the normalisation condition for a vector tangent to a geodesic reads

   $$\left(1 - \frac{r_\mathrm{s}}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_\mathrm{s}}{r}} - r^2\dot{\varphi}^2 = \beta \ ,$$

   where $\beta = 1$ for massive particles and $\beta = 0$ for massless particles.

   a) Using the constants of motion h and k, convert this equation into the form

   $$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}k^2 \ ,$$

   where $V(r)$ depends on r_s and h.

   b) Draw $V(r)$ (a typical case) for massive and massless particles.

   c) The above equation has the same form as that of a Newtonian particle moving in 1+1 dimensions with potential $V(r)$, with total energy $\frac{1}{2}k^2$. Using this analogy, explain how different values of k correspond to bound (motion limited to $r_1 \leq r \leq r_2$) and unbound (no upper limit to r) orbits, and orbits that lead towards the black hole at the center. (You don't have to solve the corresponding k values.)

d) Are there bound orbits for photons?

e) Explain how to locate stable and unstable circular (r = constant) orbits on the plot of $V(r)$.

f) Are there stable/unstable orbits for massive particles? What about photons?