Local Algorithms on Grids

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"**LCL Problems on Grids**", joint work with:

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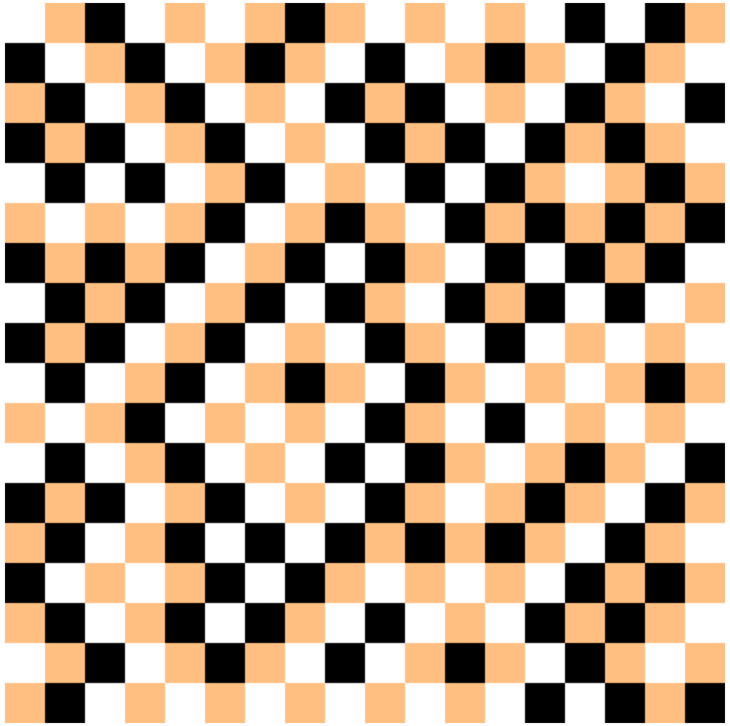
- Sebastian Brandt, Przemysław Uznański (ETH)
- Juho Hirvonen (Paris Diderot)
- Joel Rybicki (Helsinki)

2-colouring

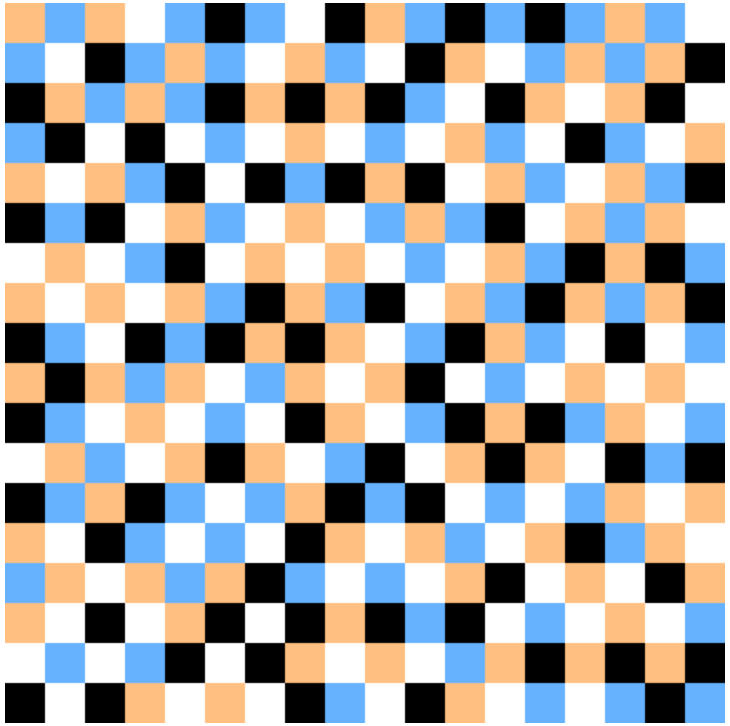
global

3-colouring



global

4-colouring



local

Introduction

Setting

- Distributed graph algorithms
- ***Input graph = computer network***
  - node = computer, edge = communication link
  - unknown topology
- Each node outputs its own part of solution
  - e.g. graph colouring: node outputs its own colour

Setting

- Deterministic distributed algorithms, LOCAL model of computing
  - unique identifiers
  - synchronous communication rounds
  - ***time = number of rounds*** until all nodes stop
  - unlimited message size,
    unlimited local computation

Setting

- Deterministic distributed algorithms, **LOCAL** model of computing
- Time = distance
- Algorithm with running time T: ***mapping from radius-T neighbourhoods to local outputs***

LCL problems

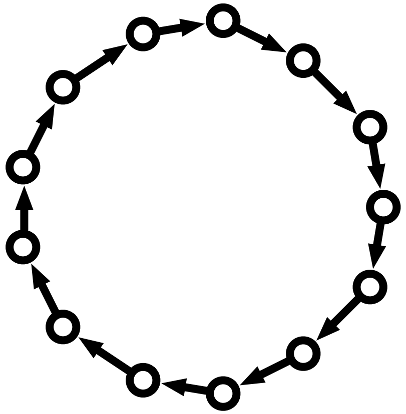
- **LCL = locally checkable labelling**
  - Naor–Stockmeyer (1995)
- Valid solution can be detected by checking $O(1)$-radius neighbourhood of each node
  - maximal independent set, maximal matching, vertex colouring, edge colouring …

LCL problems

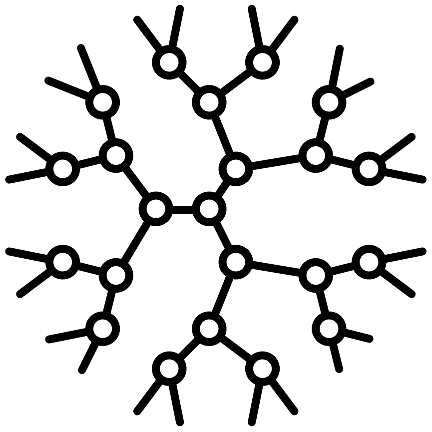
- All LCL problems can be solved with $O(1)$-round ***nondeterministic*** algorithms
  - guess a solution, verify it in $O(1)$ rounds
- Key question: how fast can we solve them with ***deterministic*** algorithms?
  - cf. P vs. NP

Traditional settings

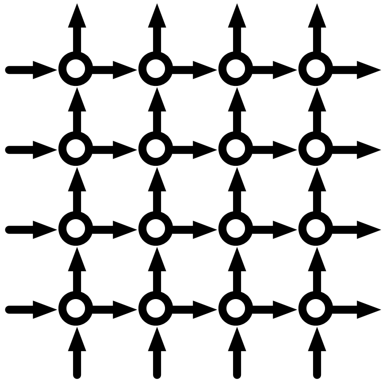
- **Directed cycles**
  - Cole–Vishkin (1986), Linial (1992)…
  - well understood



- **General (bounded-degree) graphs**
  - lots of ongoing work…
  - typical challenge:
    expander-like constructions

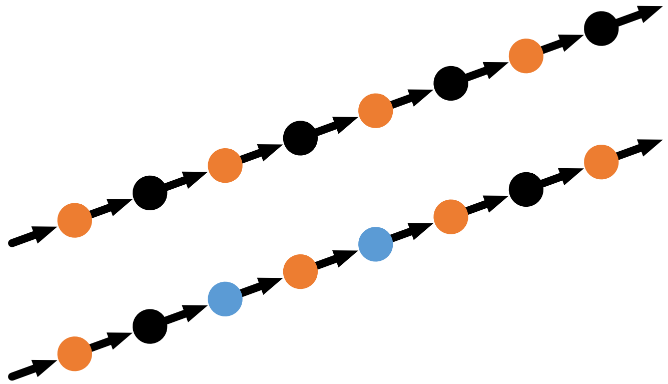


Our setting today



- **Oriented grids** (2D)
  - toroidal grid, $n \times n$ nodes, unique identifiers
  - consistent orientations north/east/south/west
- ***Generalisation of directed cycles*** (1D)
- Closer to real-world systems than expander-like worst-case constructions?

1D grids

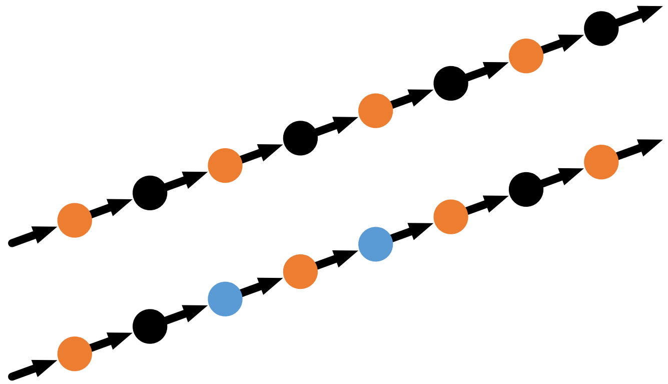


- Vertex colouring
- **2-colouring:** global, $\Theta(n)$ rounds
- **3-colouring:** local, $\Theta(\log^* n)$ rounds
  - Cole–Vishkin (1986), Linial (1992)

Why is 3-colouring $\Theta(\log^* n)$?

- Upper bound: ***one-round colour reduction***
  - **input:** colouring with **2^k colours**
  - **output:** colouring with **$2k$ colours**
- Lower bound: ***speed-up lemma***
  - **given:** algorithm for **k-colouring in time T**
  - **construct:** algorithm for **2^k-colouring in time $T - 1$**

1D grids



- Vertex colouring
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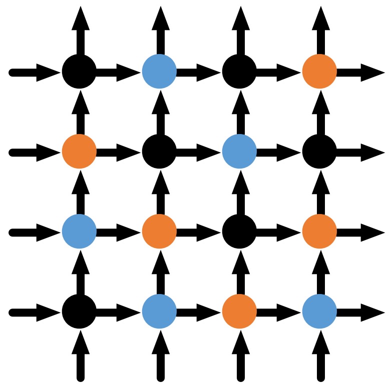
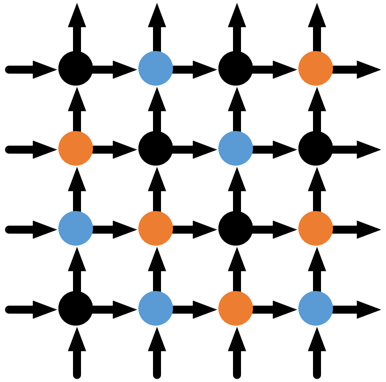
2D grids



- Vertex colouring
- **2-colouring:** global, $\Theta(n)$ rounds
- **3-colouring:** ???
- **4-colouring:** ???
- **5-colouring:** local, $\Theta(\log^* n)$ rounds

2D grids



- Vertex colouring
- **2-colouring:** global, $\Theta(n)$ rounds
- **3-colouring:** global, $\Theta(n)$ rounds
- **4-colouring:** local, $\Theta(\log^* n)$ rounds
- **5-colouring:** local, $\Theta(\log^* n)$ rounds

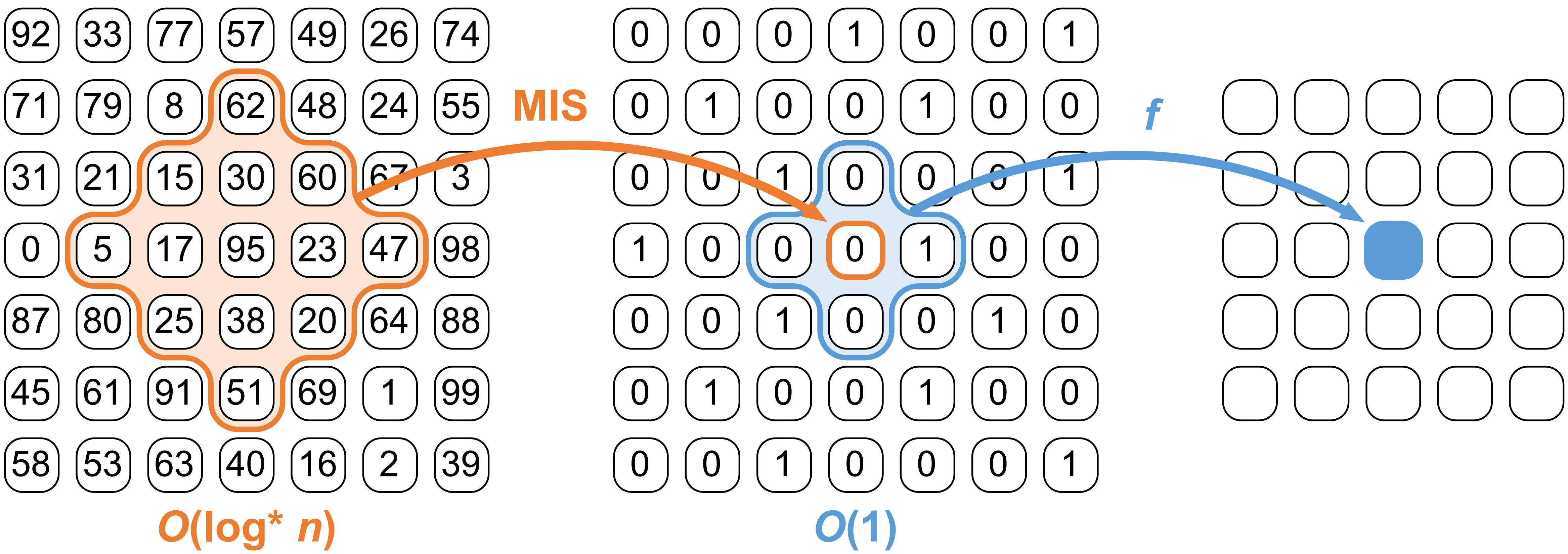
Classification of LCL problems

LCL problems on grids

- **$O(1)$** time: "**trivial**"
  - $o(\log^* n)$ time implies $O(1)$ time (Naor–Stockmeyer)
- **$\Theta(\log^* n)$** time: "**local**"
- **$\Theta(n)$** time: "**global**"
- Why ***nothing between local and global***?

Normalisation

- **Setting:** LCL problems, 2D grids
- **Theorem:** Any $o(n)$-time algorithm can be translated to a "***normal form***":
  1. fixed $\Theta(\log^* n)$-time component
  2. problem-specific $O(1)$-time component


92 33 77 57 49 26 74
71 79 8 62 48 24 55
31 21 15 30 60 67 3
0 5 17 95 23 47 98
87 80 25 38 20 64 88
45 61 91 51 69 1 99
58 53 63 40 16 2 39
MIS
0 0 0 1 0 0 1
0 1 0 0 1 0 0
0 0 1 0 0 0 1
1 0 0 0 1 0 0
0 0 1 0 0 1 0
0 1 0 0 1 0 0
0 0 1 0 0 0 1
f
O(log* n)
O(1)

Normalisation in more detail...

- For ***any problem P*** of complexity $o(n)$, there are **constants *k* and *r*** and function f such that P can be solved as follows:
  - input: 2D grid G with unique identifiers
  - find a ***maximal independent set in G^k***
  - ***discard unique identifiers***
  - apply function f to each $r \times r$ neighbourhood

Some proof ideas...

- Given: A solves P in time $o(n)$ in $n \times n$ grids
- Solving P in time $O(\log^* N)$ in $N \times N$ grids:
  - pick suitable $n = O(1)$, $k = O(1)$
  - find a maximal independent set (MIS) in G^k
  - use MIS to find ***locally unique identifiers*** for $n \times n$ neighbourhoods
  - simulate A in $n \times n$ local neighbourhoods

LCL problems on grids

- **$O(1)$** time: "**trivial**"
  - $o(\log^* n)$ time implies $O(1)$ time (Naor–Stockmeyer)
- **$\Theta(\log^* n)$** time: "**local**"
  - $o(n)$ time implies $O(\log^* n)$ time (***normalisation***)
- **$\Theta(n)$** time: "**global**"

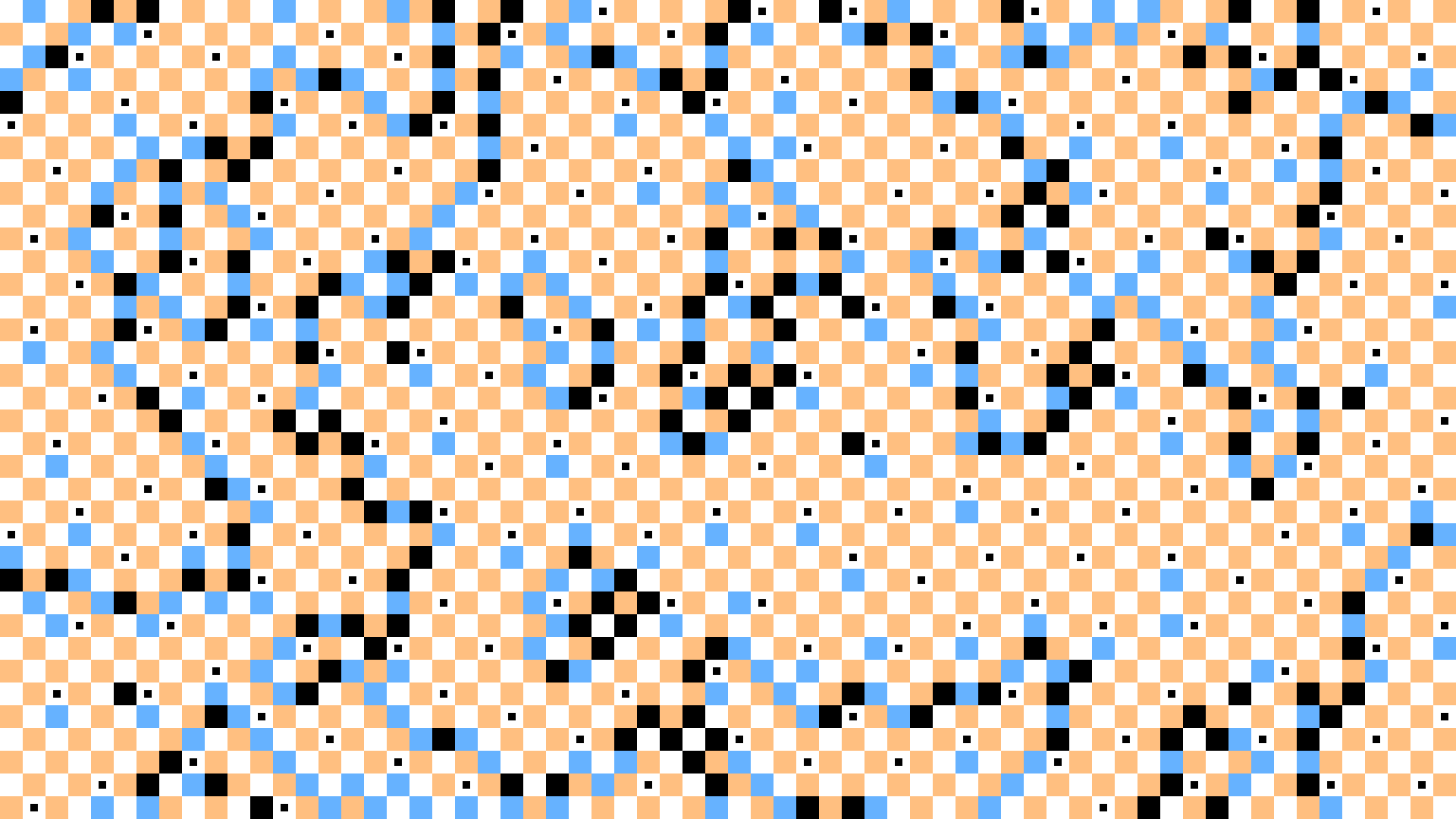
Vertex colouring

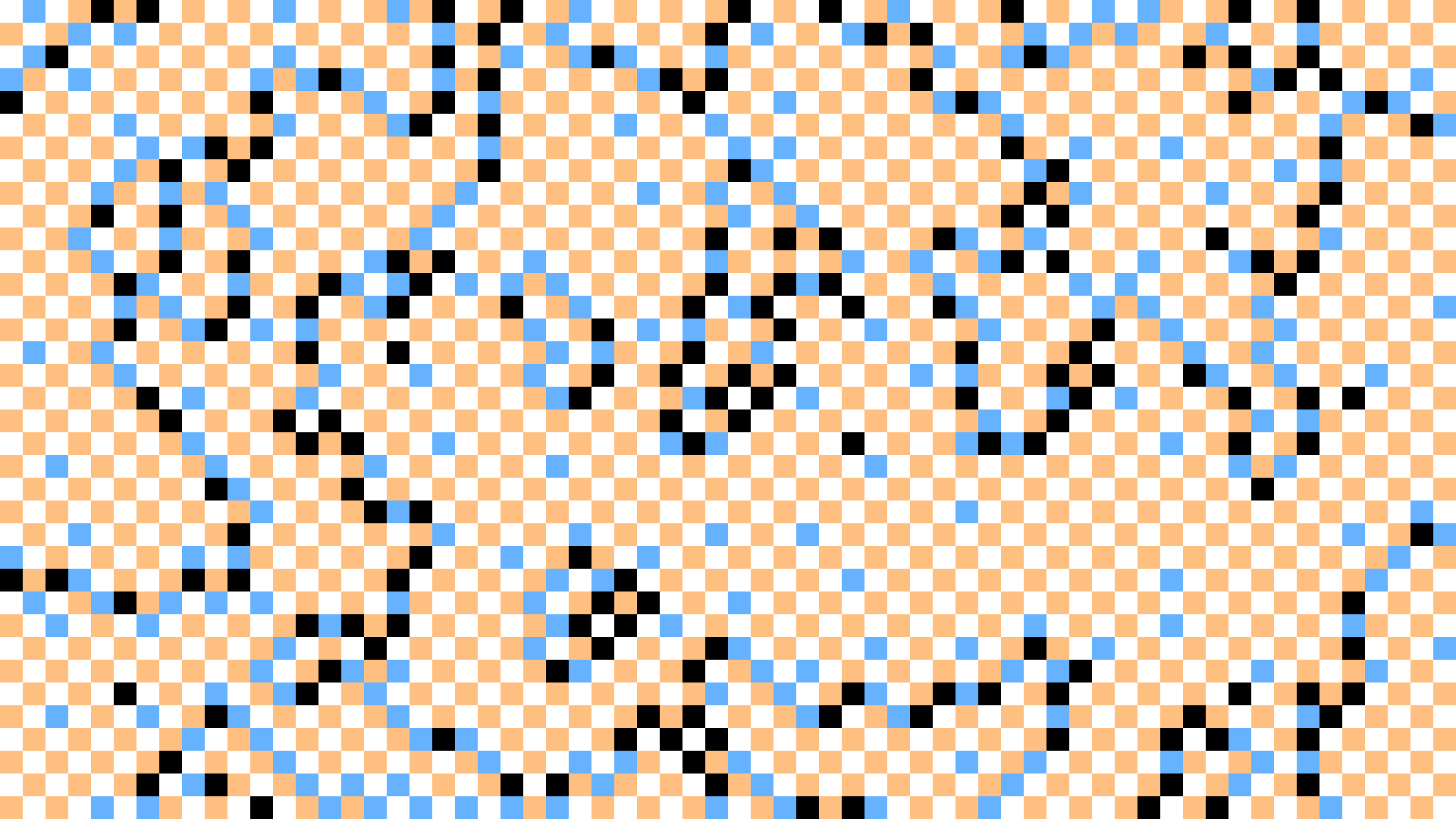
- Every LCL problem is trivial, local, or global
- Why is **4-colouring** in 2D grids “local”?
- Why is **3-colouring** in 2D grids “global”?

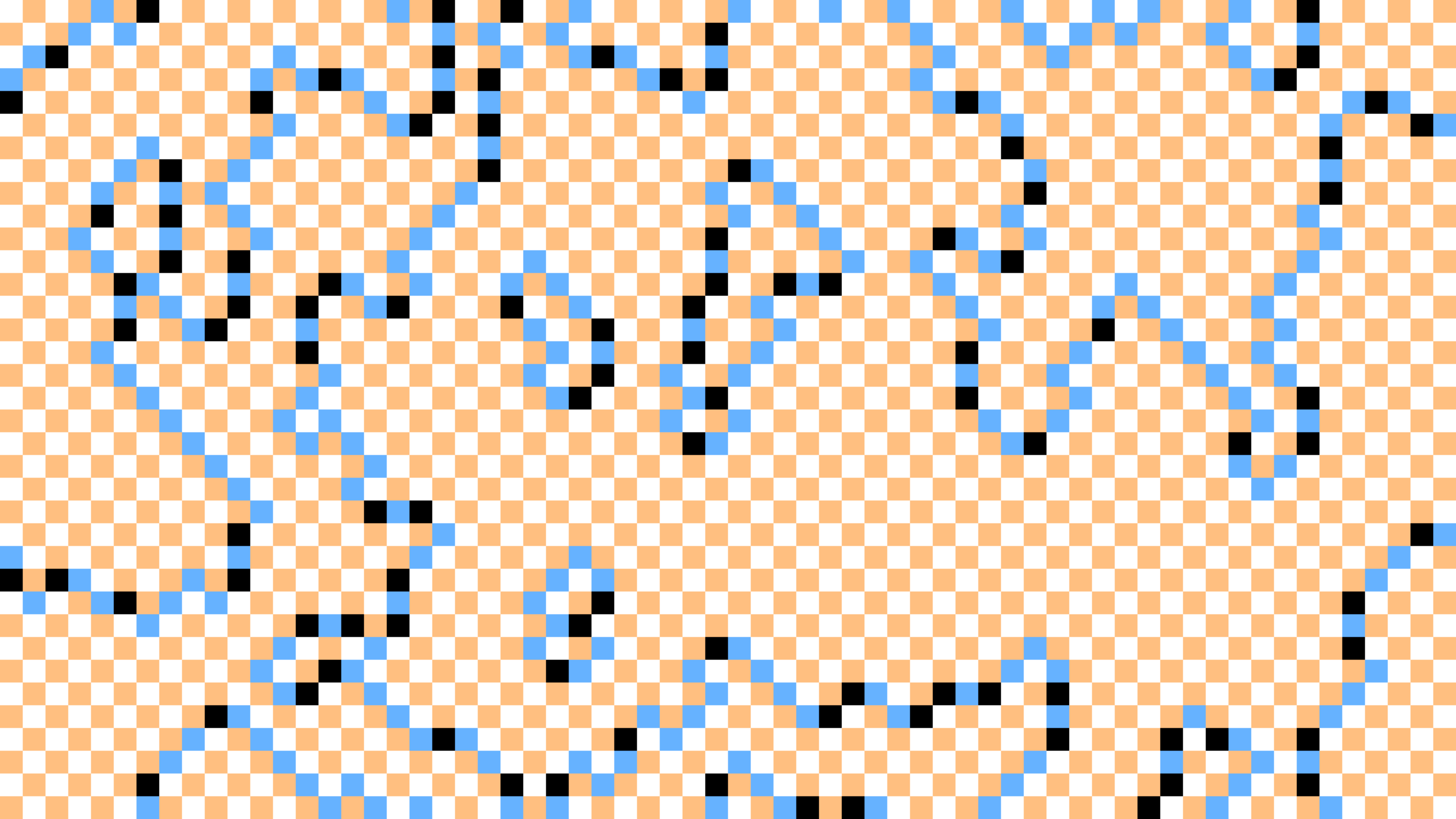
4-colouring on grids

4-colouring

- Lucky guess: **maybe it is local?**
- Try to use computers to find **normal form**
  - turns out it is enough to find an MIS in G^3, then consider **7 × 5 tiles**
  - algorithm ≈ mapping $\{0, 1\}^{7 \times 5} \to \{1, 2, 3, 4\}$
  - only **2079** possible tiles, easy to find a solution







3-colouring on grids

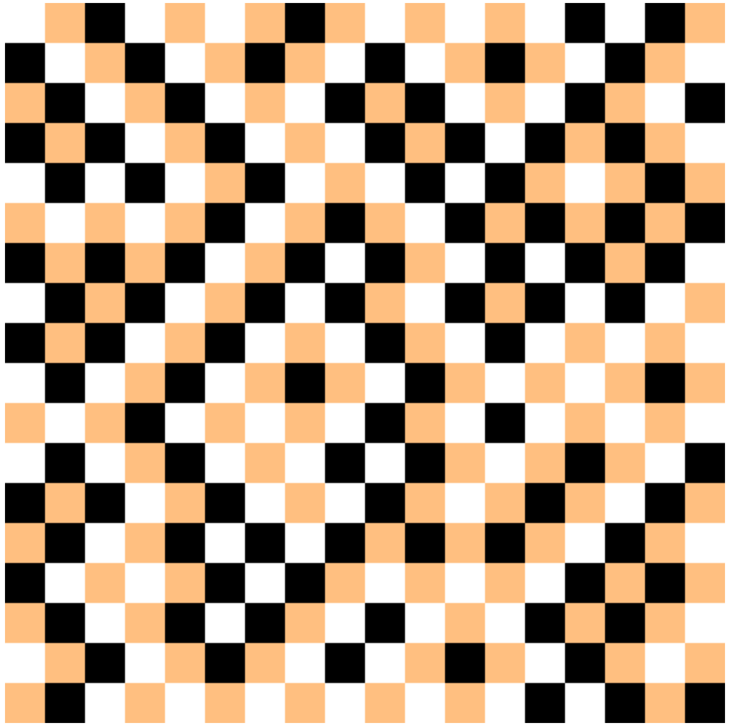
3-colouring

- Inherently different from 4-colouring:
  - cannot be solved locally
- But also different from 2-colouring:
  - nontrivial to argue that the problem is global

2-colouring

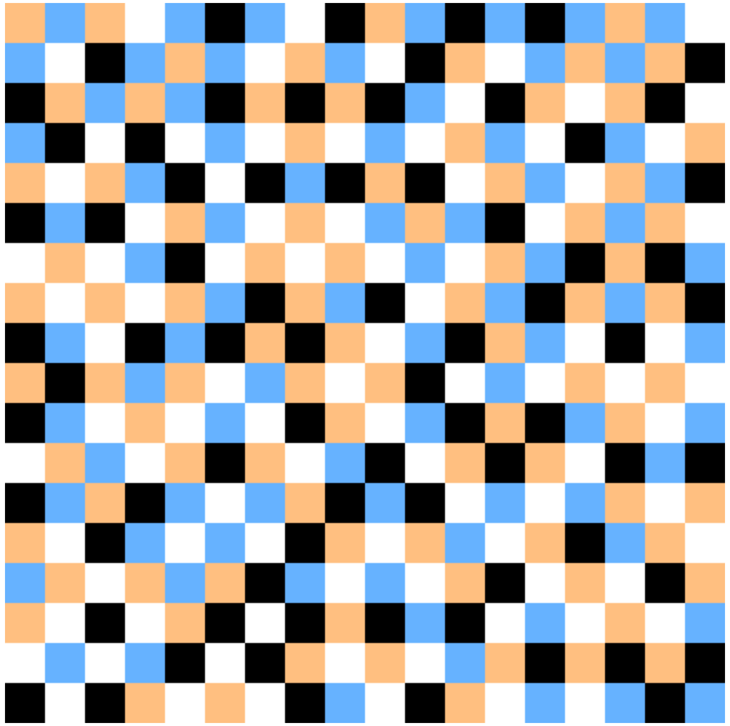
global

3-colouring

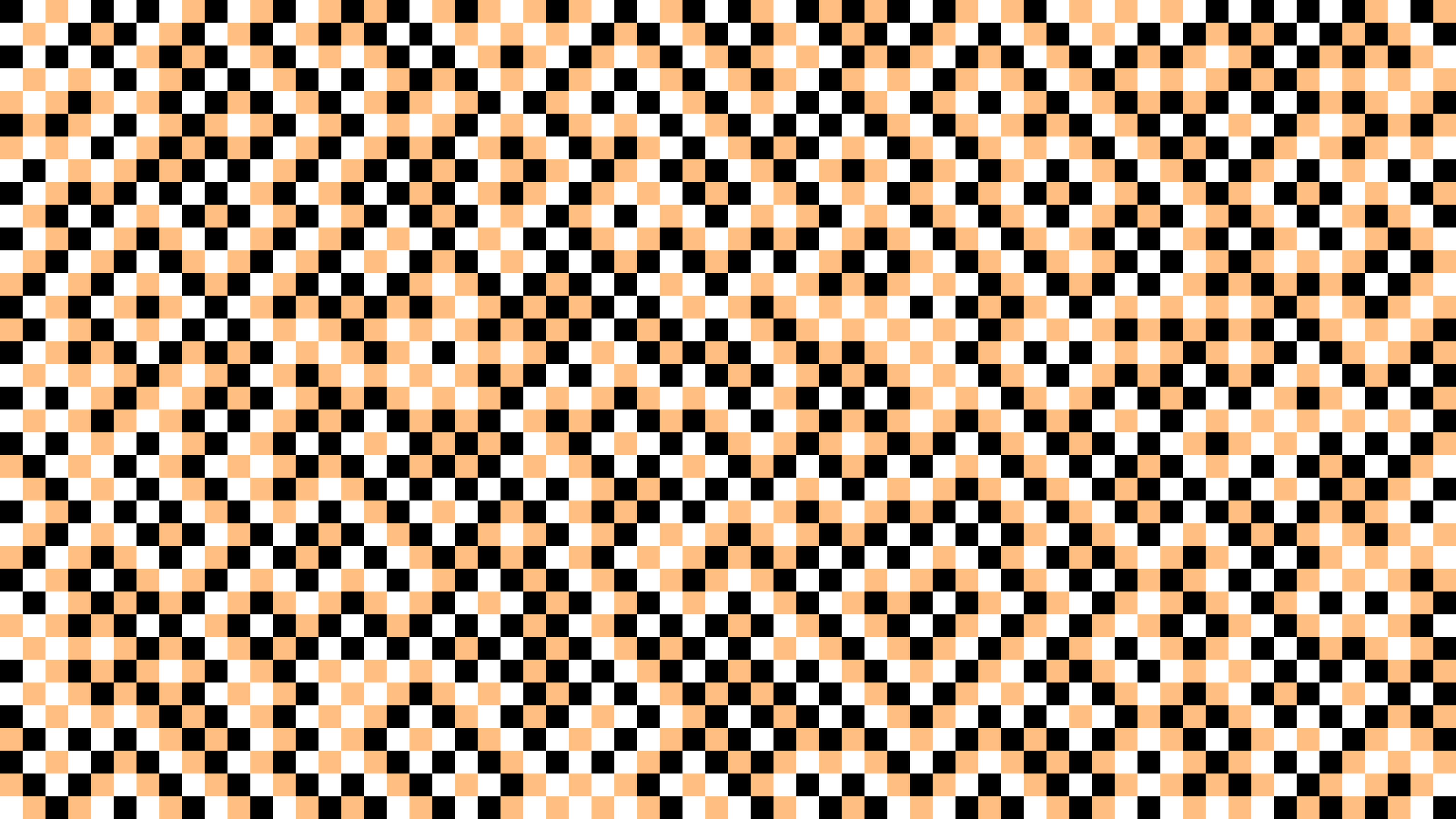


global

4-colouring



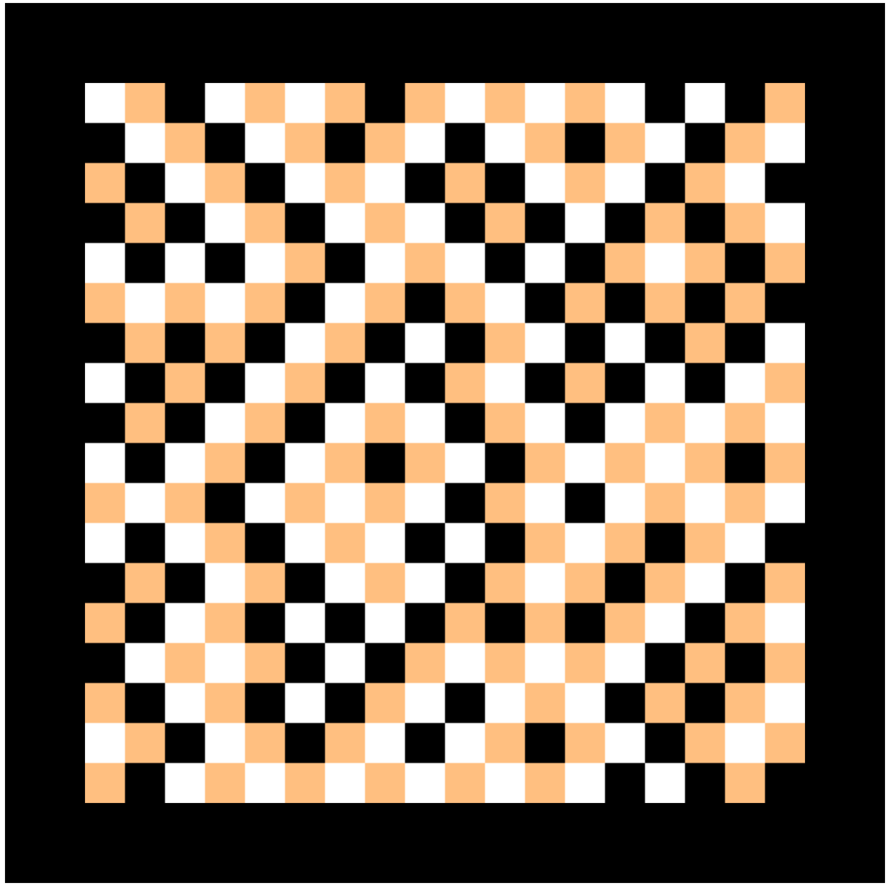
local



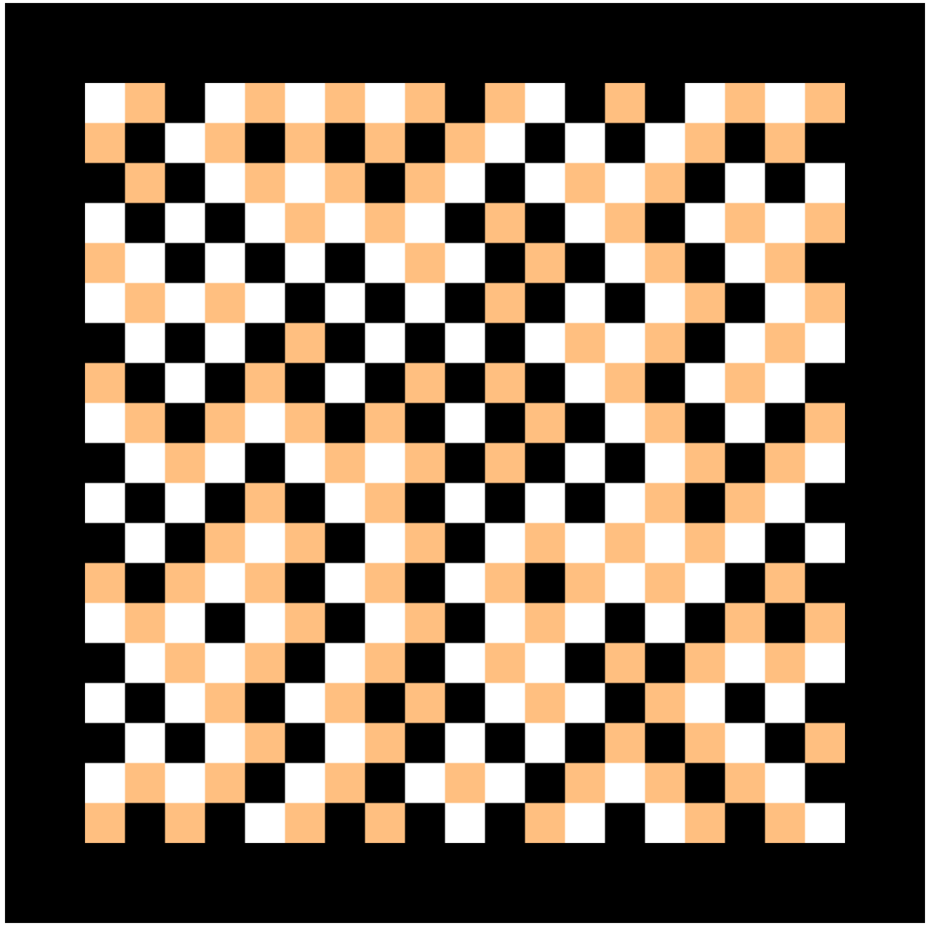
Proof idea

- **Assume:** a local algorithm
  for **3-colouring** in ***n* × *n* grids**
- **Implication:** a local algorithm
  for "**sum coordination**" in ***n*-cycles**
- But we can prove that this problem is global

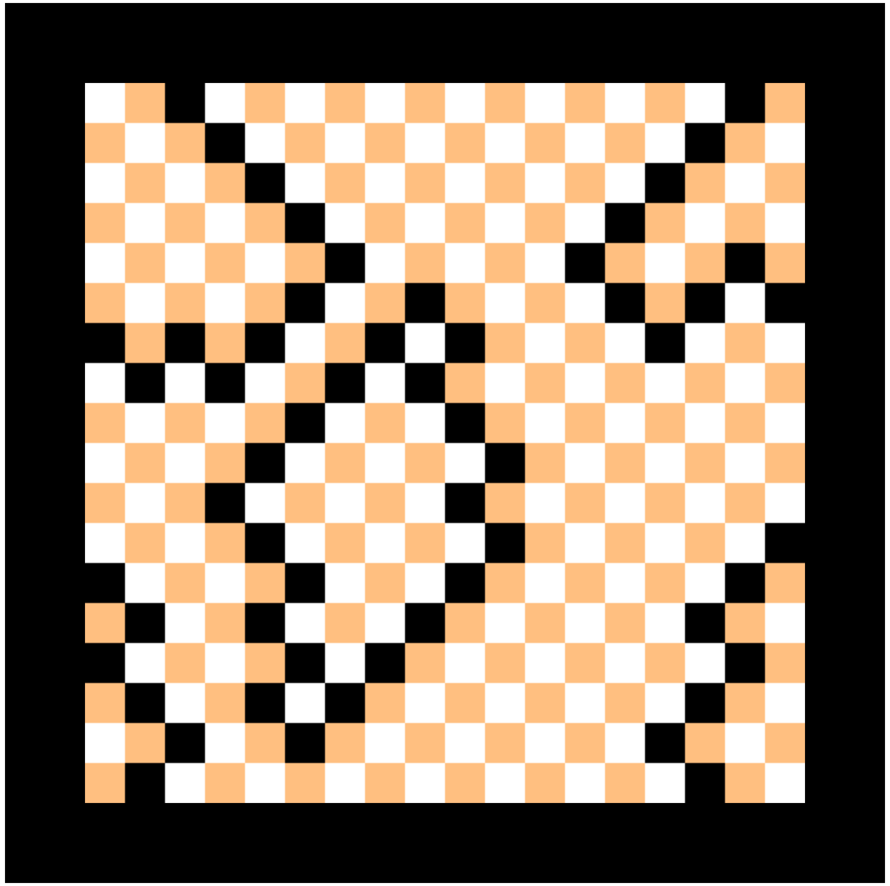
even × even

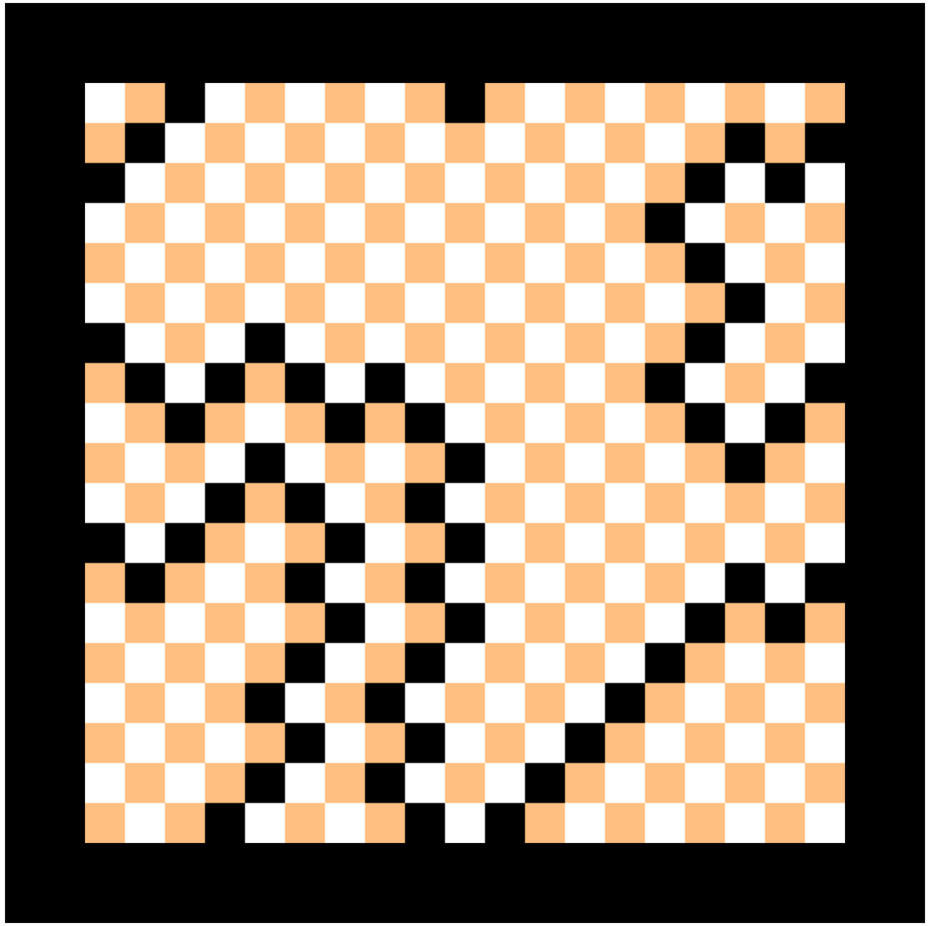


odd × odd



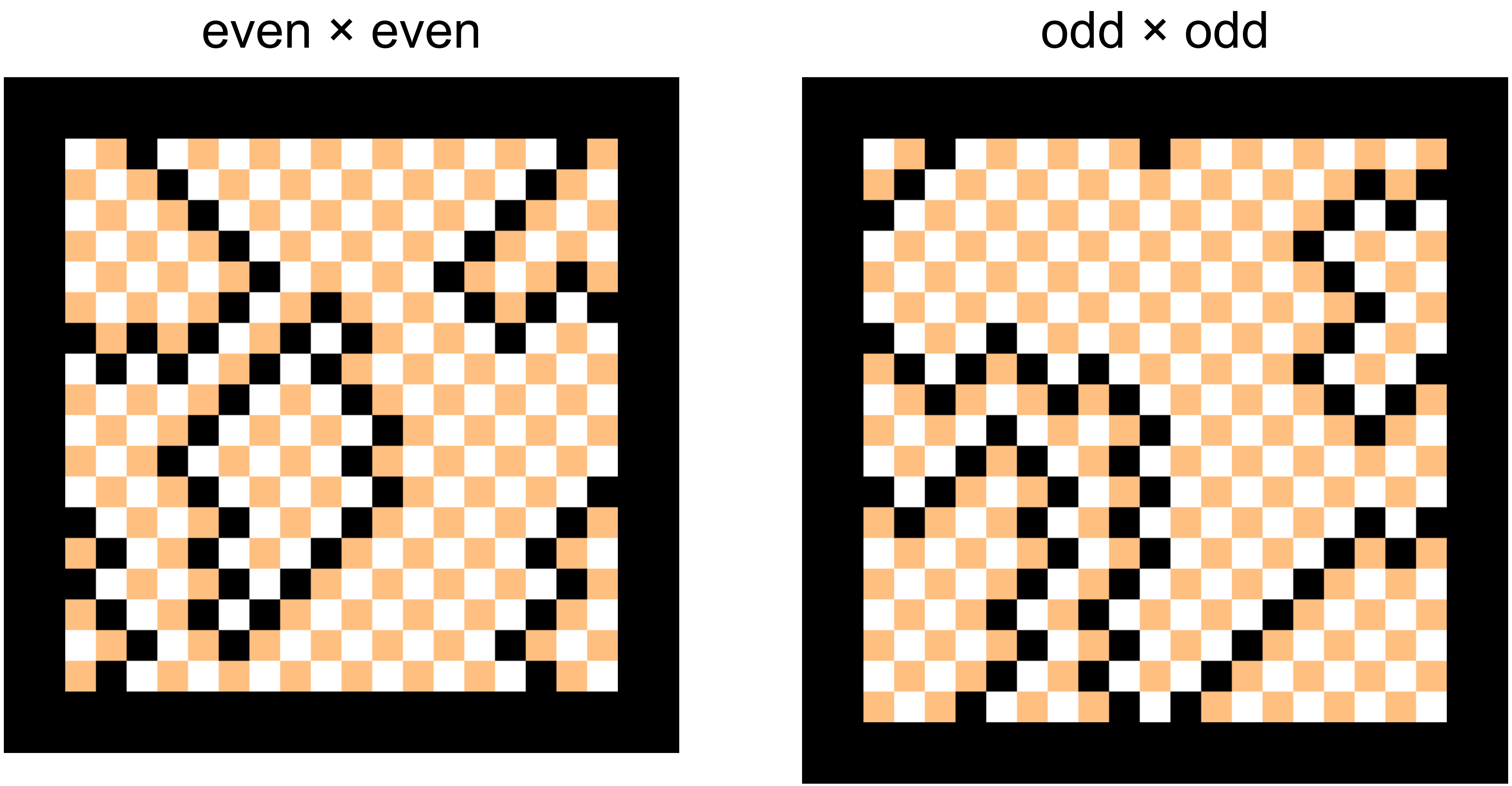
Consider any feasible 3-colouring…





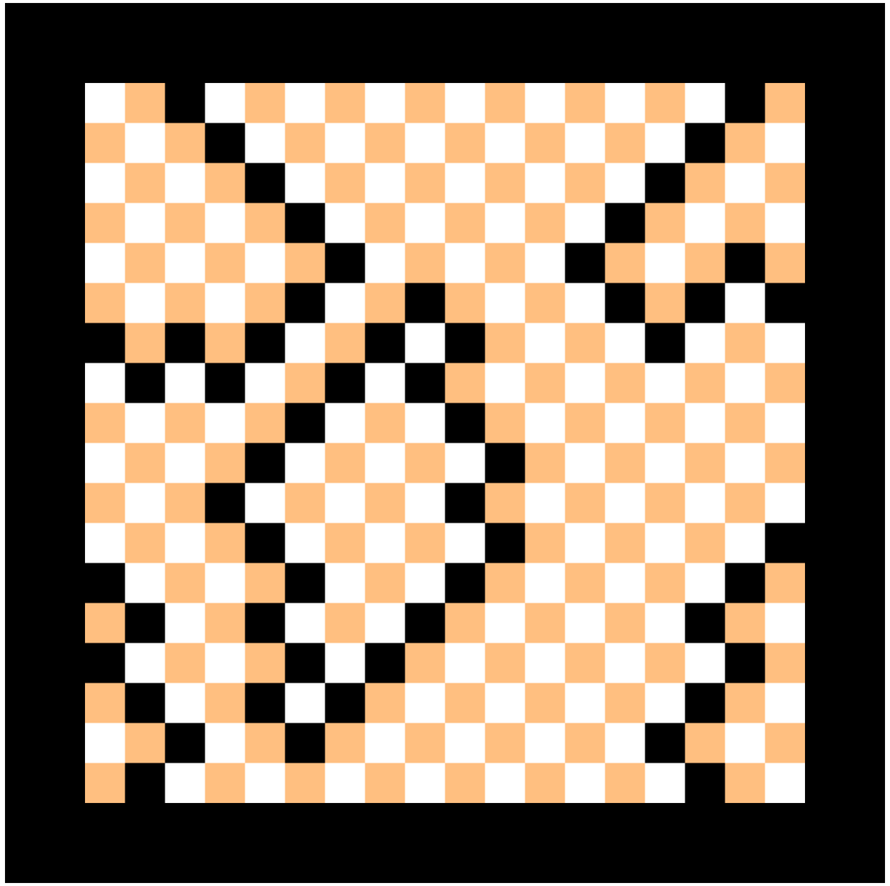
We can convert it into a ***greedy*** solution in constant time

(eliminate colour 2 whenever possible, then colour 3)

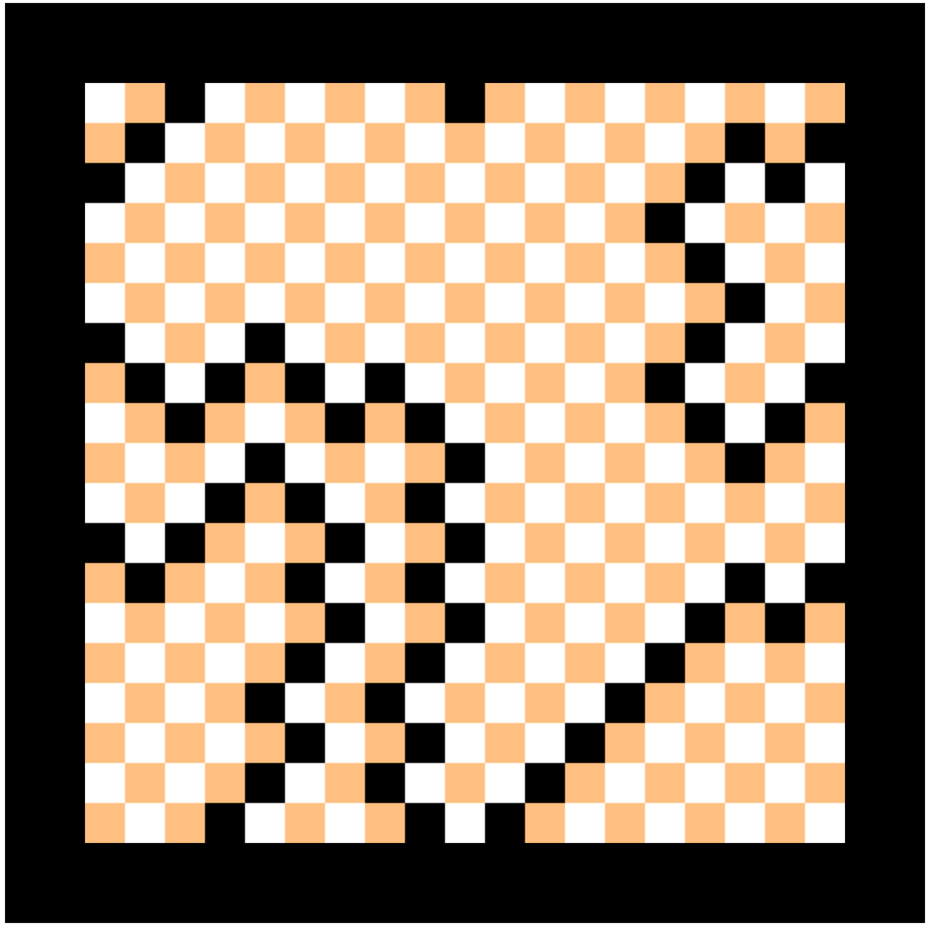


Greedy solution: ***boundaries + 2-coloured regions***

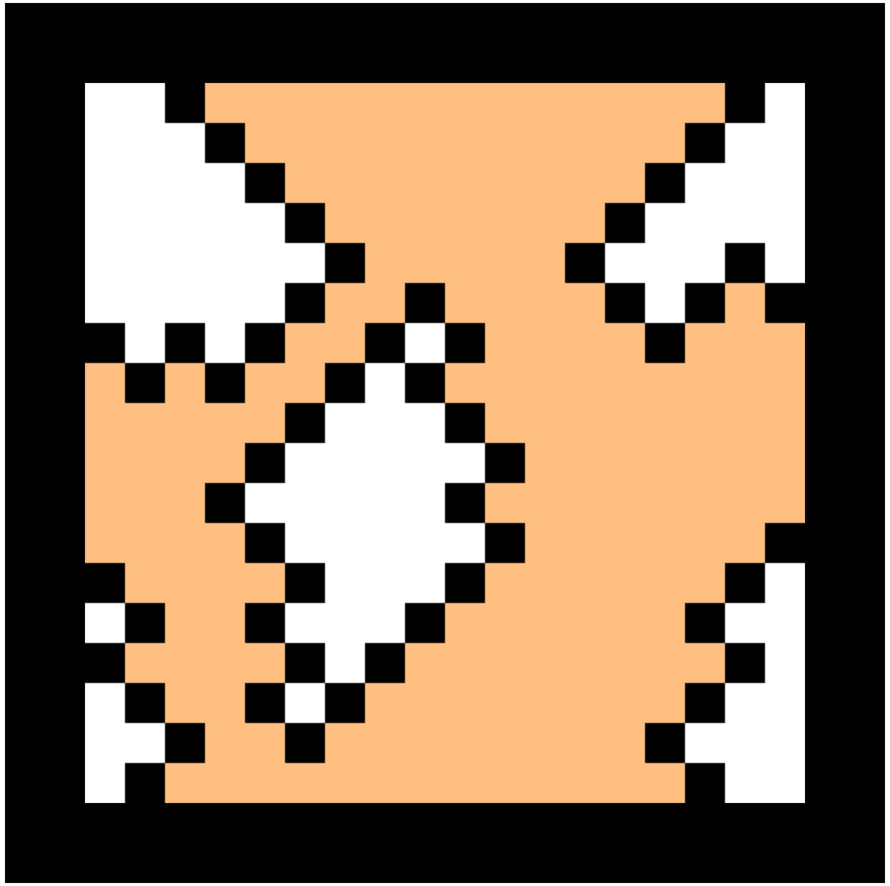
even × even

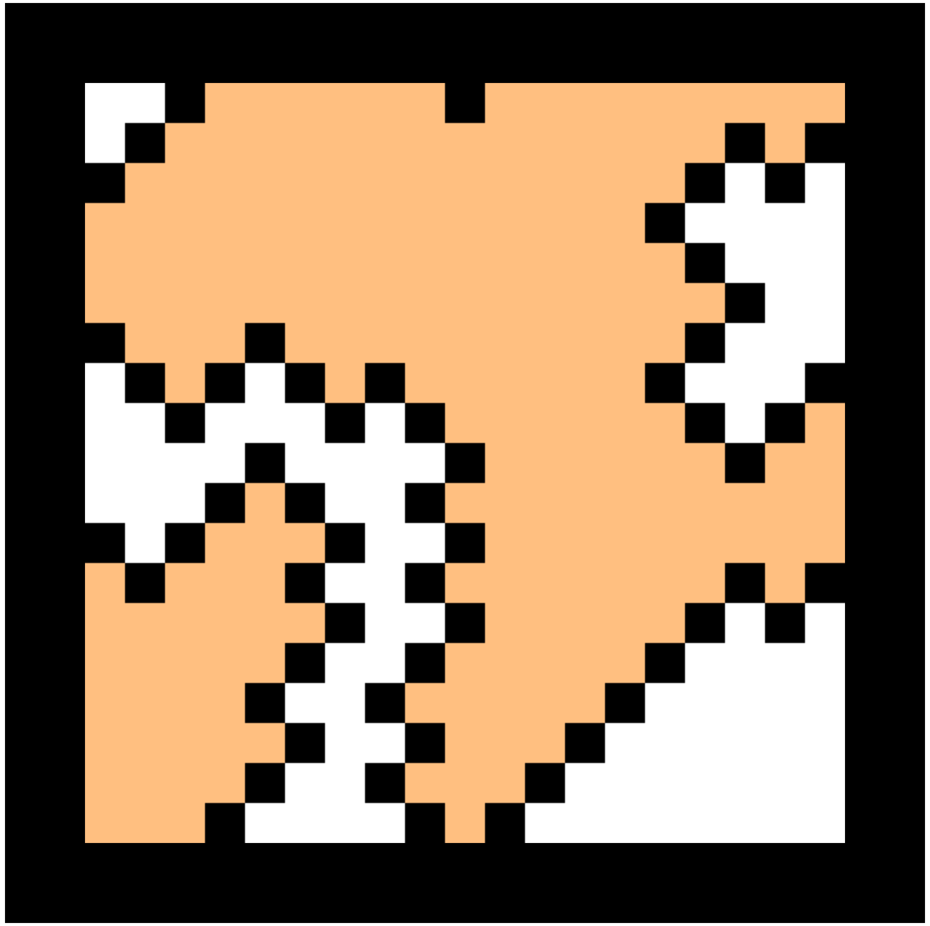


odd × odd



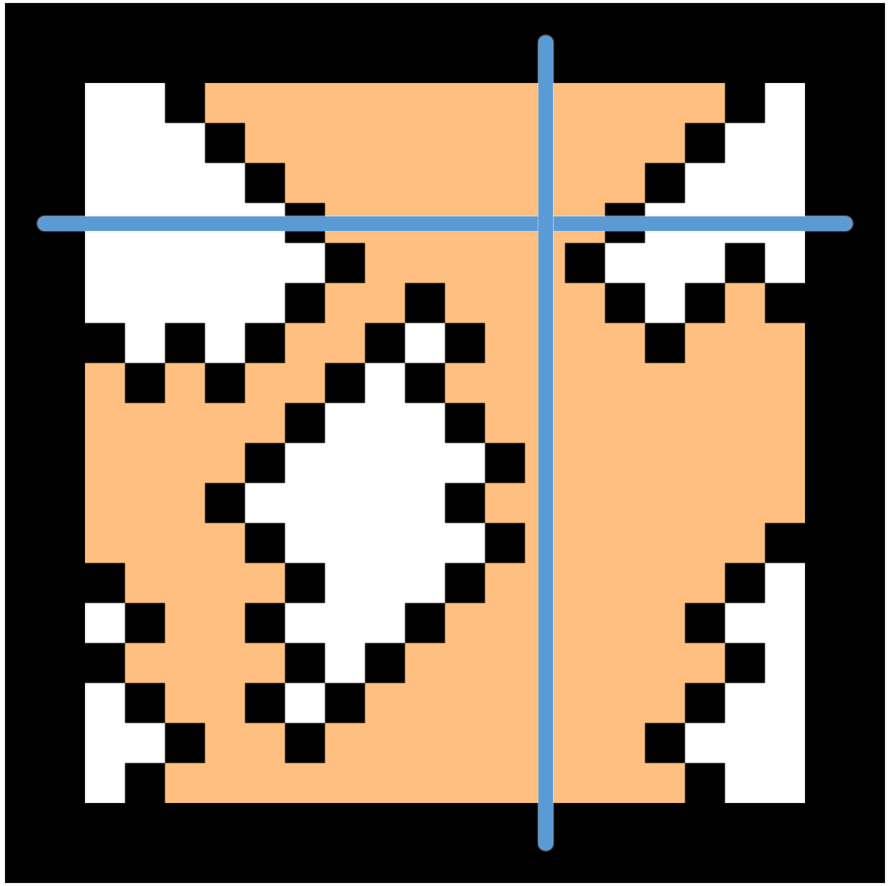
Parity changes at each boundary





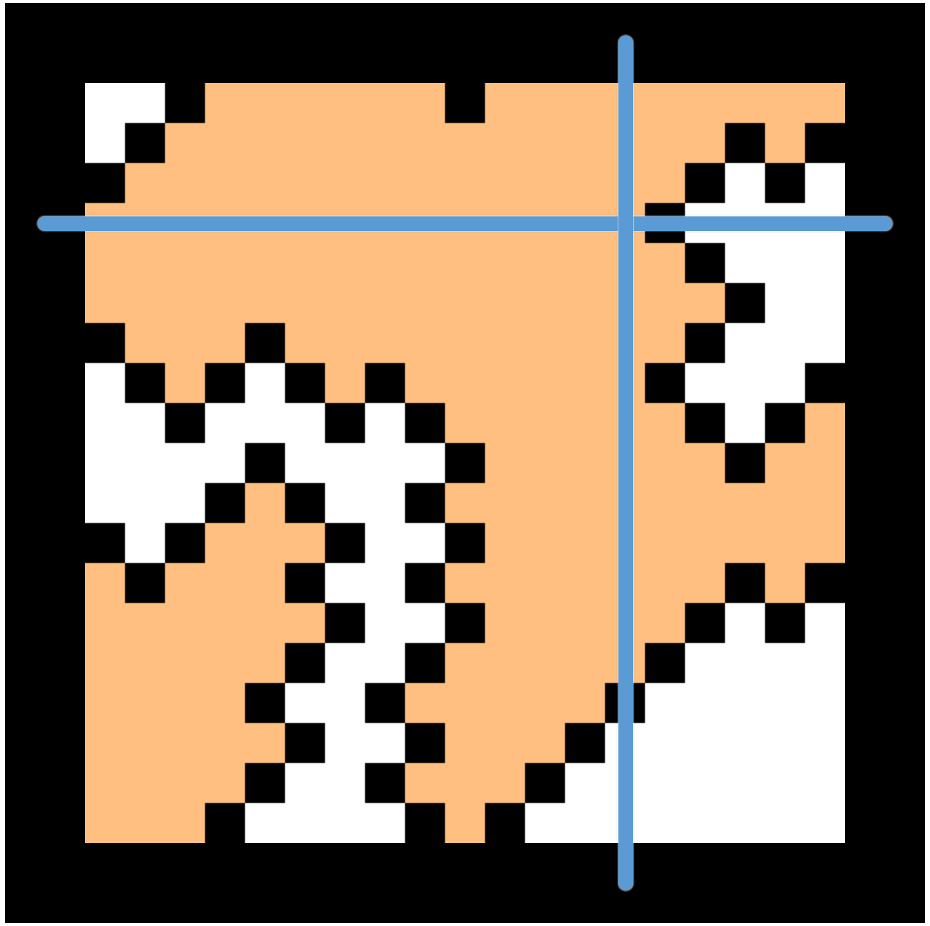
Parity changes at each boundary

even × even

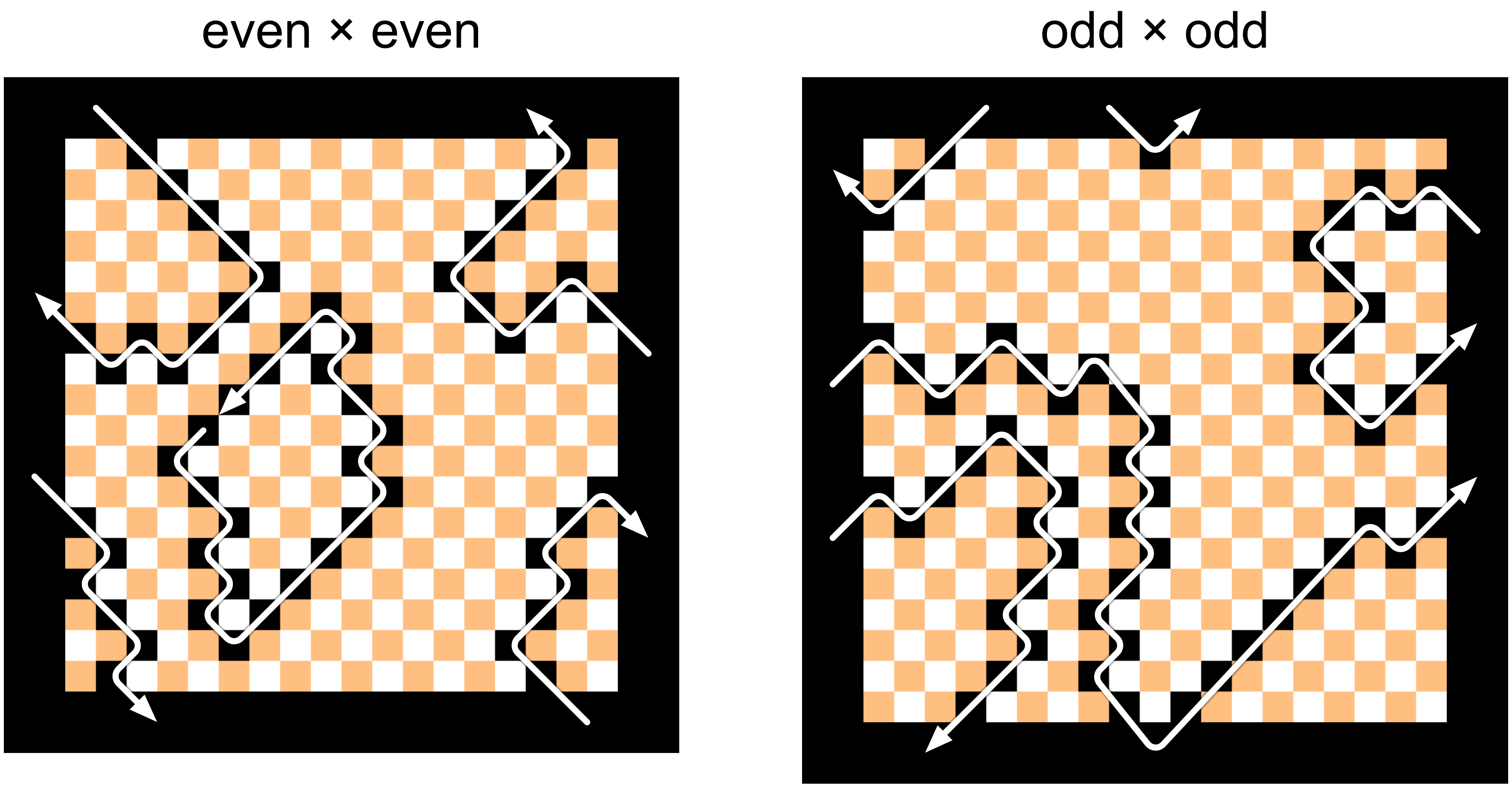


Wrap around:
same parity

odd × odd

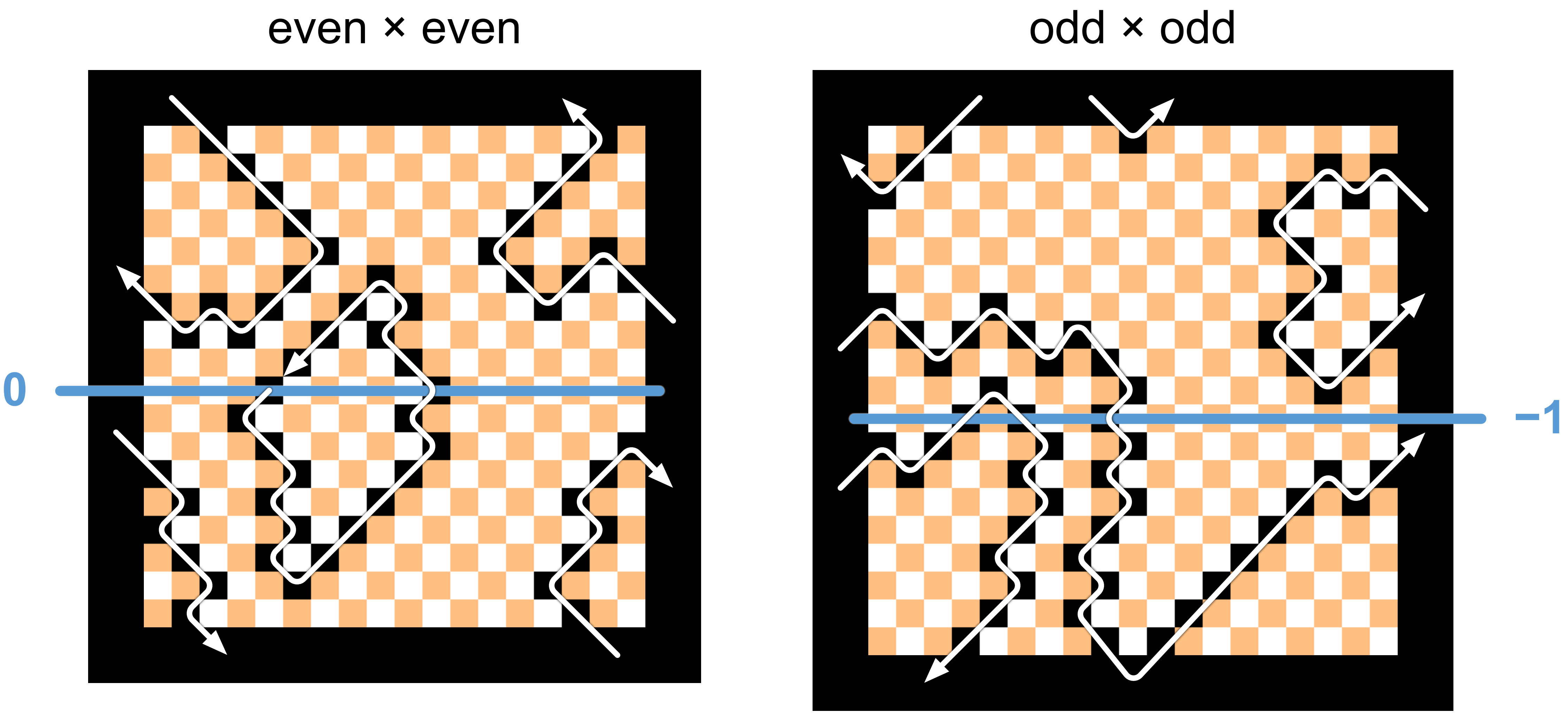


Wrap around:
opposite parity



Boundaries can be ***oriented*** with local rules

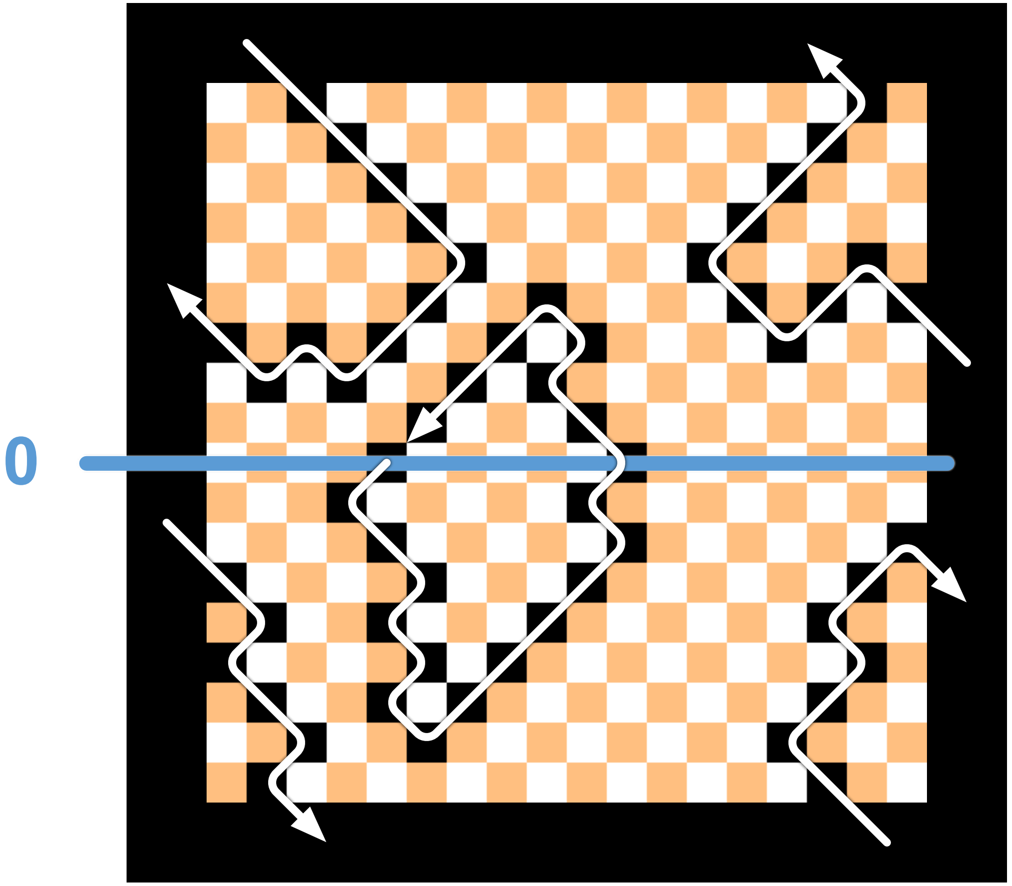
(keep orange on right, white on left)



Pick any row, label ***boundary crossings*** with +1 / −1

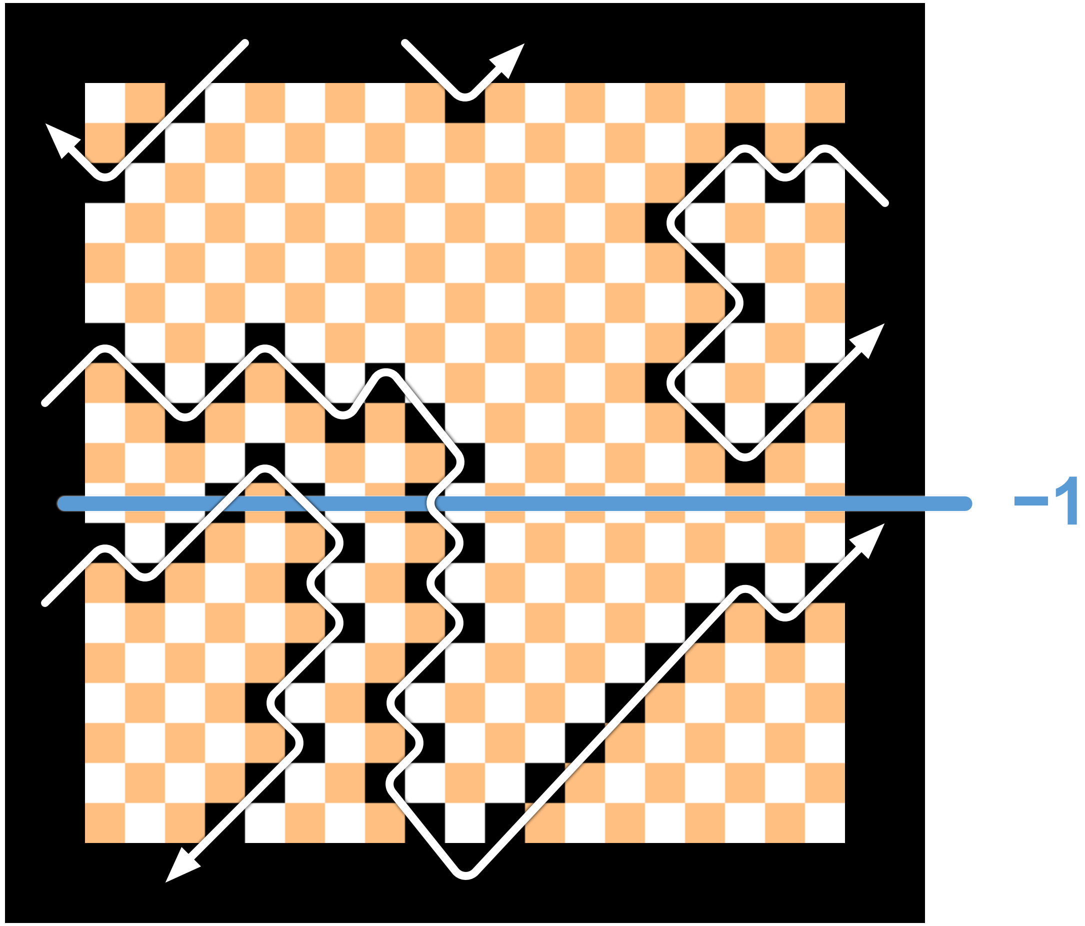
up = +1, down = −1

even × even



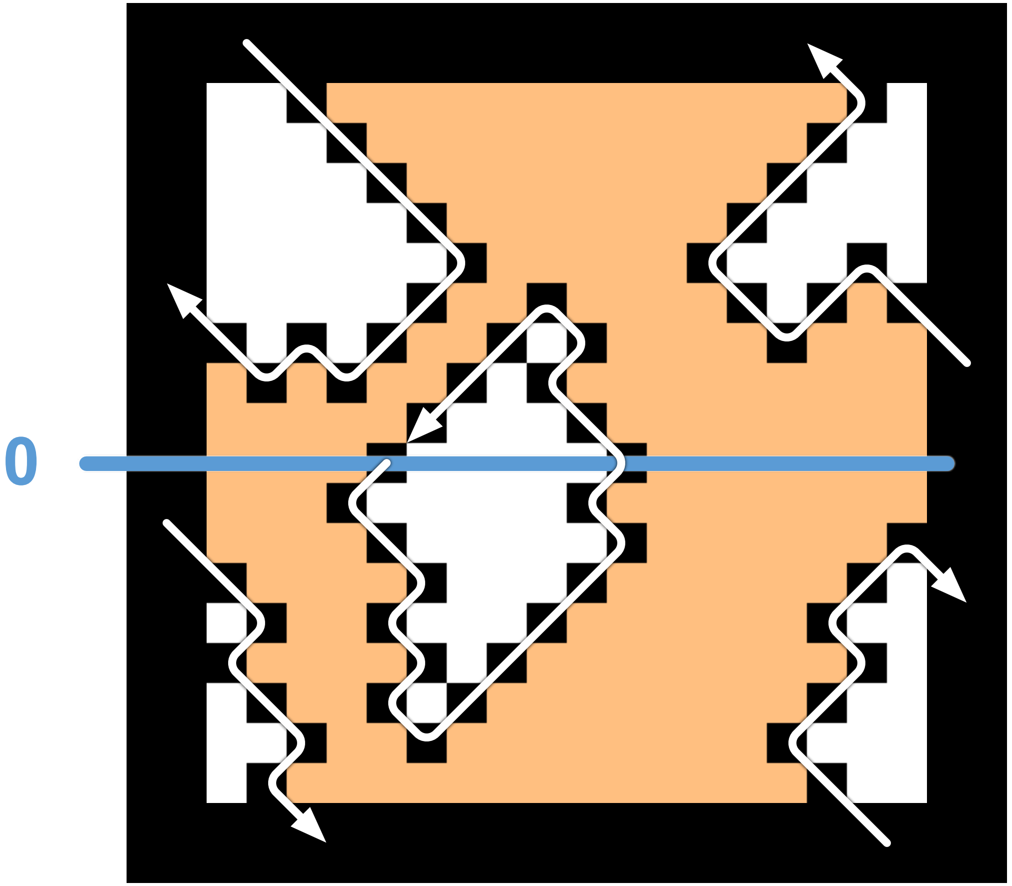
Sum of crossings:
even

odd × odd



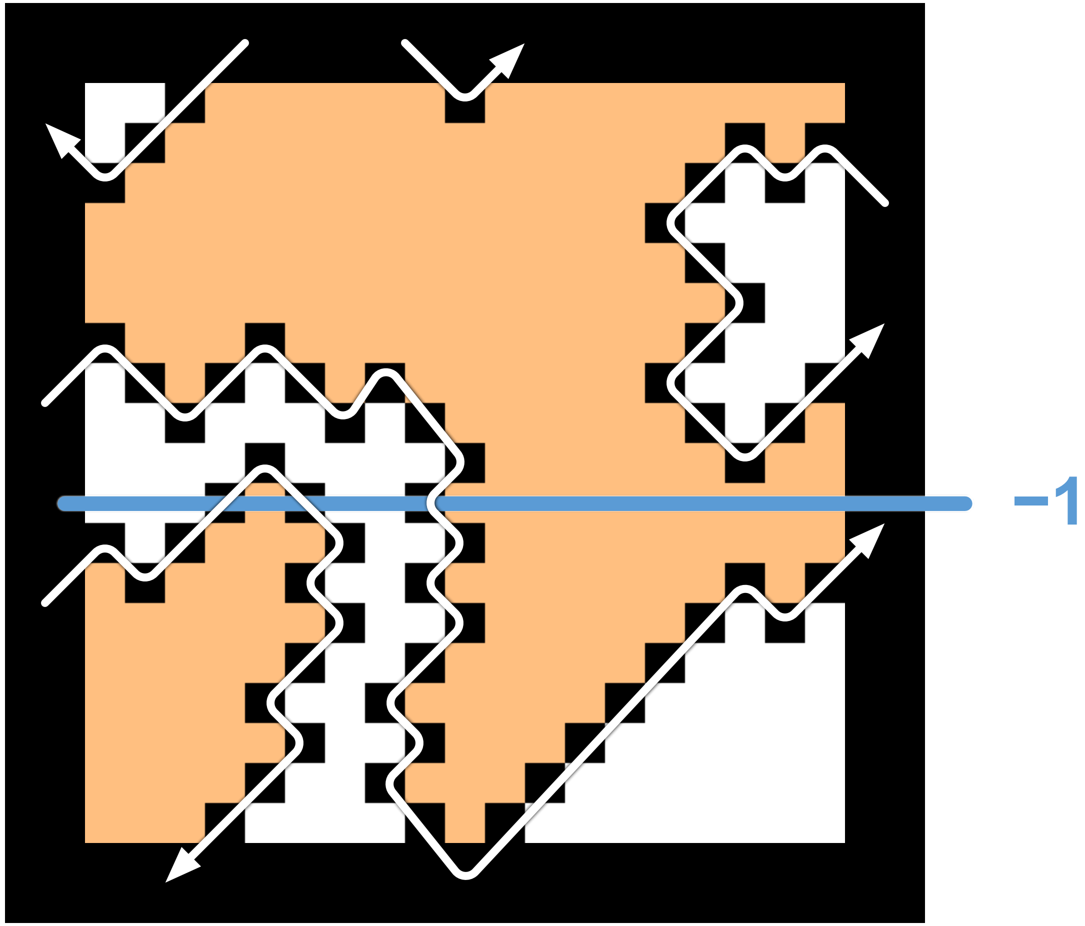
Sum of crossings:
odd

even × even

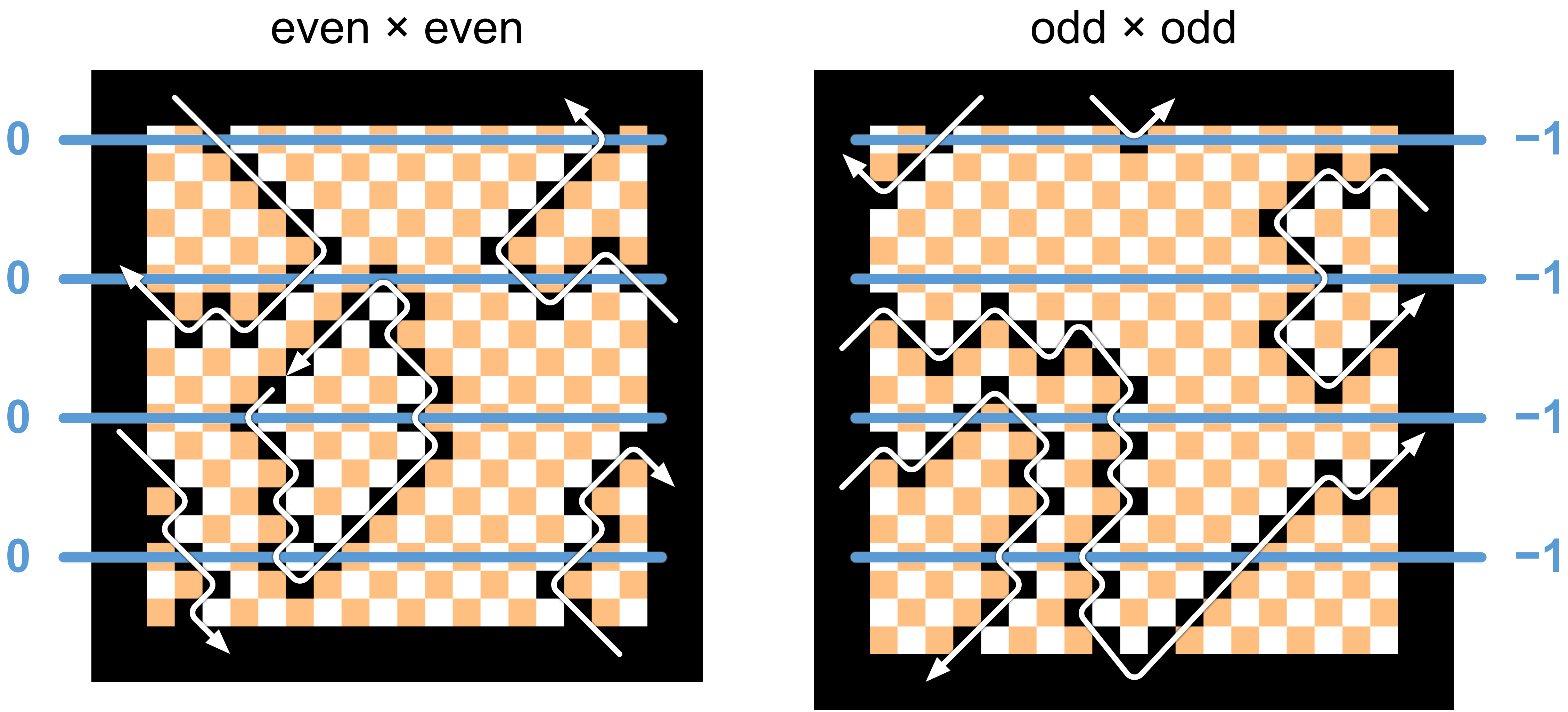


Sum of crossings:
even

odd × odd

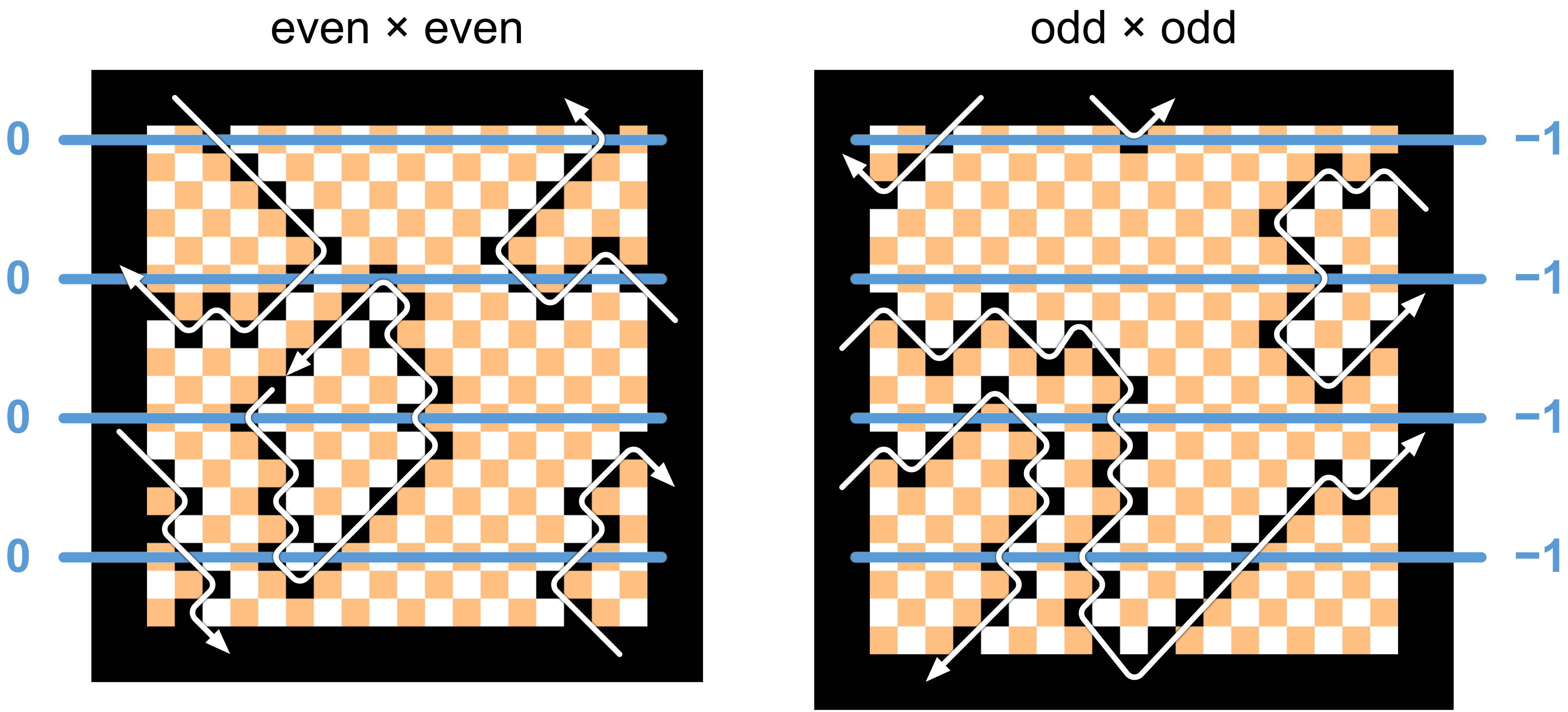


Sum of crossings:
odd



Boundaries are closed curves: ***constant sum***

up = +1, down = −1



Locality: sum only depends on ***grid dimensions***, not on IDs

(otherwise we could construct one instance with non-constant sum)

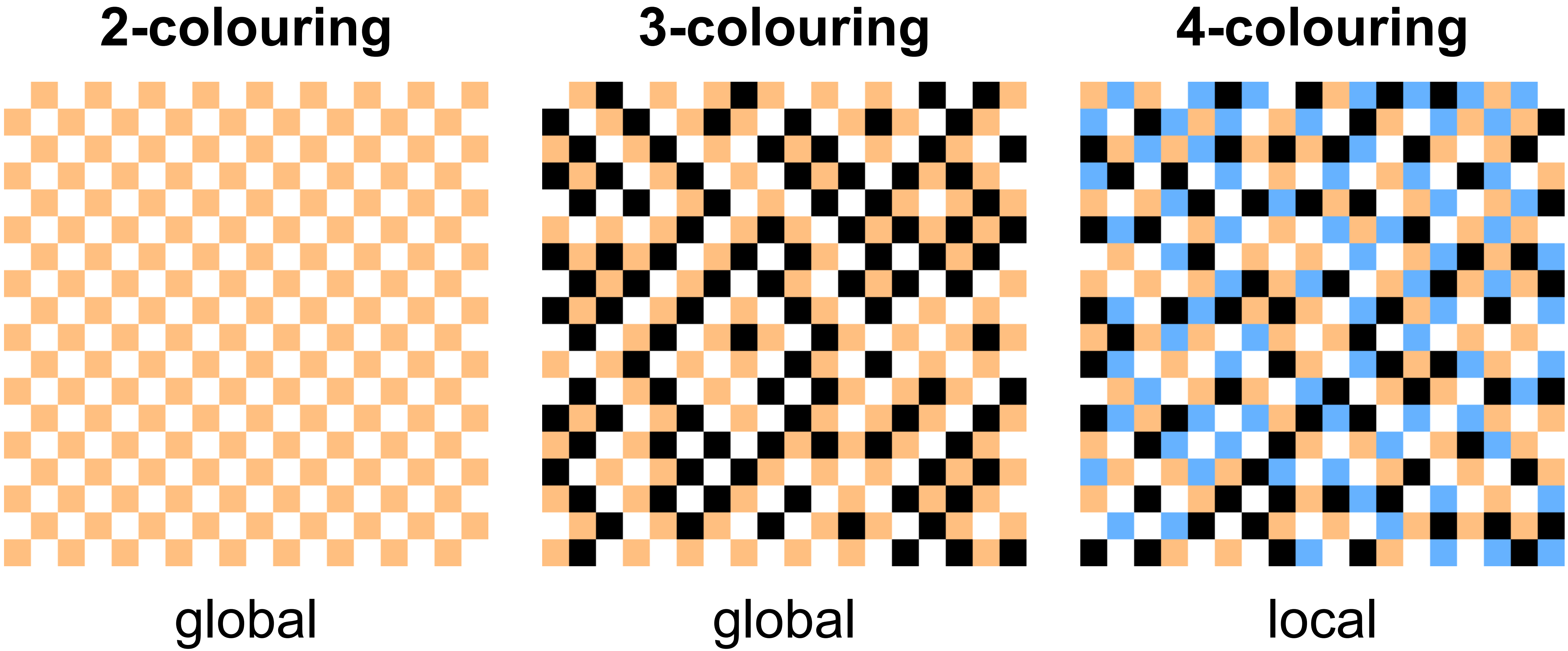
Sum coordination

- What any 3-colouring algorithms has to solve for every row of the grid:
  - label nodes with **$\{+1, 0, -1\}$**
  - there is some function ***q*** so that the ***sum of labels*** is $q(n)$ in any n-cycle, regardless of unique identifiers
  - $q(n)$ ***odd*** iff n is odd: cannot label everything with 0
  - $|q(n)|$ ***not too large***: cannot label everything with +1

Sum coordination

- What any 3-colouring algorithms has to solve for every row of the grid
- Requires global coordination

Conclusions


2-colouring
3-colouring
4-colouring
global
global
local

Conclusions: LCLs on grids

- Only three complexity classes in 2D grids: trivial $O(1)$, local $\Theta(\log^* n)$, global $\Theta(n)$
- **4-colouring is local**: algorithm synthesis
- **3-colouring is global**: sum coordination
- Can be generalised to *d*-dimensional grids!