Approximating vertex covers in anonymous networks

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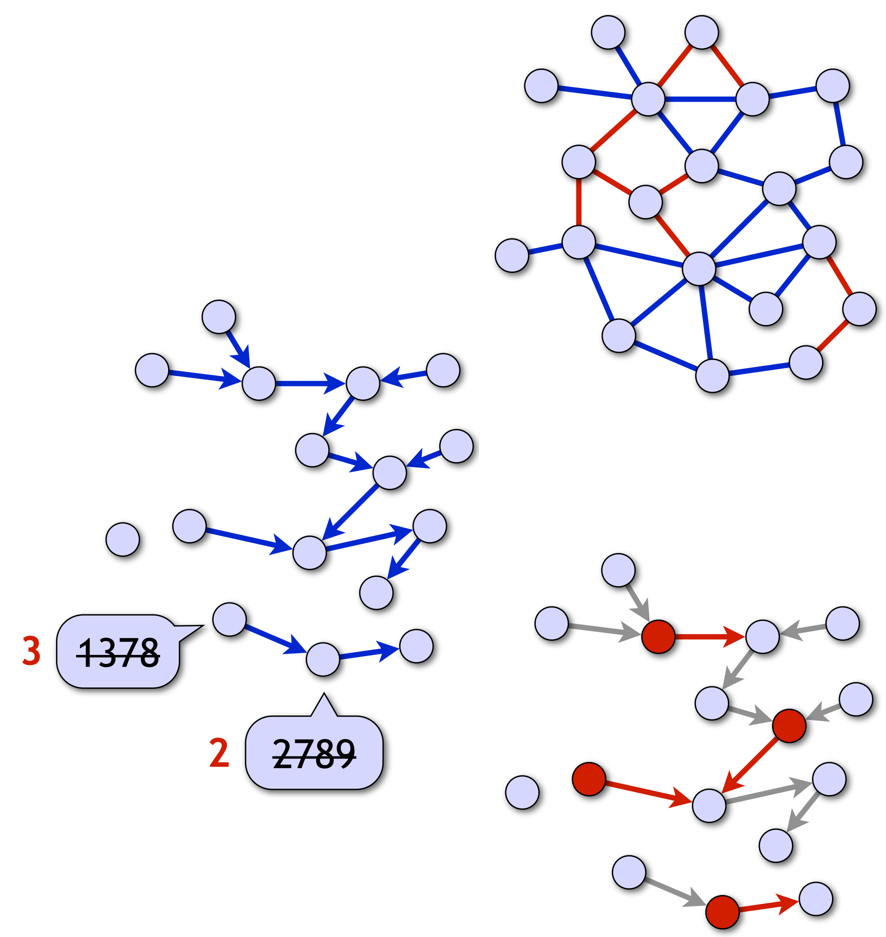
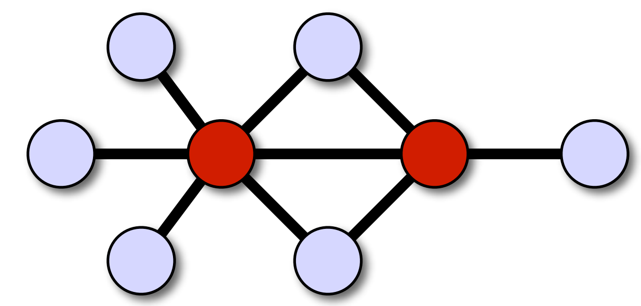


Vertex cover problem

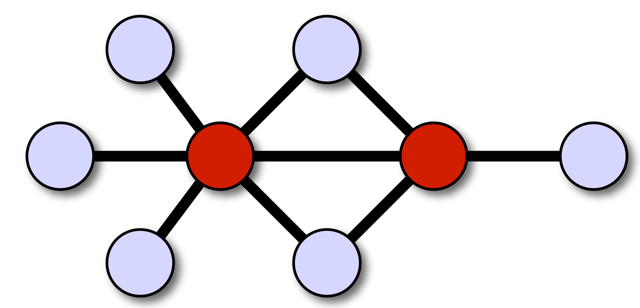
- Vertex cover for a graph G:
  - Subset C of nodes that “covers” all edges: each edge incident to at least one node in C



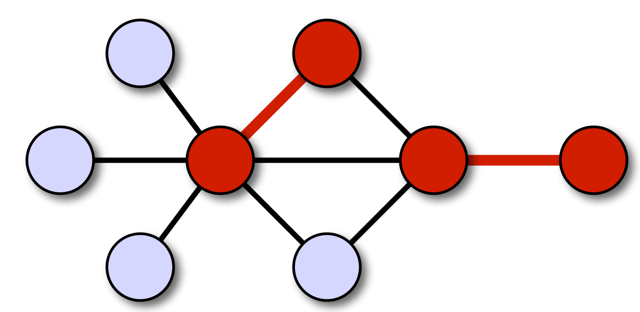
- Minimum vertex cover:
  - Vertex cover with the smallest number of nodes
- Minimum-weight vertex cover:
  - Vertex cover with the smallest total weight

Vertex cover problem

- Classical NP-hard optimisation problem: given a graph *G*, find a minimum vertex cover



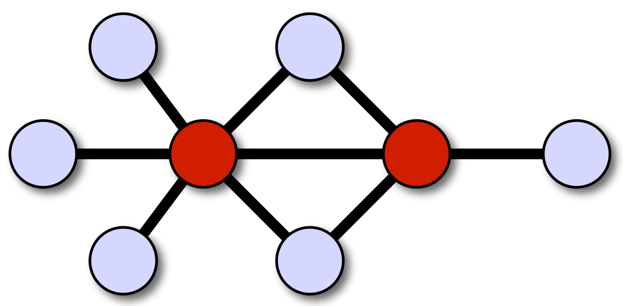
- Simple 2-approximation algorithm:
  - Find a **maximal matching**, output all endpoints
  - At most 2 times as large as minimum VC



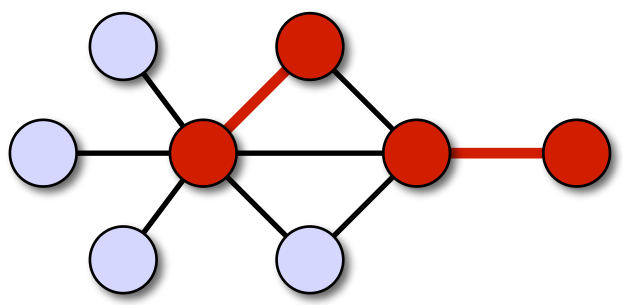
- No polynomial-time algorithm with approximation factor 1.9999 known

Research question

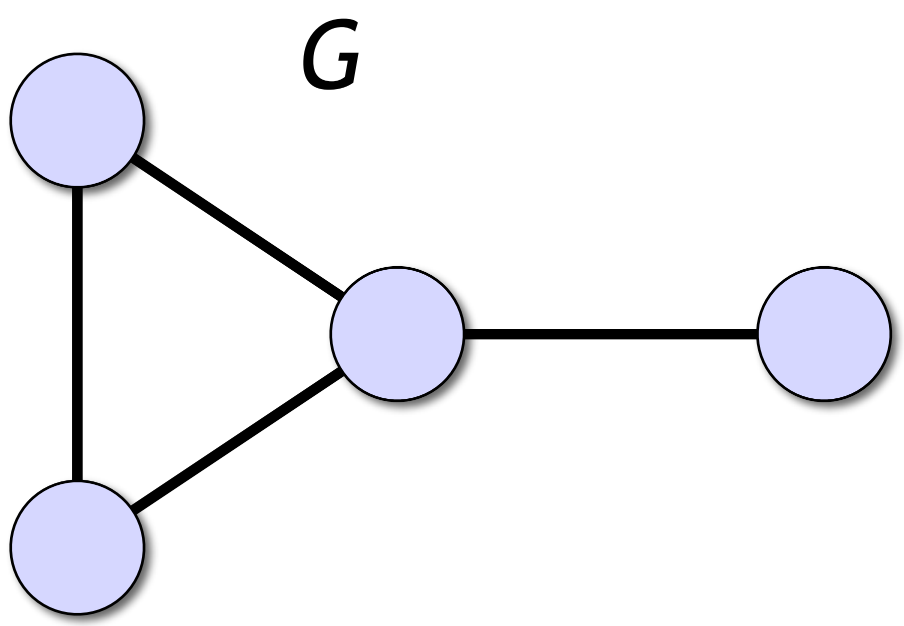
- Exactly how well can we approximate vertex cover in a **distributed setting**?



- Focus:
  - Fast, synchronous, **deterministic** distributed algorithms
  - Weakest possible models

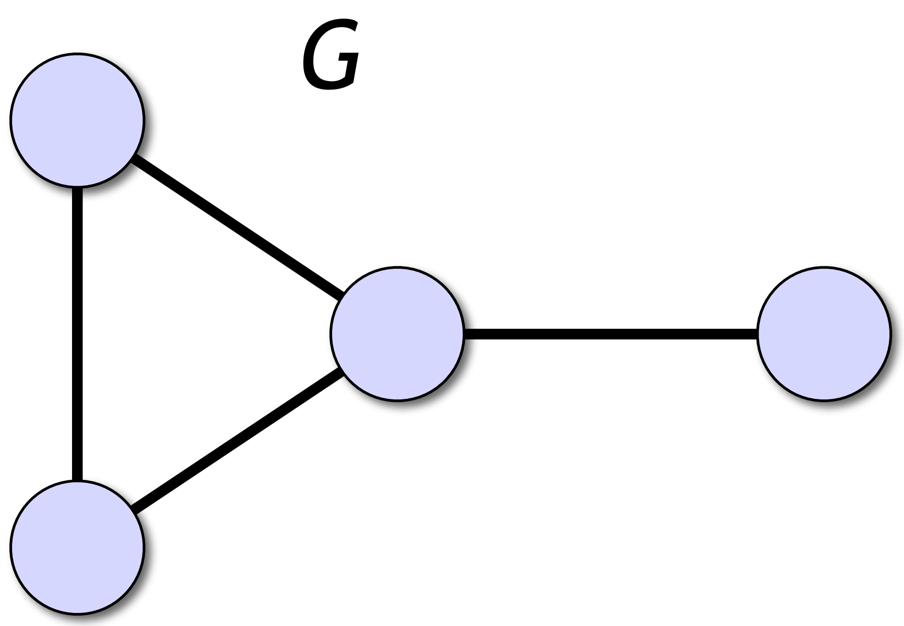


Distributed algorithms



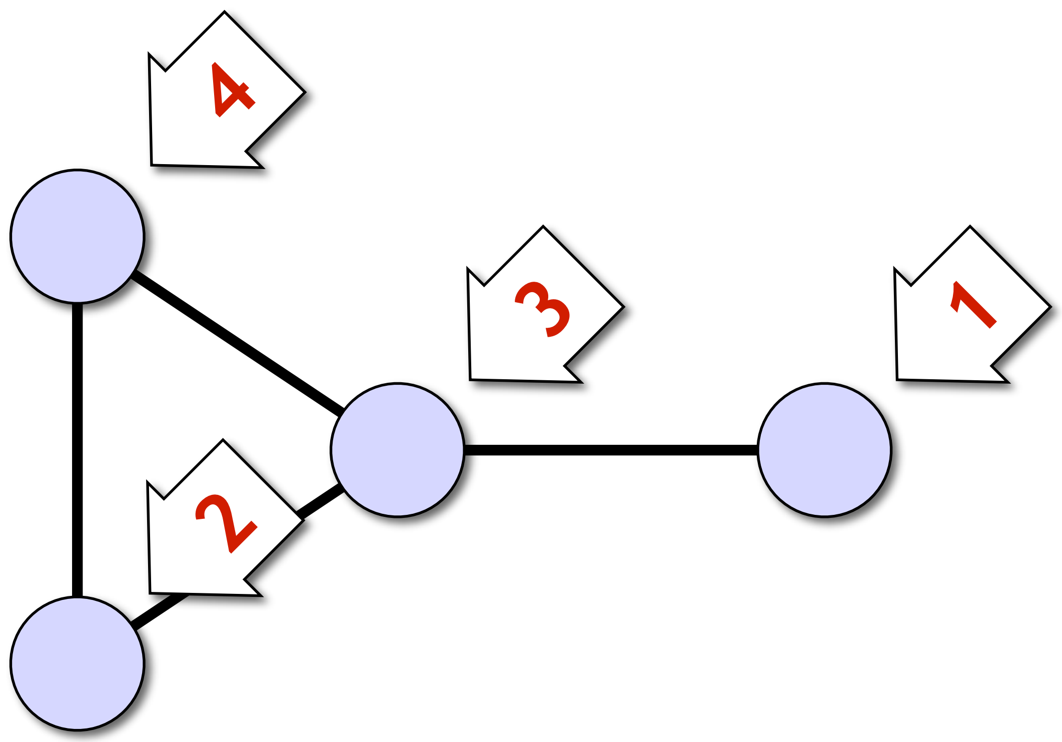
- Communication graph *G*
- Node = computer
  - e.g., Turing machine, finite state machine
- Edge = communication link
  - computers can exchange messages

Distributed algorithms



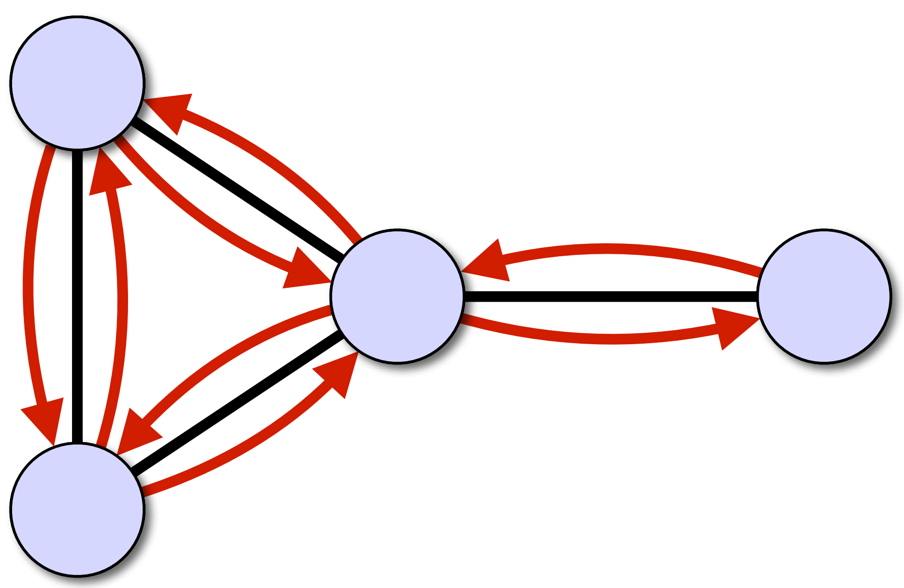
- All nodes are identical, run the same algorithm
- ***We*** can choose the algorithm
- An ***adversary*** chooses the structure of G
- Our algorithm must produce a valid vertex cover in any graph G

Synchronous distributed algorithms



1. Each node reads its own **local input**:
    - node identifier
        - if we assume unique node IDs
    - node weight
        - if we study weighted graphs

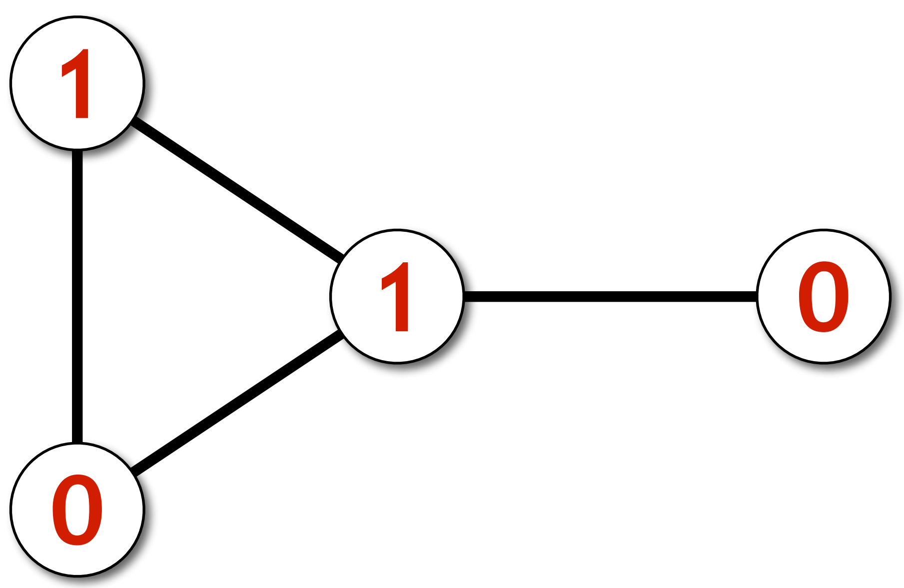
Synchronous distributed algorithms



1. Each node reads its own **local input**
2. Repeat synchronous communication rounds

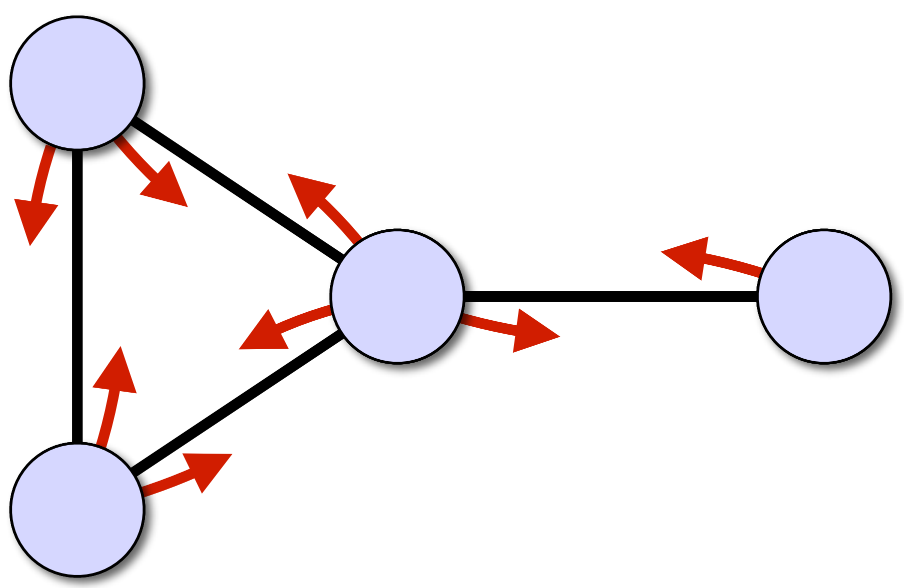
   …

Synchronous distributed algorithms



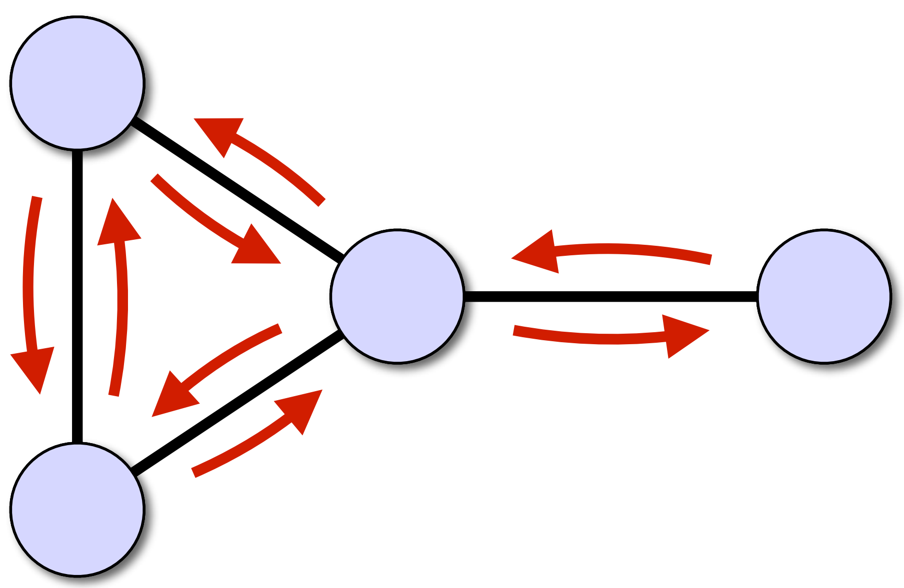
1. Each node reads its own **local input**
2. Repeat synchronous **communication rounds** until all nodes have announced their **local outputs**
   - 1 = in vertex cover

Synchronous distributed algorithms



- Communication round: each node
  1. sends a message to each neighbour

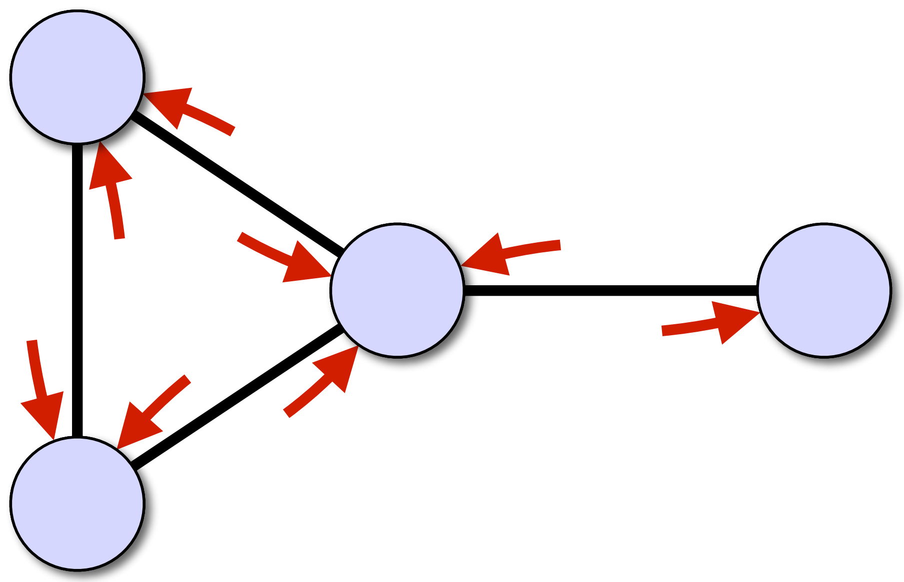
Synchronous distributed algorithms



- Communication round: each node
    1. sends a message to each neighbour

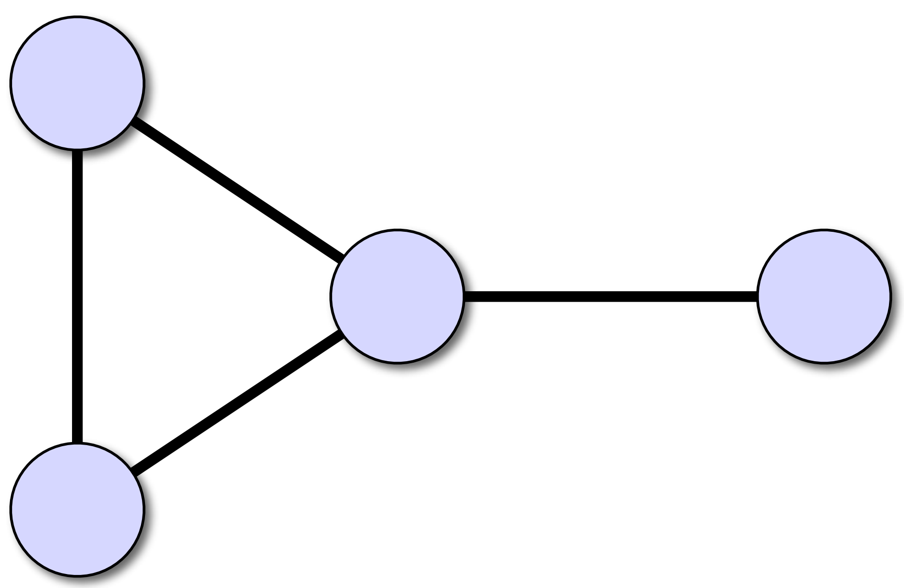
    (message propagation...)

Synchronous distributed algorithms



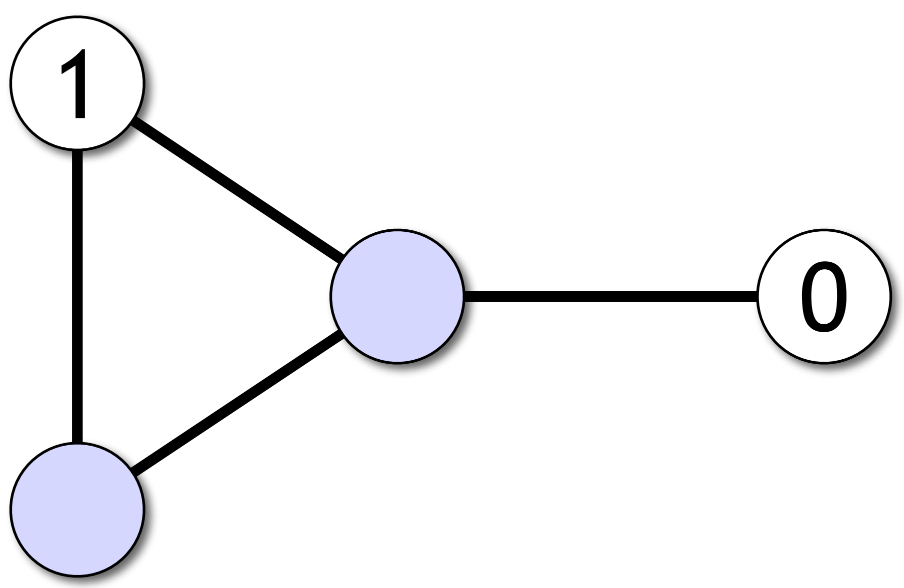
- Communication round: each node
  1. sends a message to each neighbour
  2. receives a message from each neighbour

Synchronous distributed algorithms



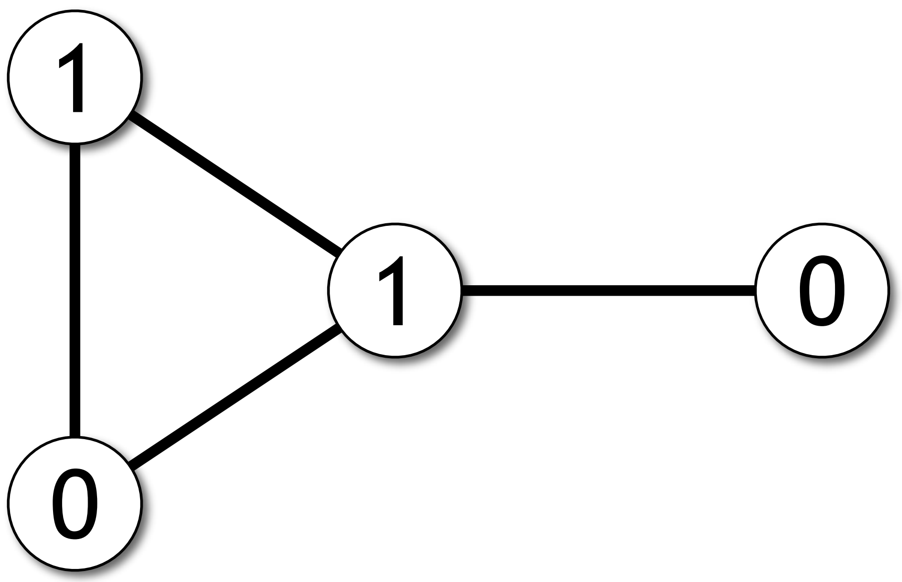
- Communication round: each node
  1. sends a message to each neighbour
  2. receives a message from each neighbour
  3. updates its own state

Synchronous distributed algorithms



- Communication round: each node
  1. sends a message to each neighbour
  2. receives a message from each neighbour
  3. updates its own state
  4. possibly stops and announces its output

Synchronous distributed algorithms

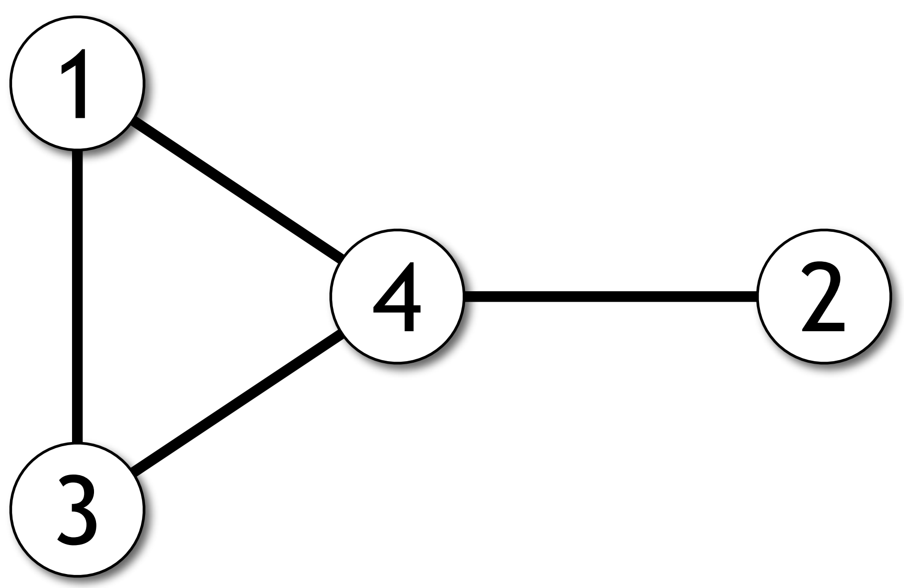


- Communication rounds are repeated until all nodes have stopped and announced their outputs
- Running time = **number of rounds**
- Worst-case analysis

Distributed algorithms: three models

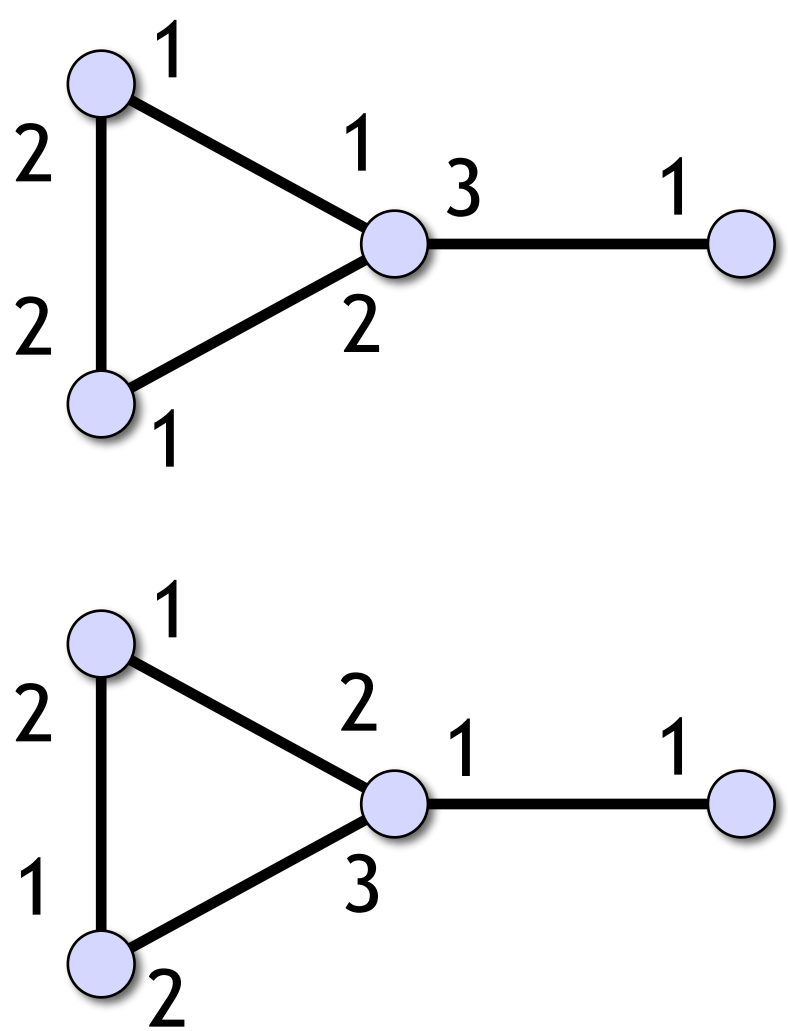
1. Unique identifiers
2. Port-numbering model
3. Broadcast model

Model 1: Unique identifiers



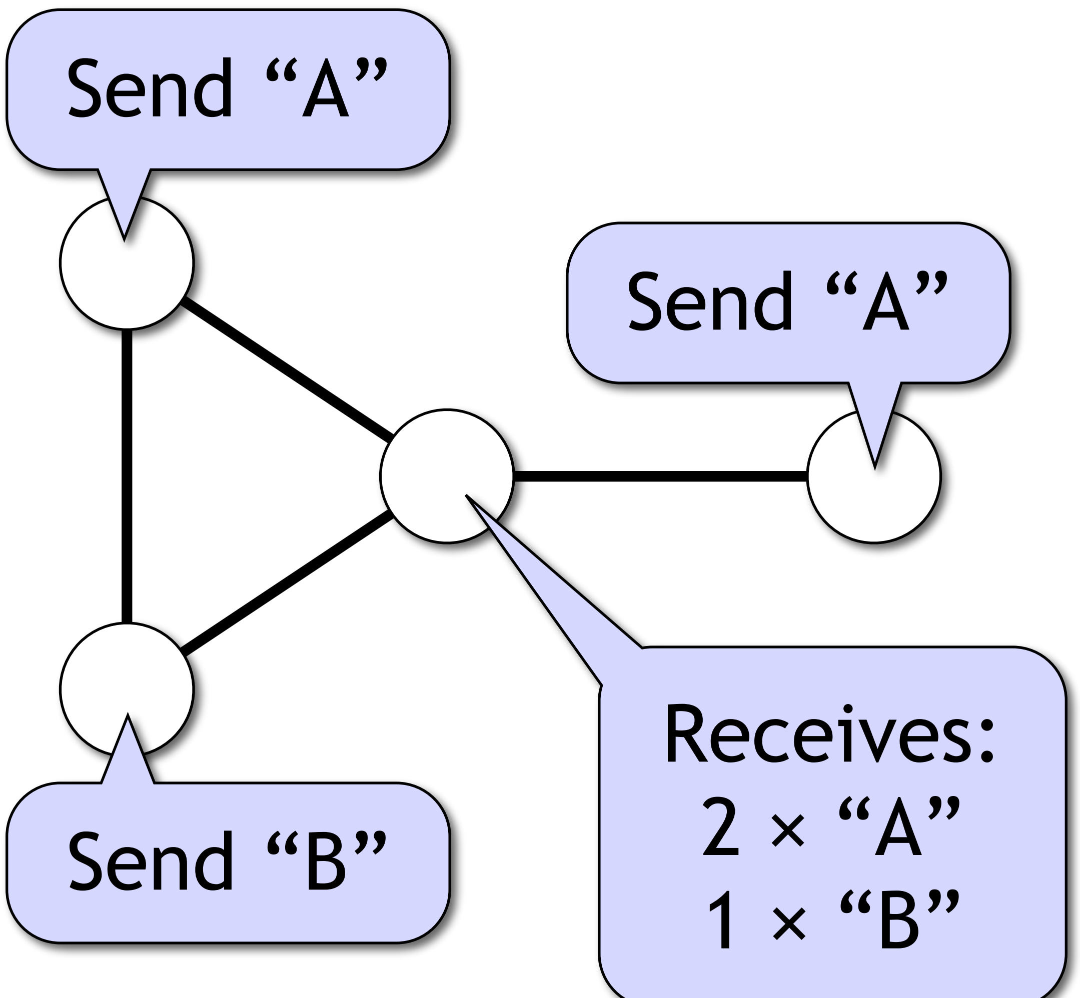
- Node identifiers are a permutation of 1, 2, …, n
- Permutation chosen by adversary

Model 2:
Port-numbering model



- No unique identifiers
- A node of degree *d* can refer to its neighbours by integers 1, 2, ..., *d*
- Port-numbering chosen by adversary

Model 3: Broadcast model



- No identifiers, no port numbers
- A node has to send the **same message** to each neighbour
- A node does not know which message was received from which neighbour

Distributed algorithms: three models

1. Unique identifiers
2. Port-numbering model
    - Vector with deg(v) outgoing messages
    - **Vector** with deg(v) incoming messages
3. Broadcast model
    - Only one outgoing message
    - **Multiset** with deg(v) incoming messages

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$						
$f(\Delta)$ + polylog(n)						
$f(\Delta) + O(\log^* n)$						
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

log* = iterated logarithm
≈ inverse of power tower

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$						**1**
$f(\Delta)$ + polylog(n)						
$f(\Delta) + O(\log^* n)$						
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

Trivial algorithm

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$						1
$f(\Delta)$ + polylog(n)						**2**
$f(\Delta) + O(\log^* n)$						**2**
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

Maximal matching

(Panconesi & Rizzi 2001)

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$				**2**		1
$f(\Delta)$ + polylog(n)				**2**		2
$f(\Delta) + O(\log^* n)$						2
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

Near-maximal edge packing

(Khuller et al. 1994)

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$				2		1
$f(\Delta)$ + polylog(n)				2		2
$f(\Delta) + O(\log^* n)$				**2 + ε**		2
$f(\Delta)$				**2 + ε**		**2 + ε**
	Broadcast model		Port numbering		Unique identifiers	

Deterministic LP rounding

(Kuhn et al. 2006)

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$				2		1
$f(\Delta)$ + polylog(n)				2		2
$f(\Delta) + O(\log^* n)$				$2 + \varepsilon$		2
$f(\Delta)$	**2**		**2**	$2 + \varepsilon$	**2**	$2 + \varepsilon$
	Broadcast model		Port numbering		Unique identifiers	

Czygrinow et al. 2008

Lenzen & Wattenhofer 2008

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	**2**		**2**	2		1
$f(\Delta)$ + polylog(n)	**2**		**2**	2		2
$f(\Delta) + O(\log^* n)$	**2**		**2**	2 + ε		2
$f(\Delta)$	**2**		**2**	2 + ε	2	2 + ε
	Broadcast model		Port numbering		Unique identifiers	

Trivial (cycles)

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2		2	2		1
$f(\Delta)$ + polylog(n)	2		2	2		2
$f(\Delta) + O(\log^* n)$	2		2	$2 + \varepsilon$		2
$f(\Delta)$	2		2	$2 + \varepsilon$	2	$2 + \varepsilon$
	Broadcast model		Port numbering		Unique identifiers	

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2		2	2		1
$f(\Delta)$ + polylog(n)	2		2	2		
$f(\Delta) + O(\log^* n)$	2		2	**2 + ε**		
$f(\Delta)$	2		2	**2 + ε**	2	**2 + ε**
	Broadcast model		Port numbering		Unique identifiers	

Could we have 2?

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2	**?**				1
$f(\Delta)$ + polylog(n)	2	**?**				
$f(\Delta) + O(\log^* n)$	2	**?**	2	**2 + ε**		
$f(\Delta)$	2	**?**	2	**2 + ε**	2	**2 + ε**
	Broadcast model		Port numbering		Unique identifiers	

Anything here?

Could we have 2?

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2	?	2	2		1
$f(\Delta)$ + polylog(n)	2	?	2	2		
$f(\Delta) + O(\log^* n)$	2	?	2	2		
$f(\Delta)$	2	?	2	$\boxed{2}$	2	2
	Broadcast model		Port numbering		Unique identifiers	

DISC 2009

Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2	**2**				1
$f(\Delta)$ + polylog(n)	2	**2**				
$f(\Delta) + O(\log^* n)$	2	**2**	2	**2**		
$f(\Delta)$	2	**2**	2	**2**	2	**2**
	Broadcast model		Port numbering		Unique identifiers	

Latest results

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Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2	**2**	2	2		1
$f(\Delta)$ + polylog(n)	2	**2**	2	2		2
$f(\Delta) + O(\log^* n)$	2	**2**	2	**2**		2
$f(\Delta)$	2	**2**	2	**2**	2	**2**
	Broadcast model		Port numbering		Unique identifiers	

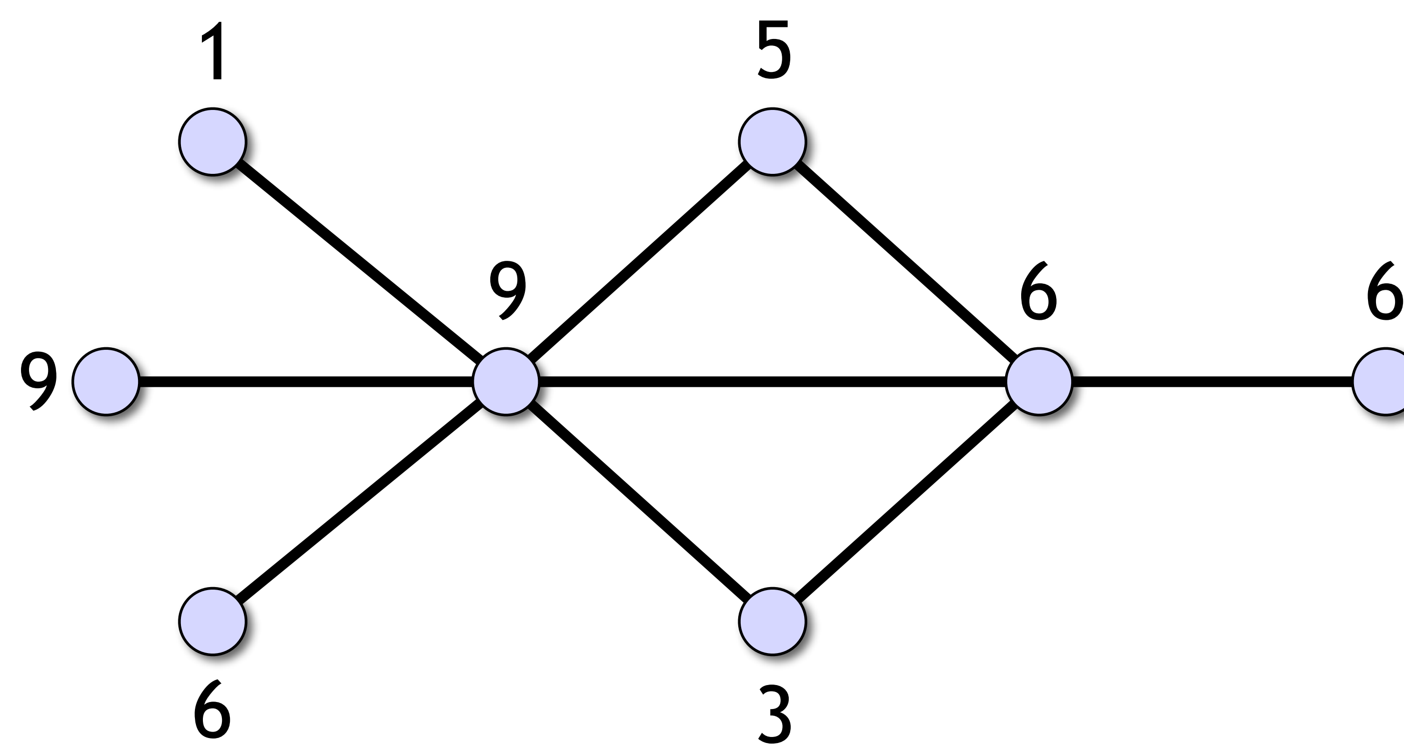
Deterministic distributed algorithms for vertex cover: approximation ratios

	lower	upper	lower	upper	lower	upper
$O(n)$	2	**2**	2	2		1
$f(\Delta)$ + polylog(n)	2	**2**	2	2		
$f(\Delta) + O(\log^* n)$	2	**2**	2	**2**		
$f(\Delta)$	2	**2**	2	**2**	2	**2**
	Broadcast model		Port numbering		Unique identifiers	

Let's study this case first...

Vertex cover in the port-numbering model

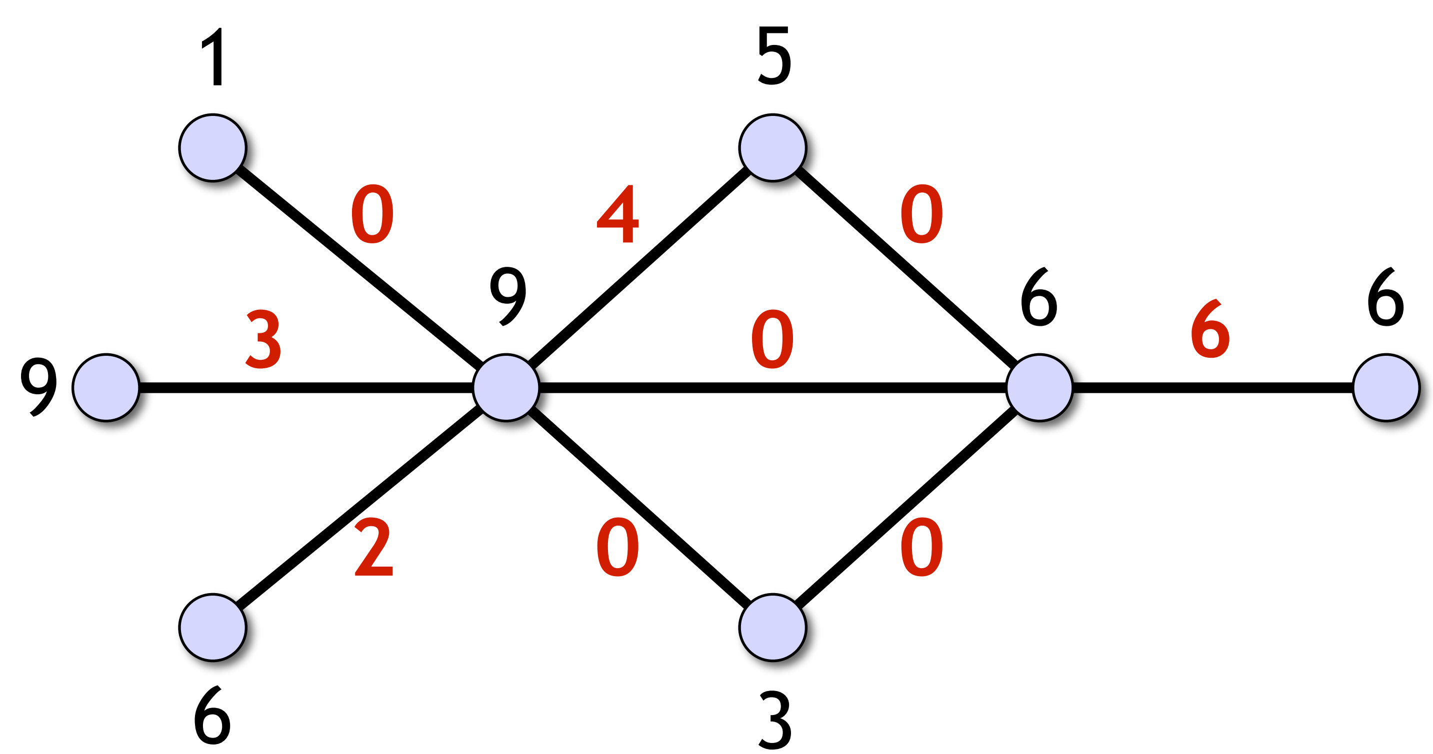
- Convenient to study a more general problem: minimum-weight vertex cover
  - More general problems are sometimes easier to solve!



Notation:
$w(v)$ = weight of v

Edge packings and vertex covers

- **Edge packing**: weight $y(e) \geq 0$ for each edge e
  - Packing constraint: for each node v, the total weight of edges incident to v is at most $w(v)$



Edge packings and vertex covers

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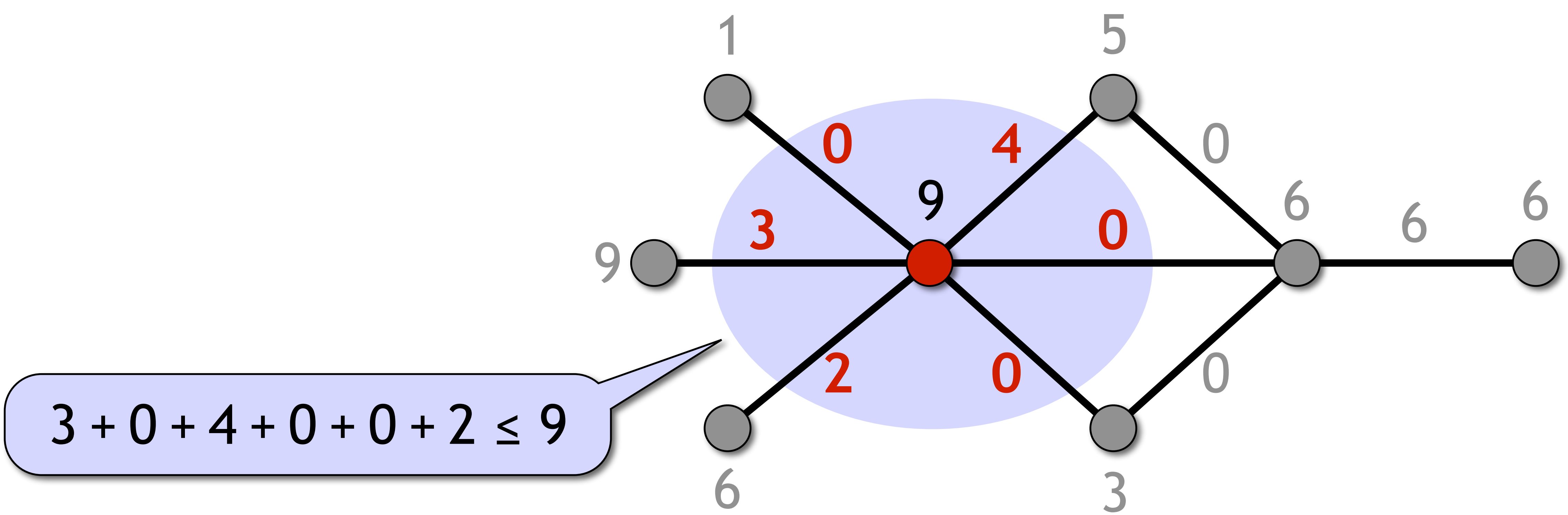
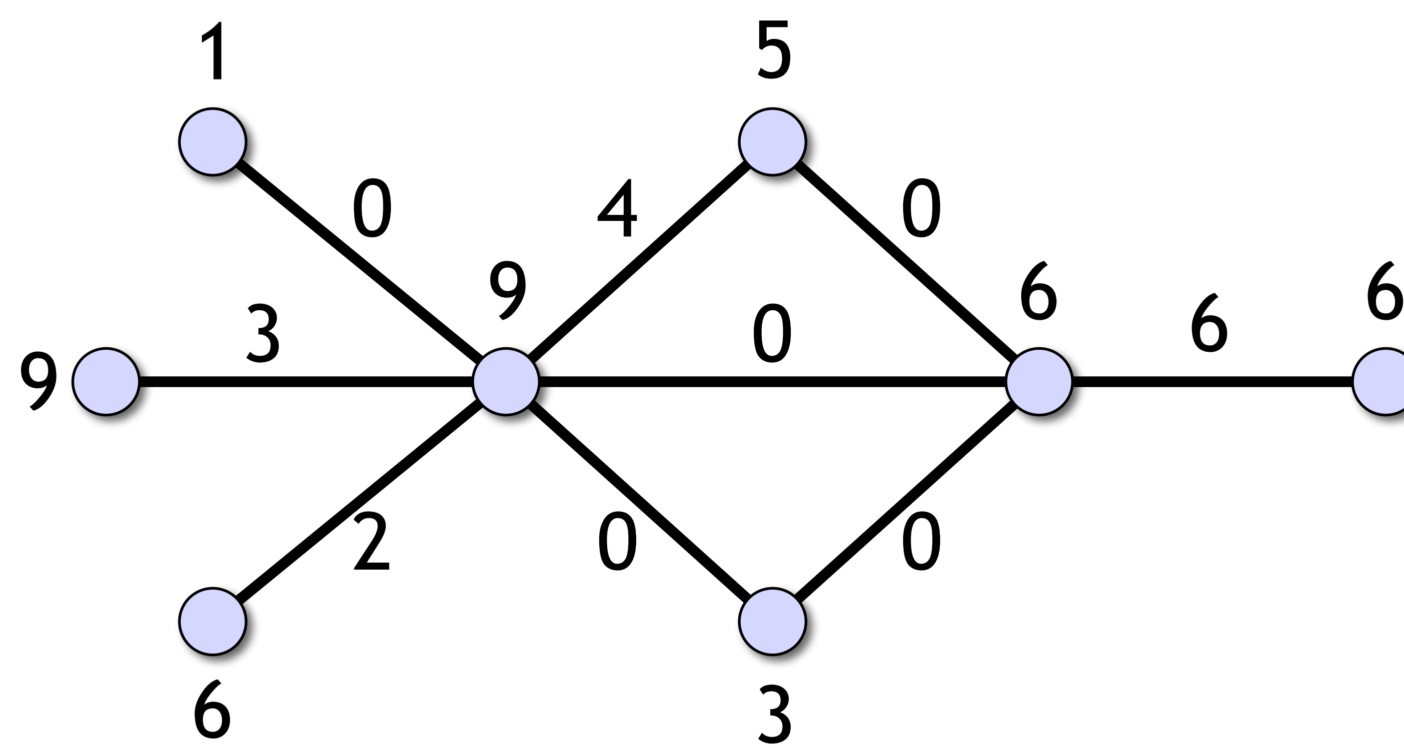


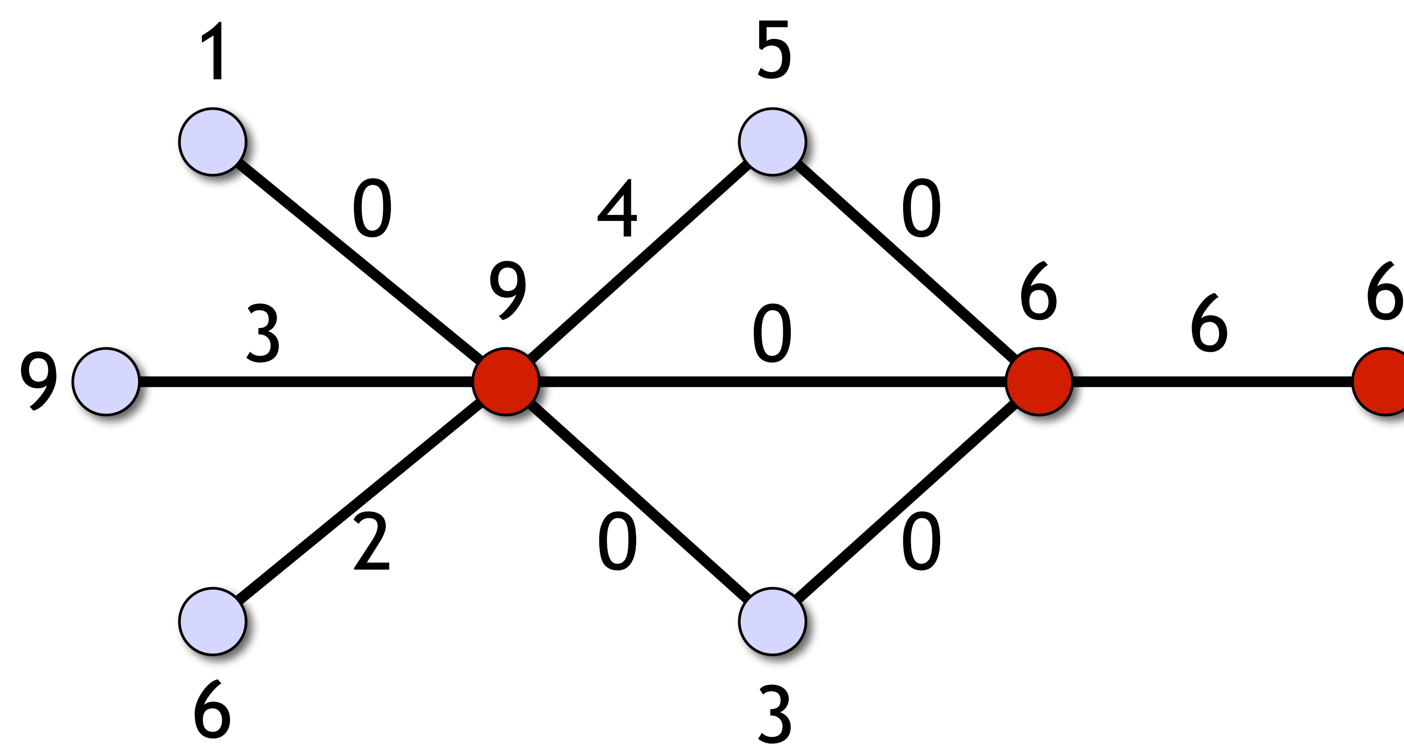
Edge packings and vertex covers

- In linear programming, these are dual problems:
  - minimum-weight (fractional) vertex cover
  - maximum-weight edge packing



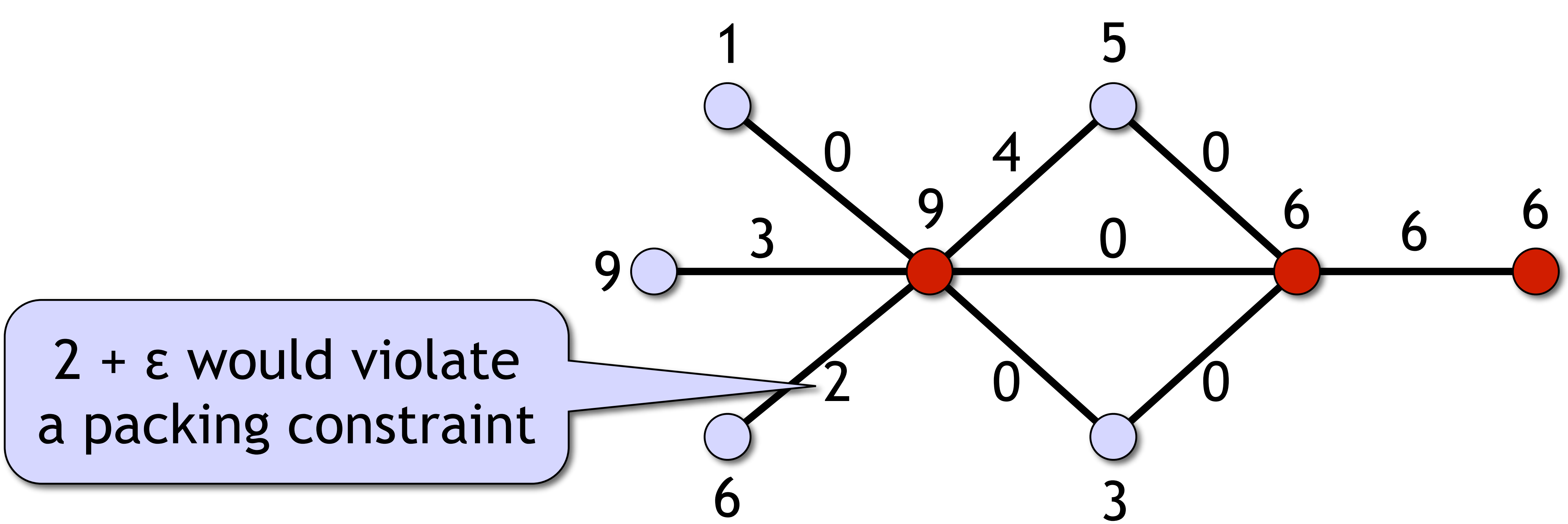
Edge packings and vertex covers

- **Saturated node** *v*: the total weight on edges incident to *v* is equal to $w(v)$



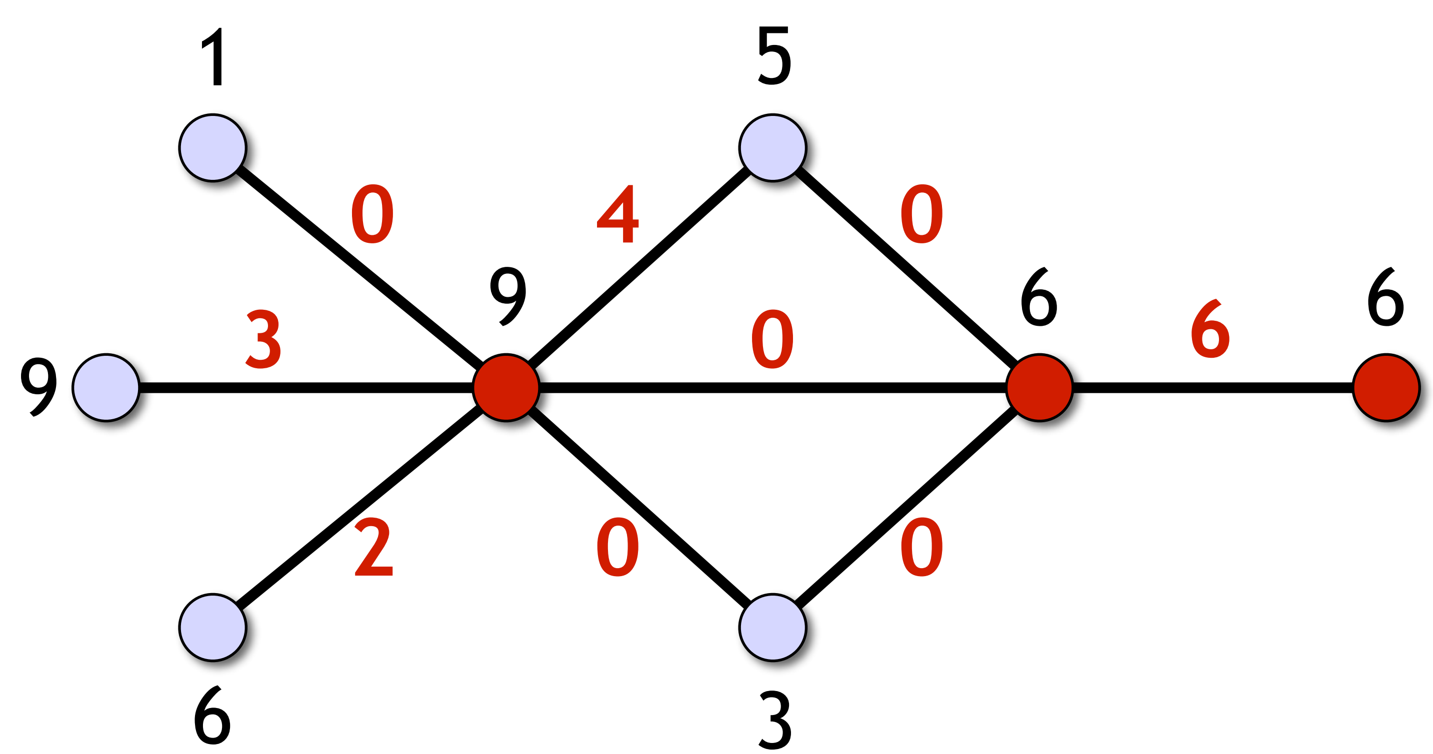
Edge packings and vertex covers

- **Saturated edge** *e*:
  at least one endpoint of *e* is saturated
  $\Longleftrightarrow$ edge weight $y(e)$ can't be increased



Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  $\Longleftrightarrow$ none of the edge weights $y(e)$ can be increased
  $\Longleftrightarrow$ saturated nodes form a vertex cover



Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find:
  - Centralised setting: NP-hard
  - Distributed setting: integer problem, symmetry-breaking issues
- Maximal edge packing y easy to find:
  - Centralised setting: trivial greedy algorithm
  - Distributed setting: linear problem, no symmetry-breaking issues (?)

Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find
- Maximal edge packing y easy to find?
- Saturated nodes $C(y)$ in y: **2-approximation** of C^*
  - $w(C(y)) \leq 2w(C^*)$
  - Notation: $w(C)$ = total weight of the nodes $v \in C$
  - Proof: LP-duality, relaxed complementary slackness

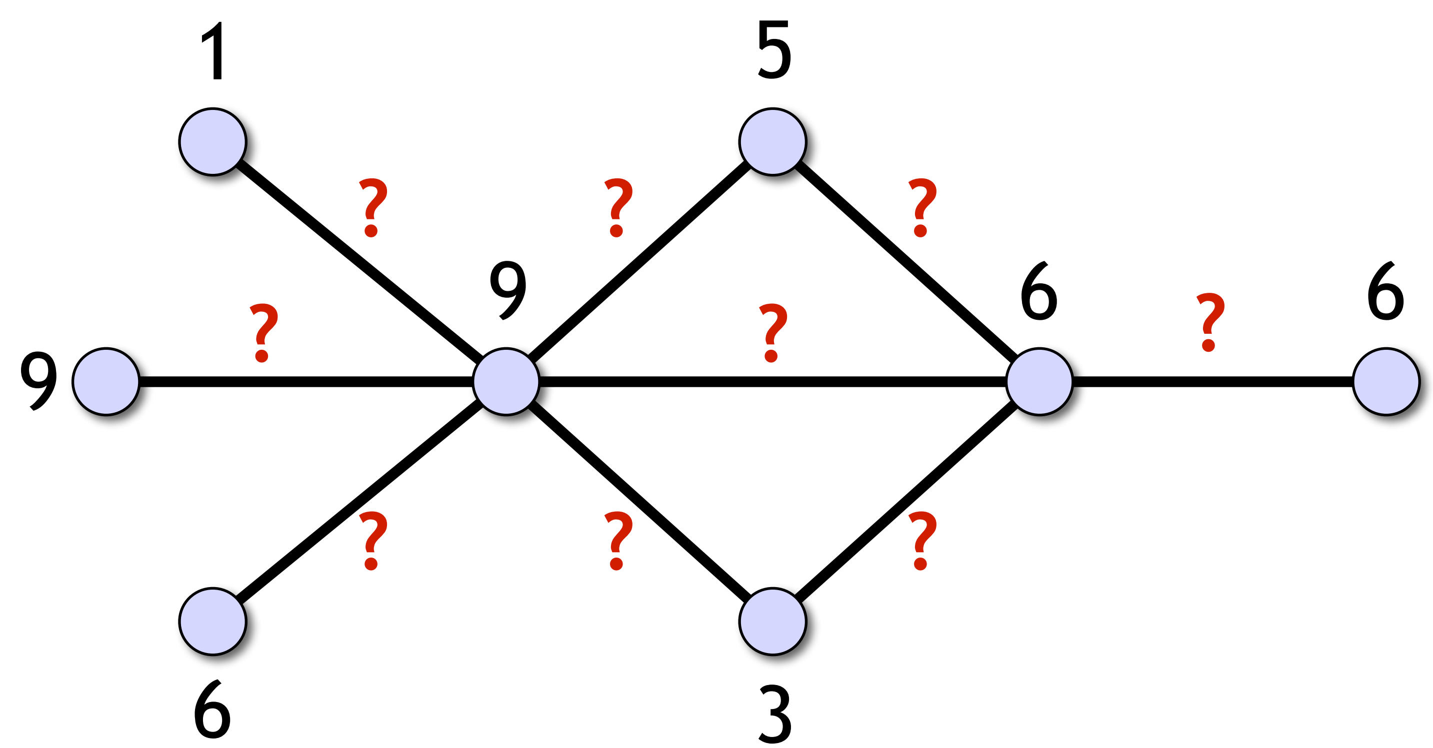
Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find
- Maximal edge packing y easy to find?
- Saturated nodes $C(y)$ in y: **2-approximation** of C^*
  - $w(C(y)) \leq 2w(C^*)$
  - Constant 2: $C(y)$ covers edges at most **twice**, C^* at least **once**
  - Immediate generalisation to hypergraphs

$$w(C(y)) = \sum_{v \in C(y)} y[v] = \sum_{e \in E} y(e)\,|e \cap C(y)| \leq 2 \sum_{e \in E} y(e)\,|e \cap C^*| = 2 \sum_{v \in C^*} y[v] \leq 2w(C^*)$$

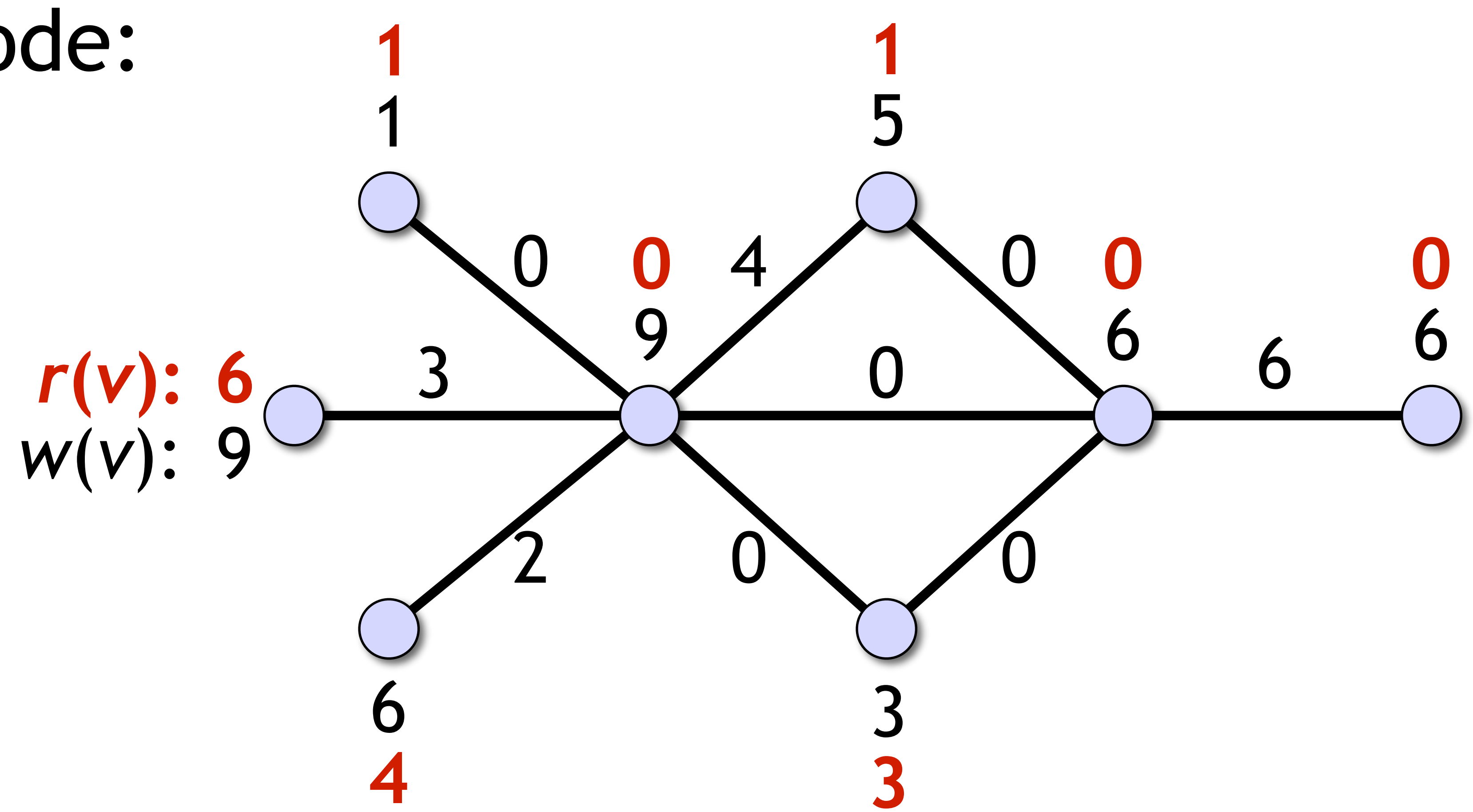
Finding a maximal edge packing

- Basic idea from Khuller et al. (1994) and Papadimitriou and Yannakakis (1993)



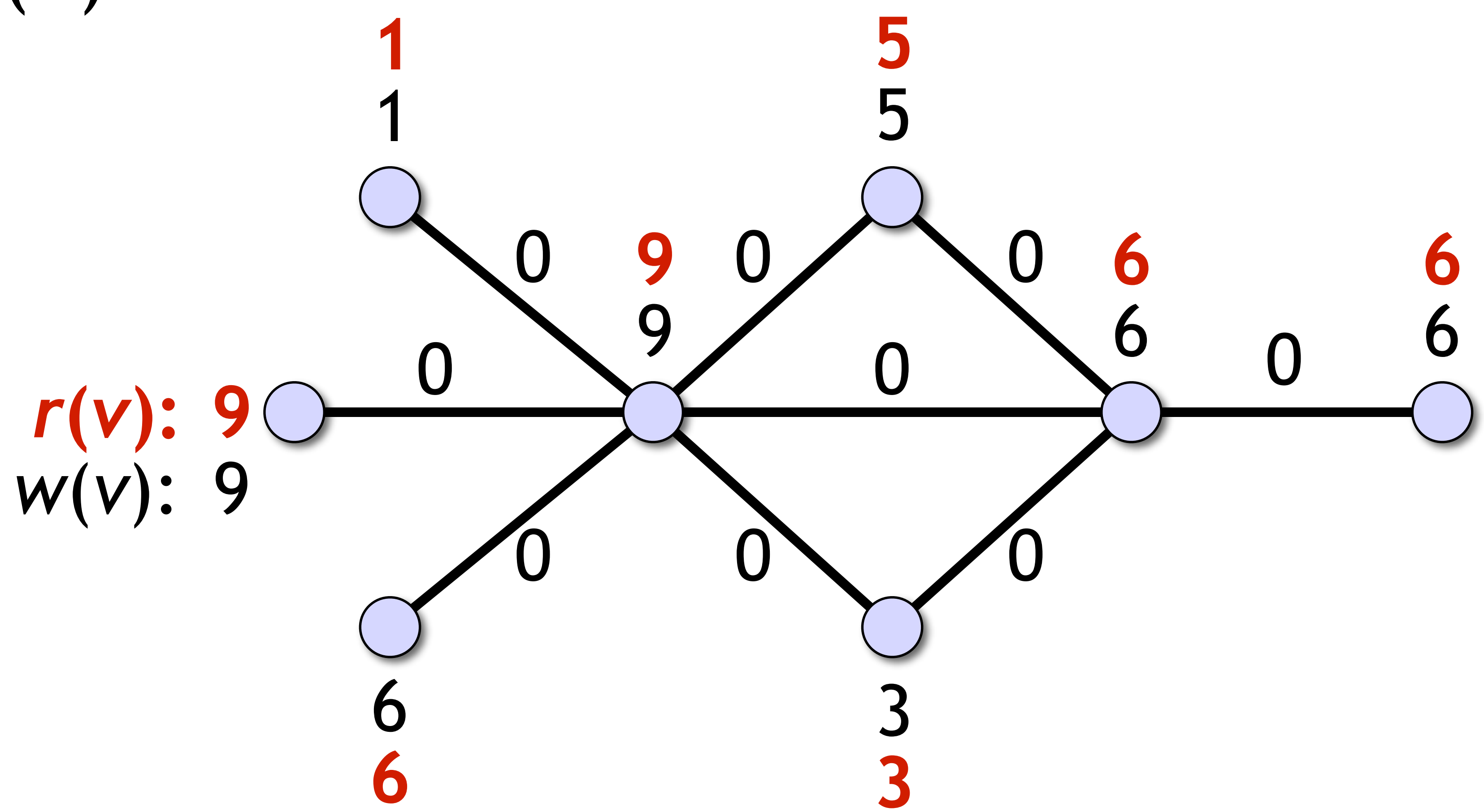
Finding a maximal edge packing: basic idea

- $y[v]$ = total weight of edges incident to node v
- **Residual capacity** of node v: $r(v) = w(v) - y[v]$
- Saturated node: $r(v) = 0$



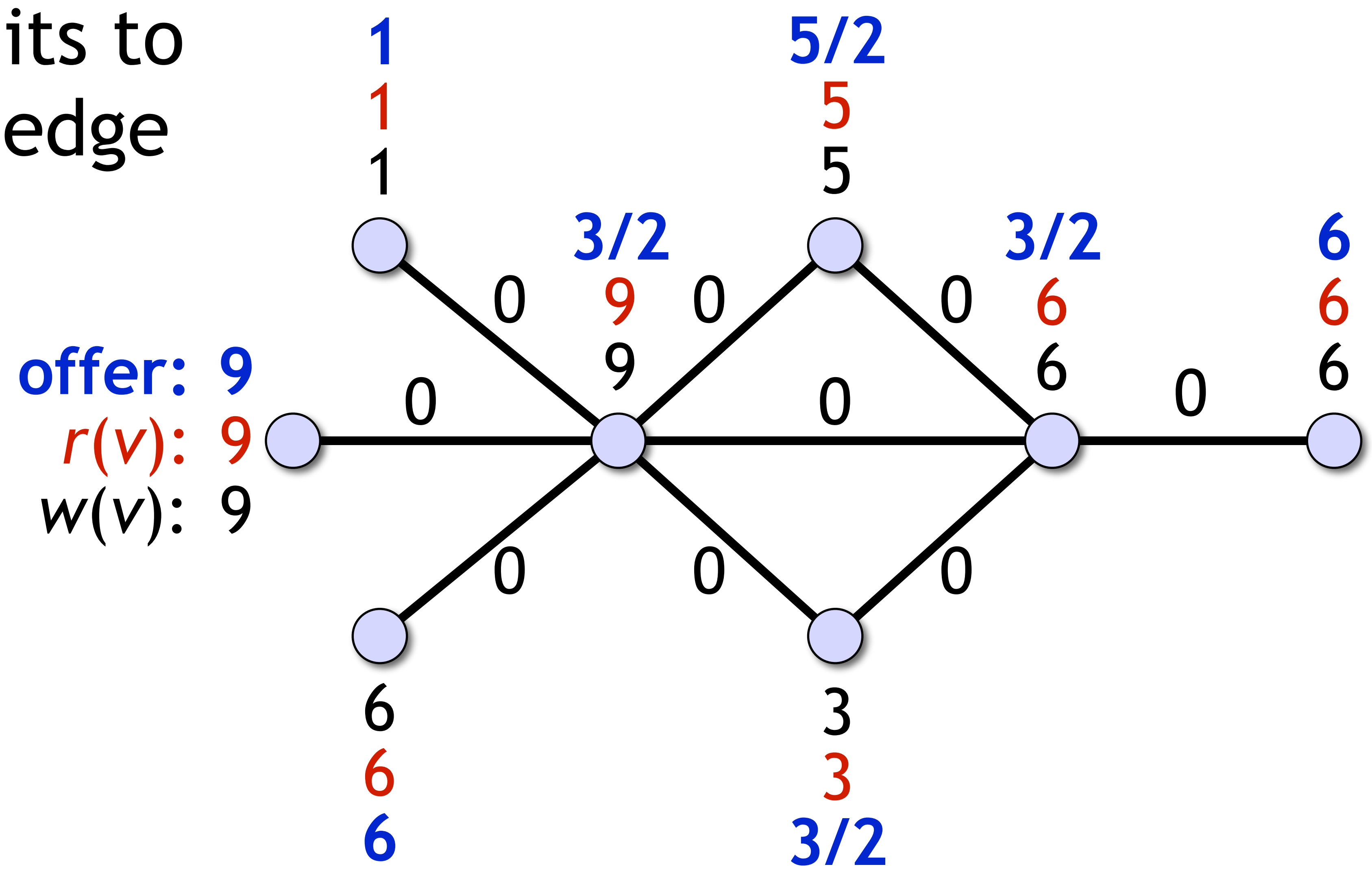
Finding a maximal edge packing: basic idea

Start with a trivial edge packing $y(e) = 0$



Finding a maximal edge packing: basic idea

Each node v **offers** $r(v)/\deg(v)$ units to each incident edge

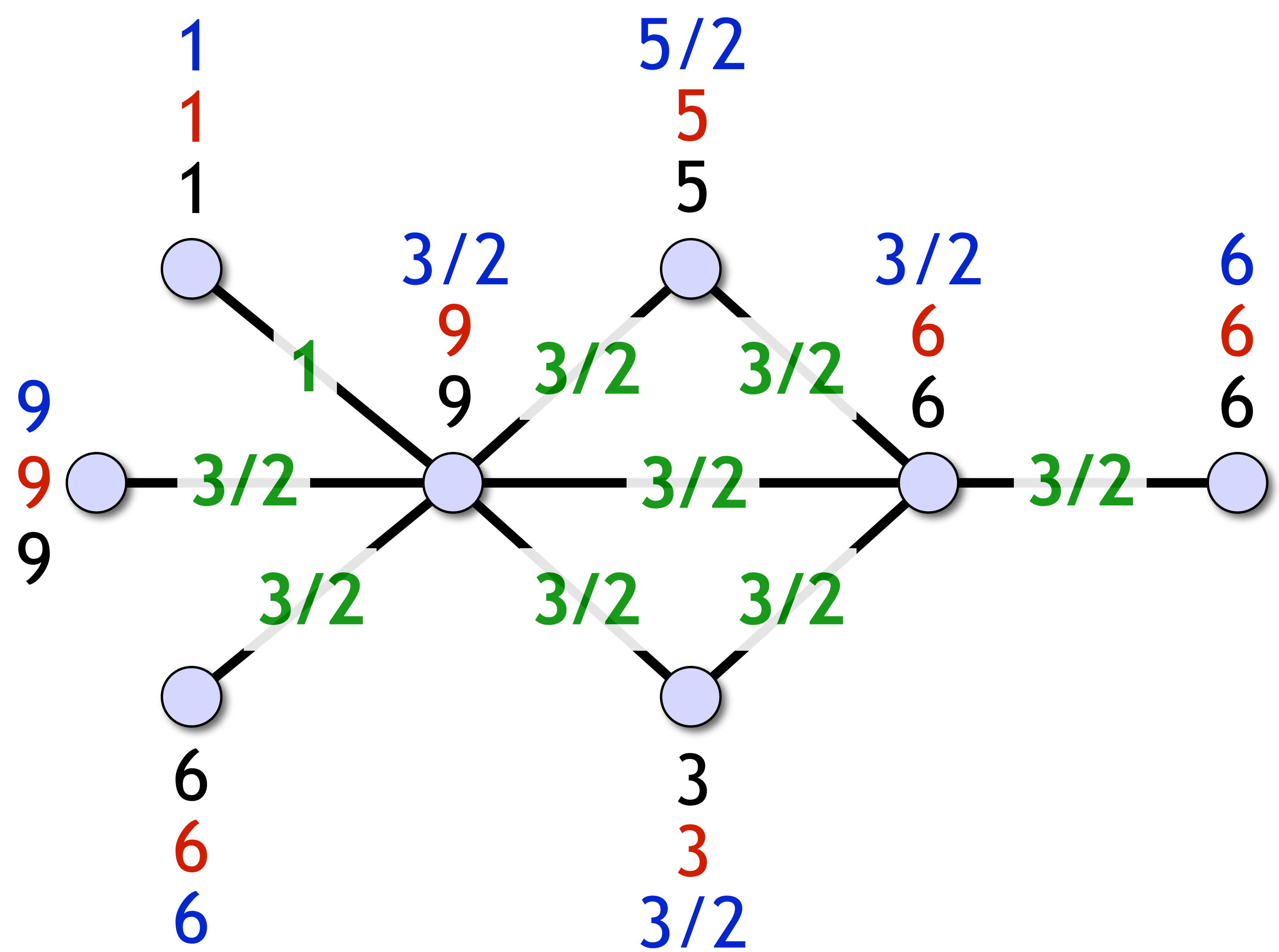


Finding a maximal edge packing: basic idea

Each edge **accepts** the smallest of the 2 offers it received

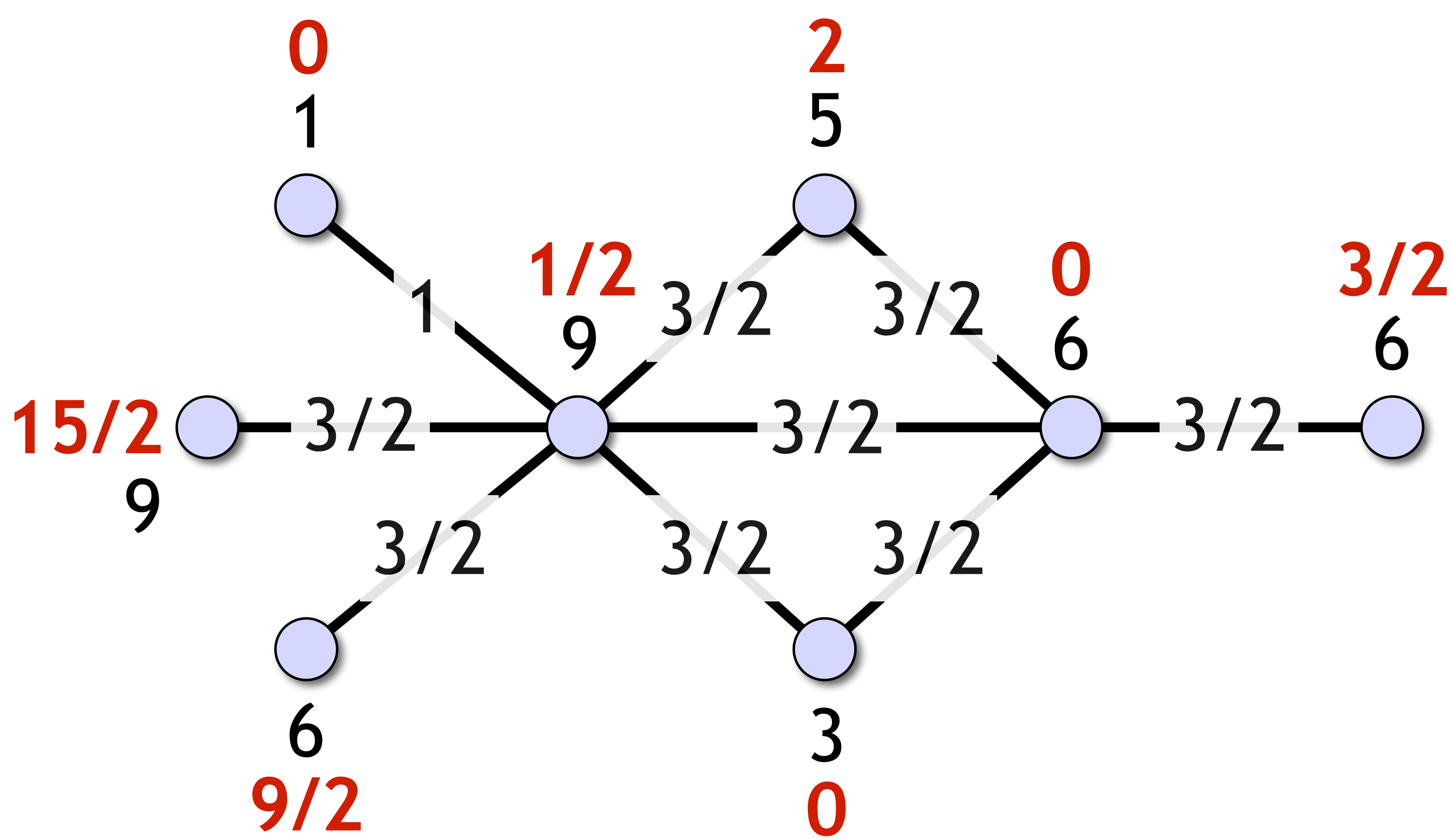
Increase *y*(*e*) by this amount

- Safe, can't violate packing constraints



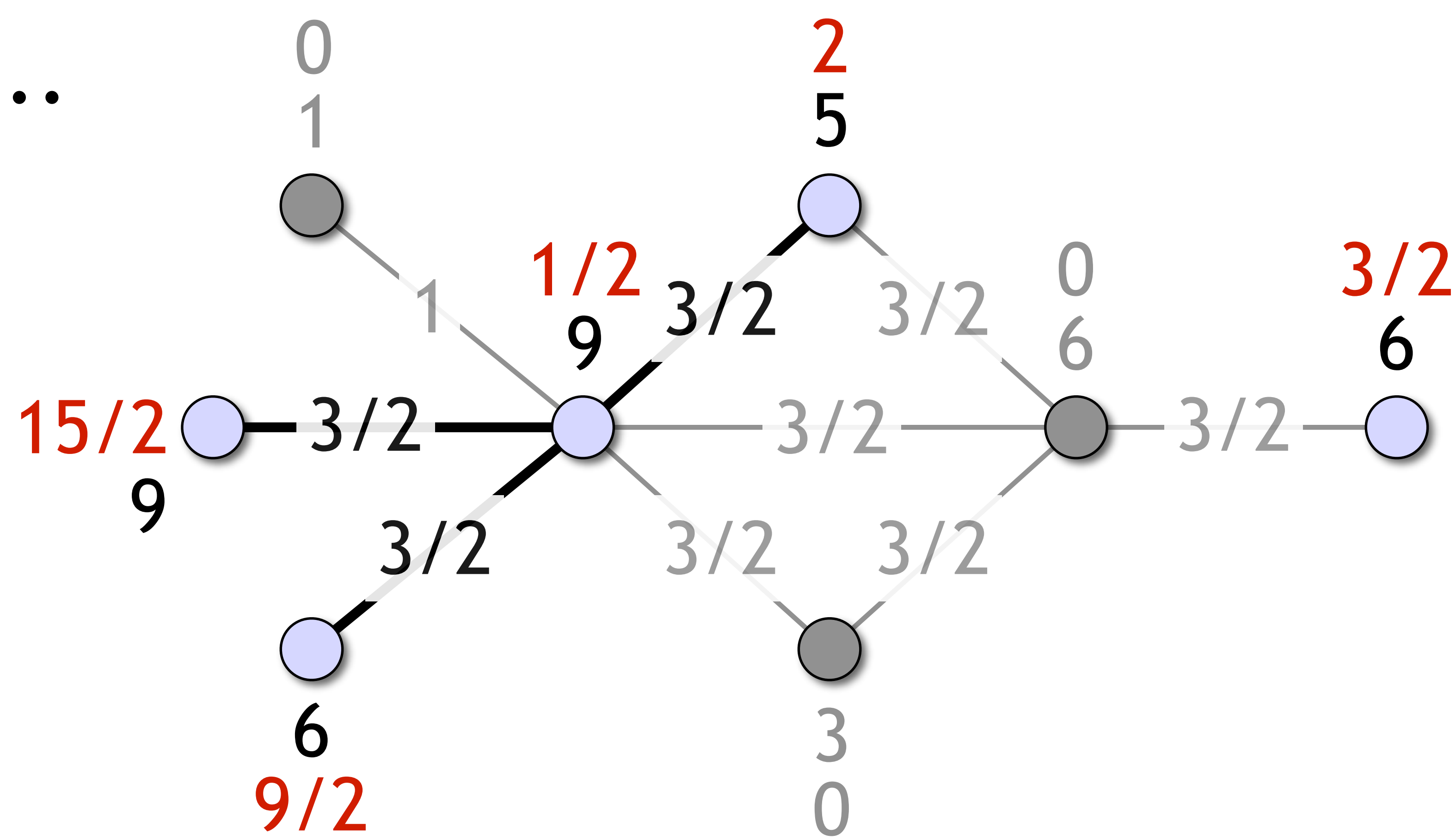
Finding a maximal edge packing: basic idea

Update **residuals**...



Finding a maximal edge packing: basic idea

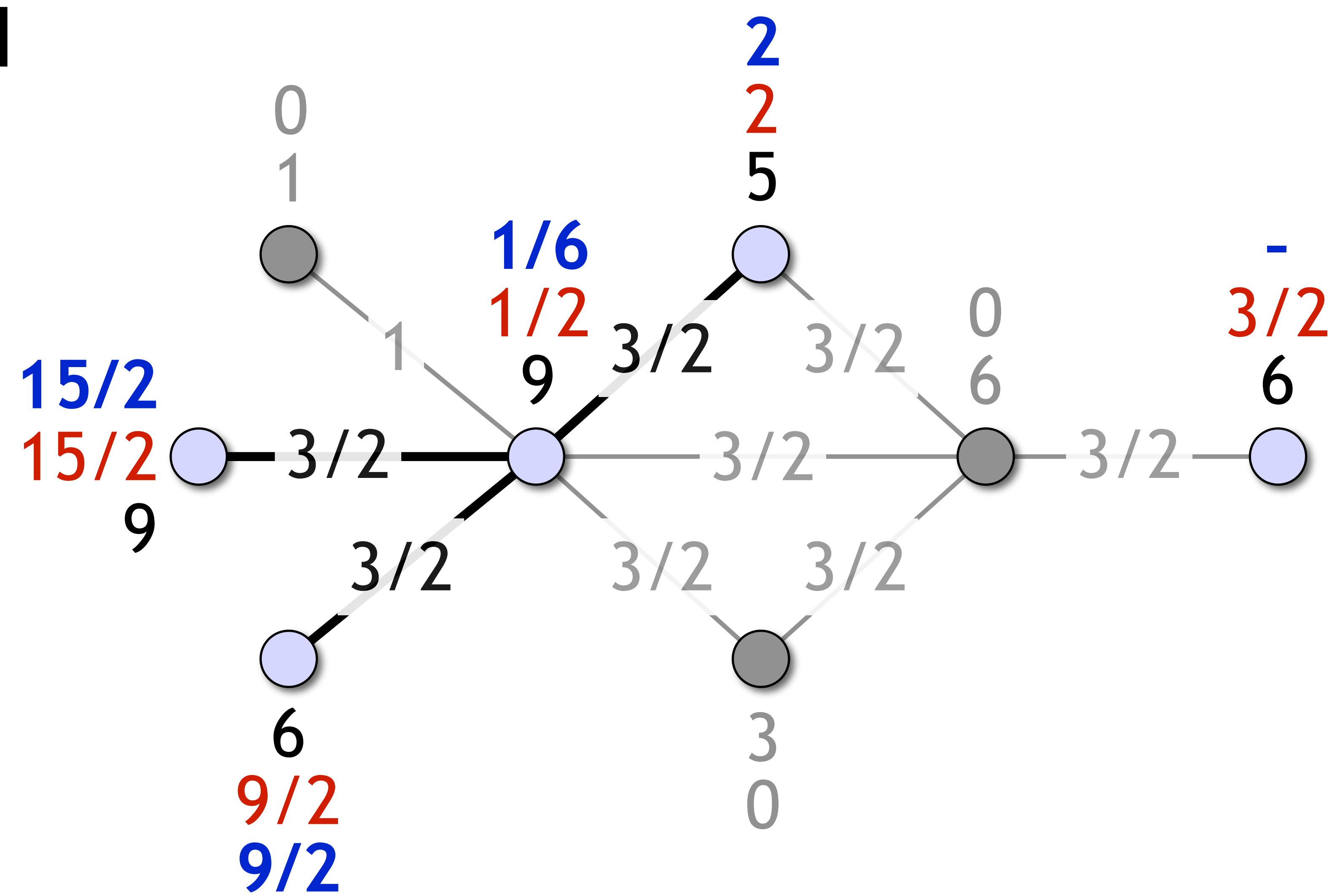
Update residuals, discard saturated nodes and edges...



Finding a maximal edge packing: basic idea

Update residuals, discard saturated nodes and edges, repeat...

Offers...

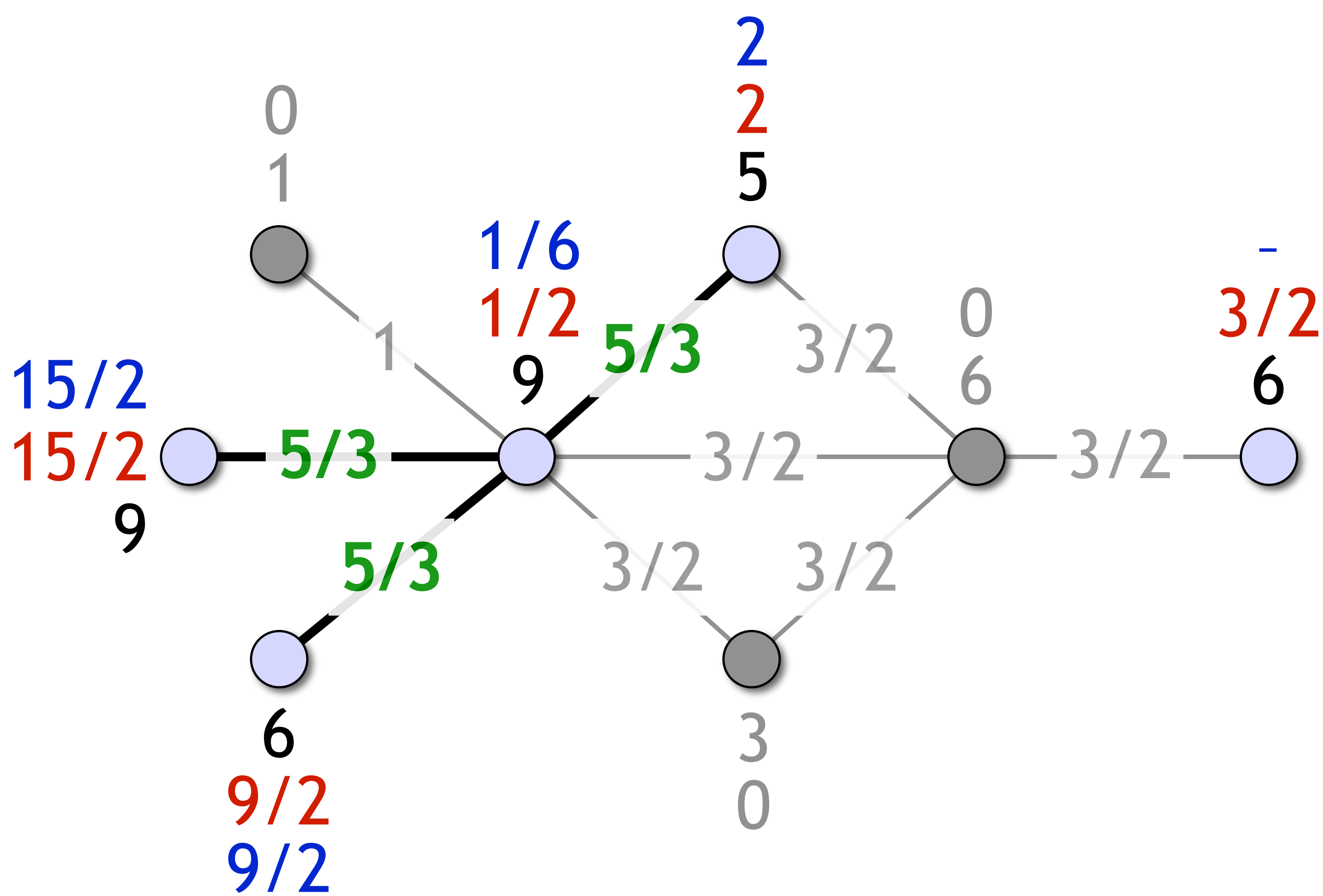


Finding a maximal edge packing: basic idea

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...



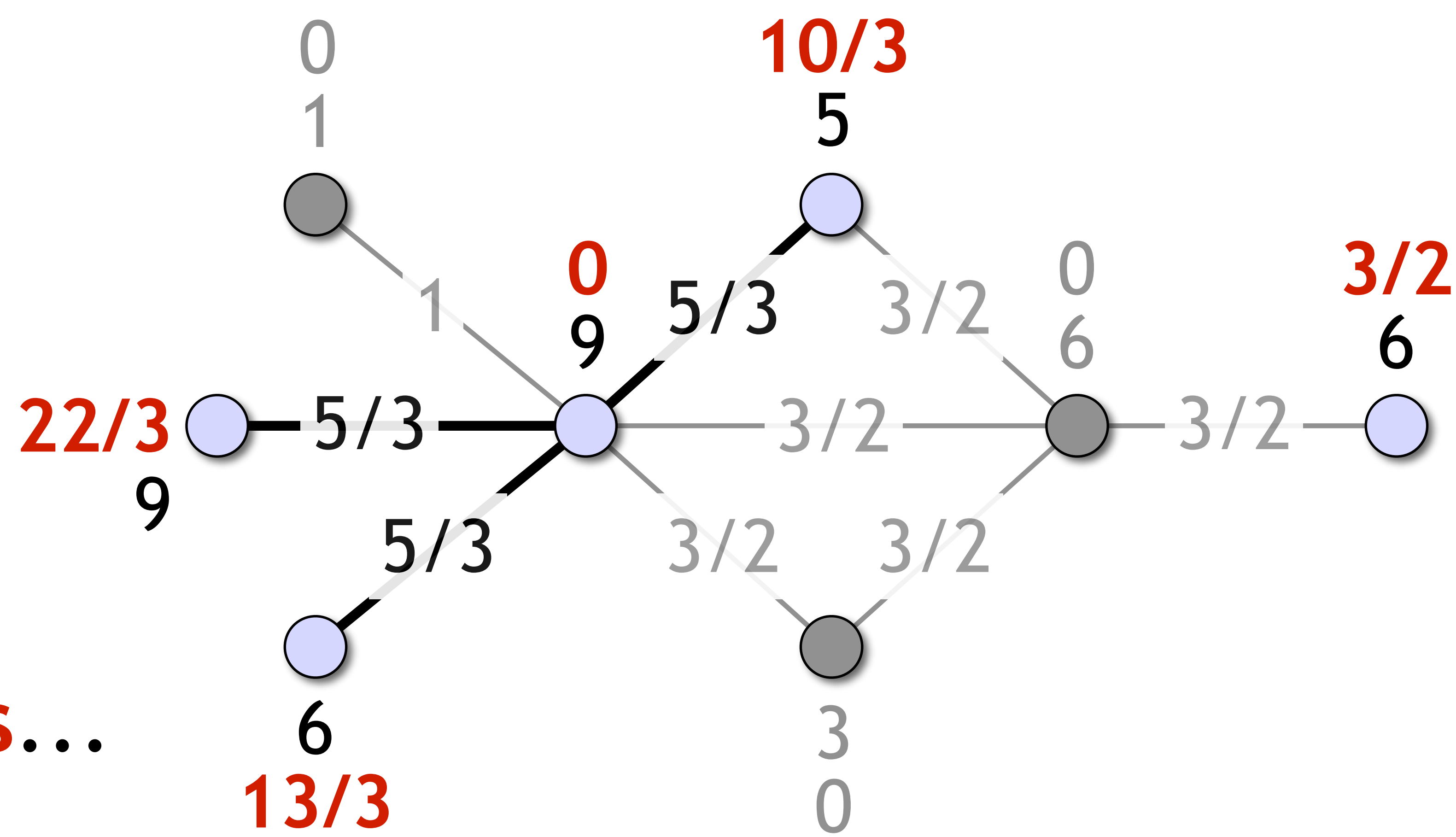
Finding a maximal edge packing: basic idea

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

Update residuals...



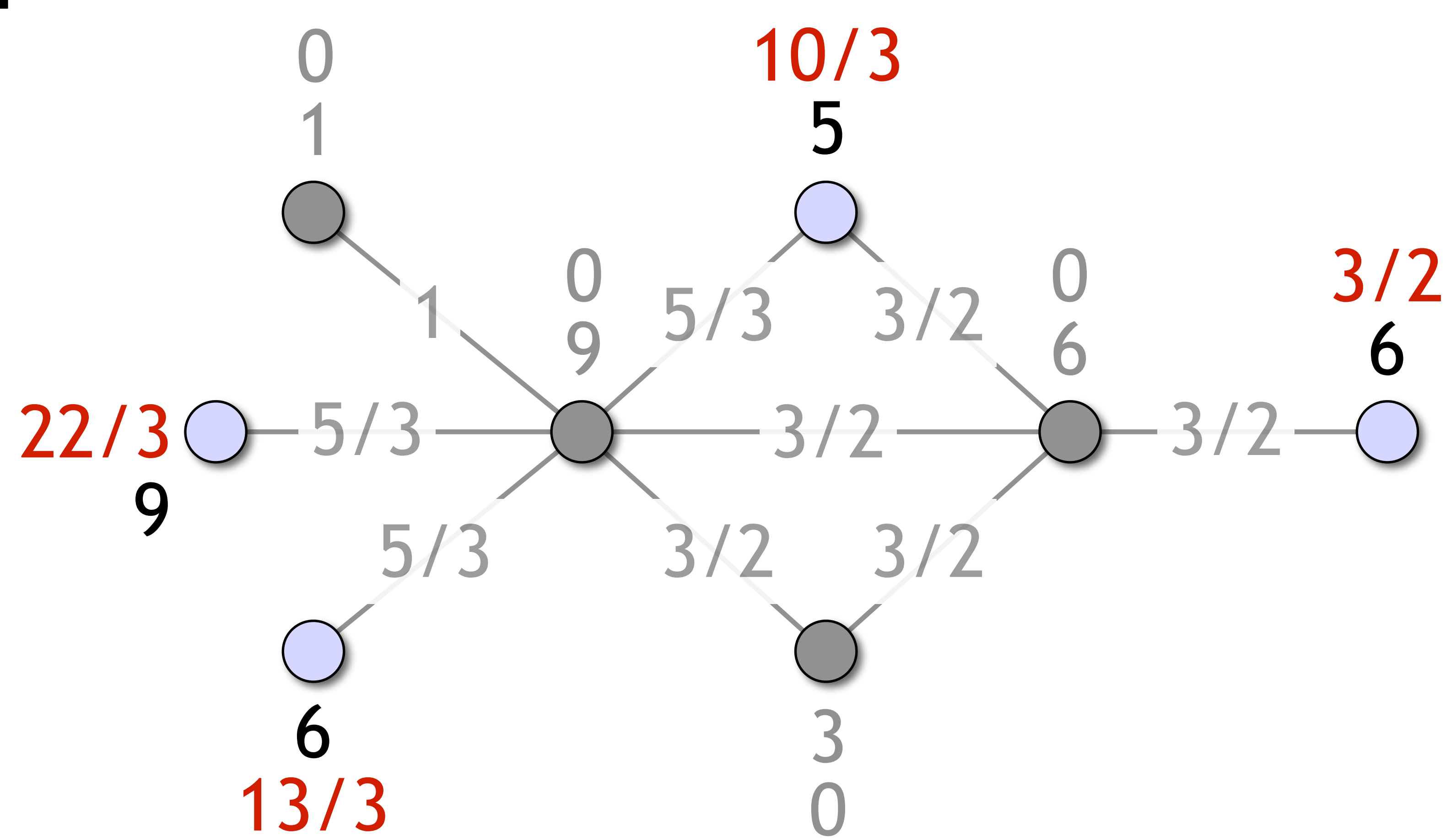
Finding a maximal edge packing: basic idea

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

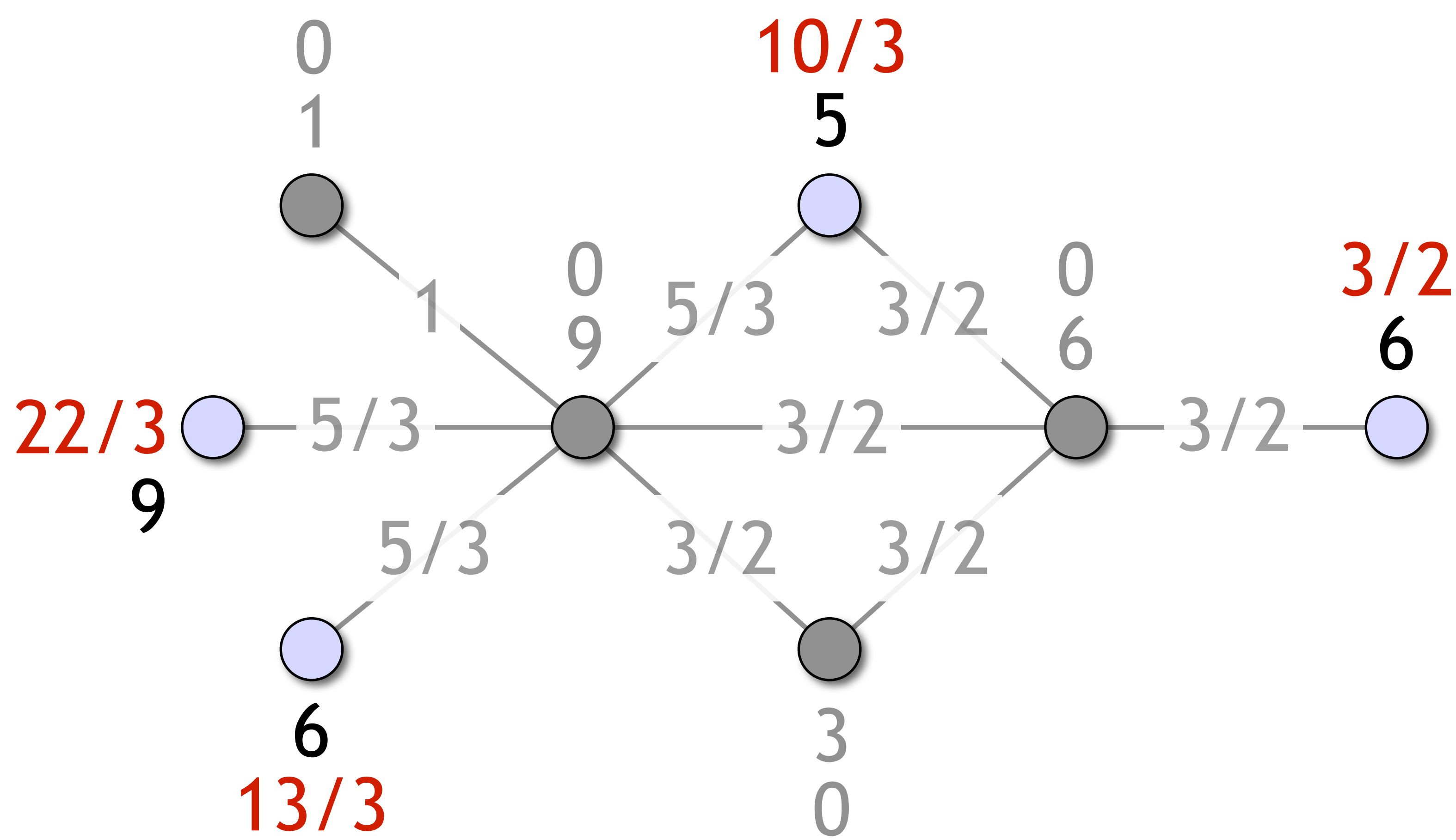
Update residuals and graph, etc.



Finding a maximal edge packing: basic idea

This is a simple deterministic distributed algorithm

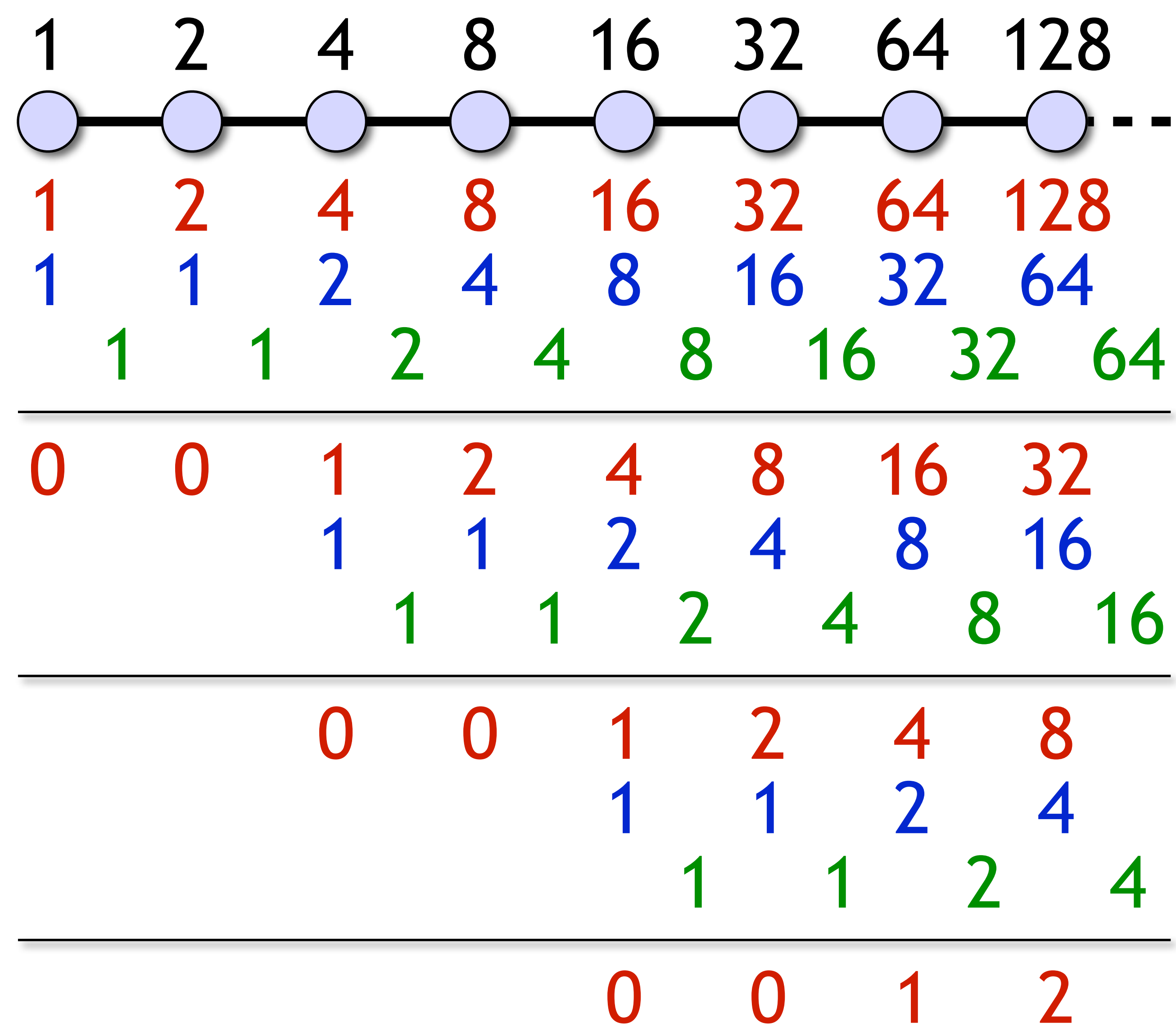
We are making some progress towards finding a maximal edge packing — but...



Finding a maximal edge packing: basic idea

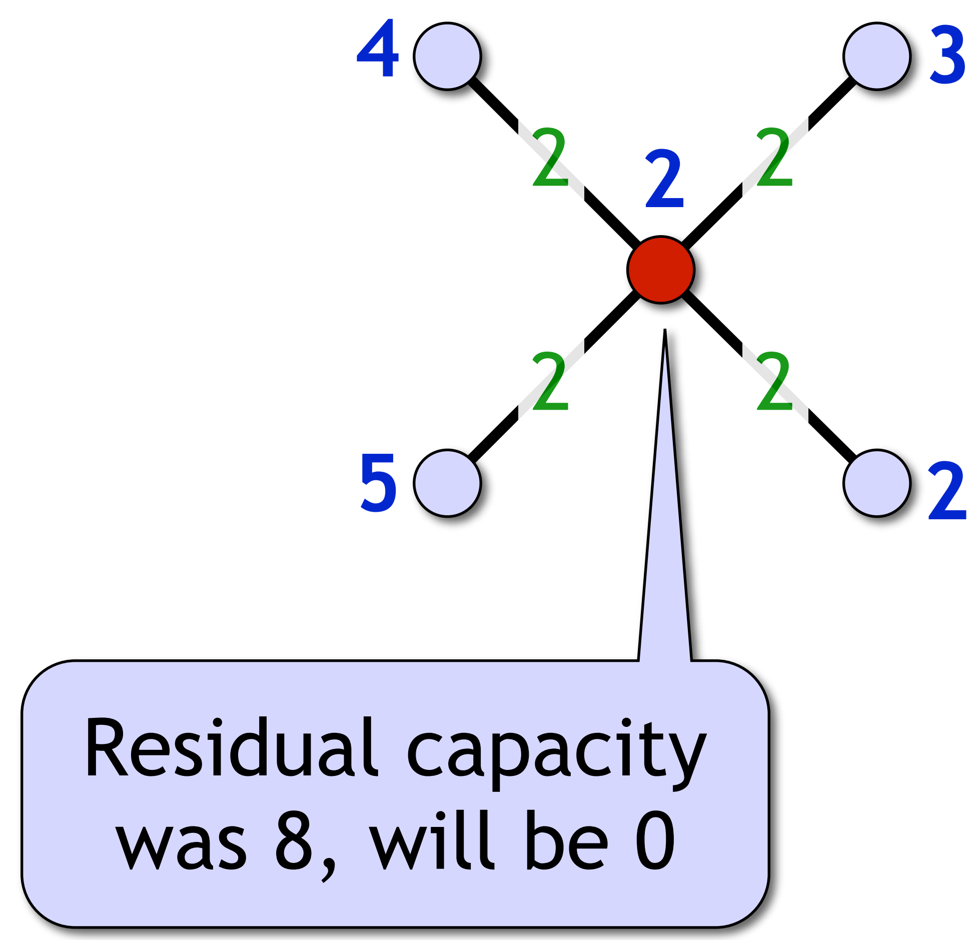
This is a simple deterministic distributed algorithm

We are making some progress towards finding a maximal edge packing — but this is **too slow**!



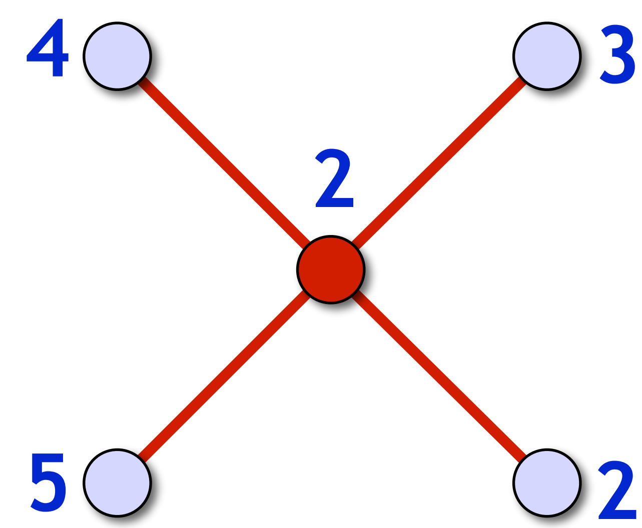
Finding a maximal edge packing: colouring trick

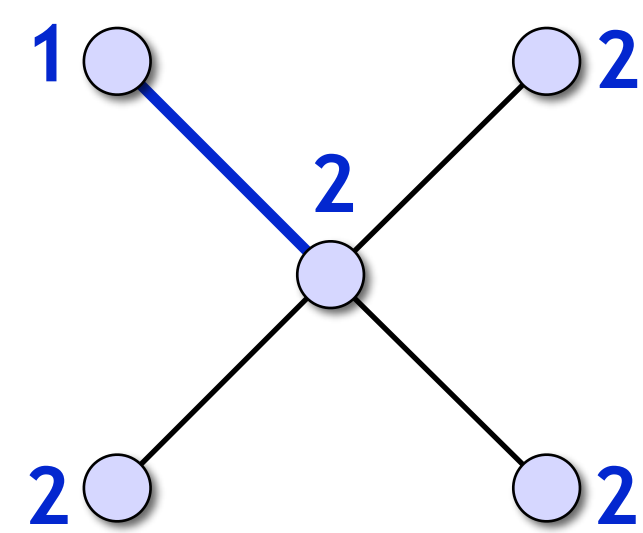
- Offer is a local minimum:
  - Node will be **saturated**
  - And all edges incident to it will be saturated as well



Finding a maximal edge packing: colouring trick

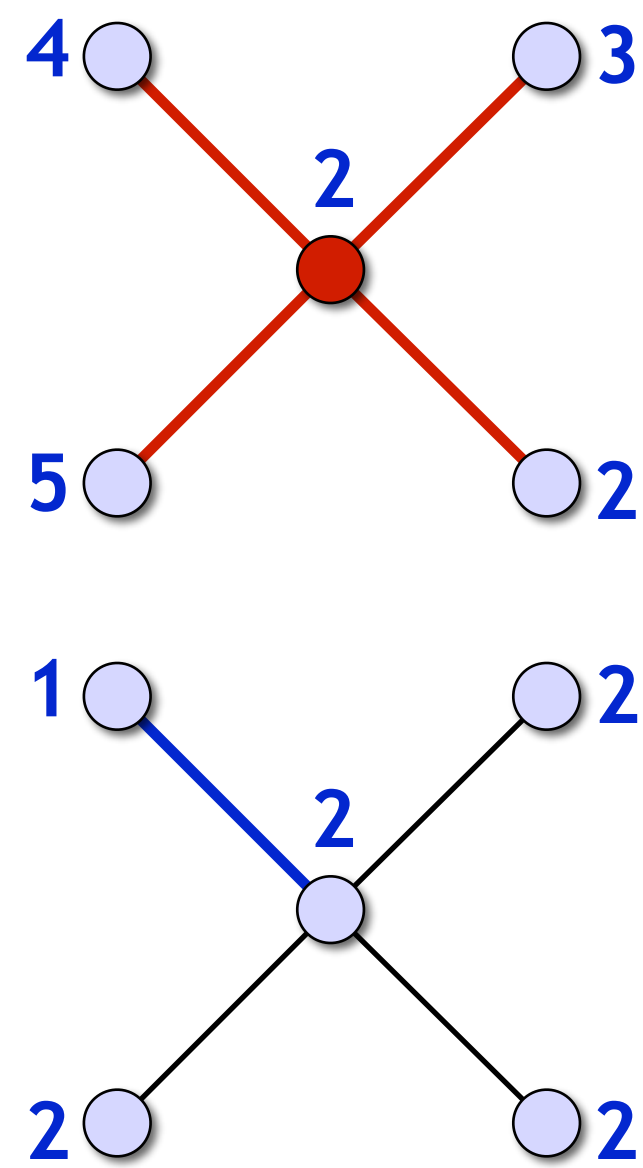
- Offer is a local minimum:
  - Node will be **saturated**
- Otherwise there is a neighbour with a different offer:
  - Interpret the offer sequences as colours
  - Nodes *u* and *v* have different colours: {*u*, *v*} is **multicoloured**





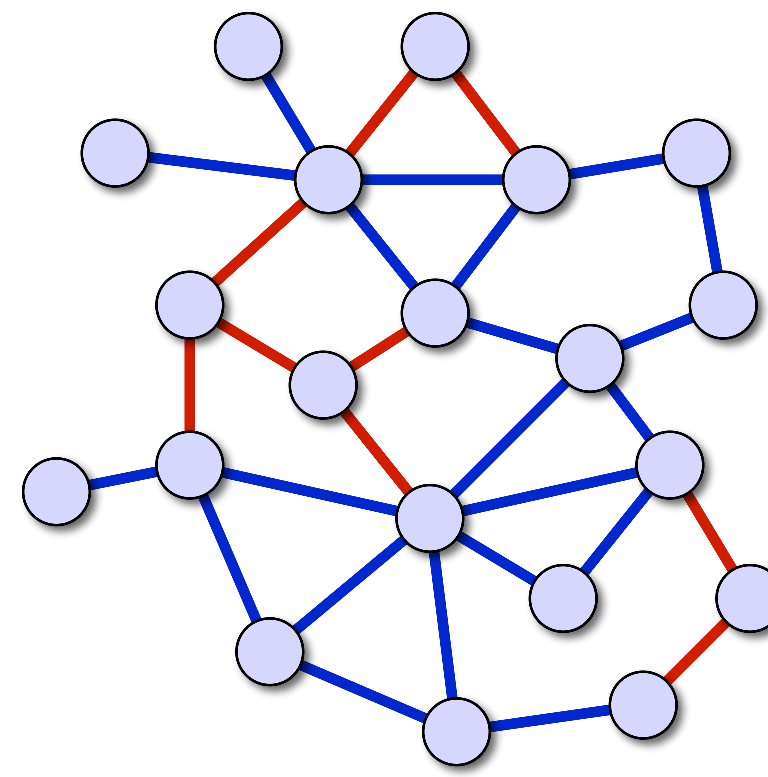
Finding a maximal edge packing: colouring trick

- Progress guaranteed:
  - On each iteration, for each node, at least one incident edge becomes **saturated** or **multicoloured**
  - Such edges are be discarded; maximum degree Δ decreases by at least one
  - Hence in Δ rounds all edges are saturated or multicoloured



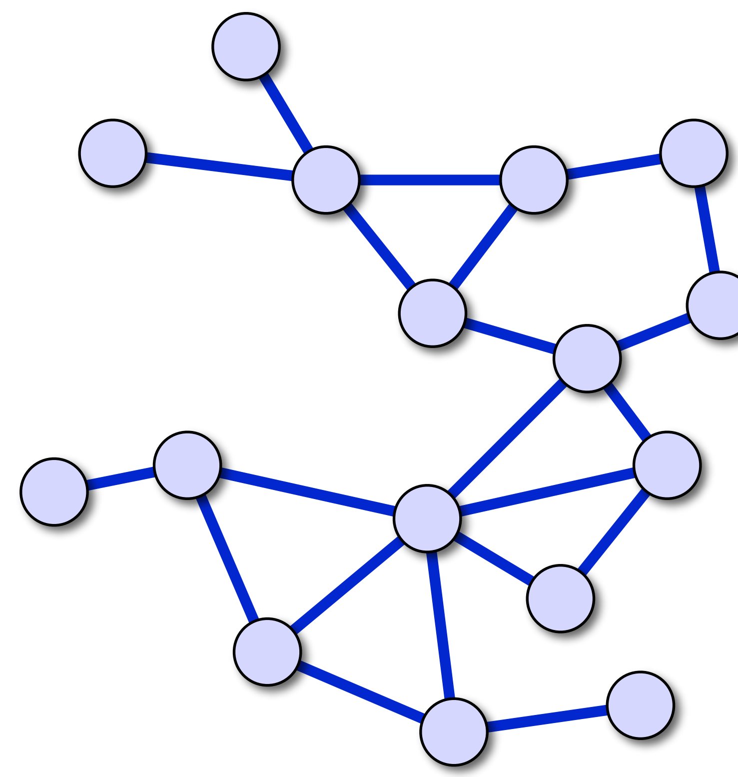
Finding a maximal edge packing: colouring trick

- In Δ rounds all edges are **saturated** or **multicoloured**
  - Saturated edges are good — we're trying to construct a maximal edge packing
  - Why are the multicoloured edges useful?



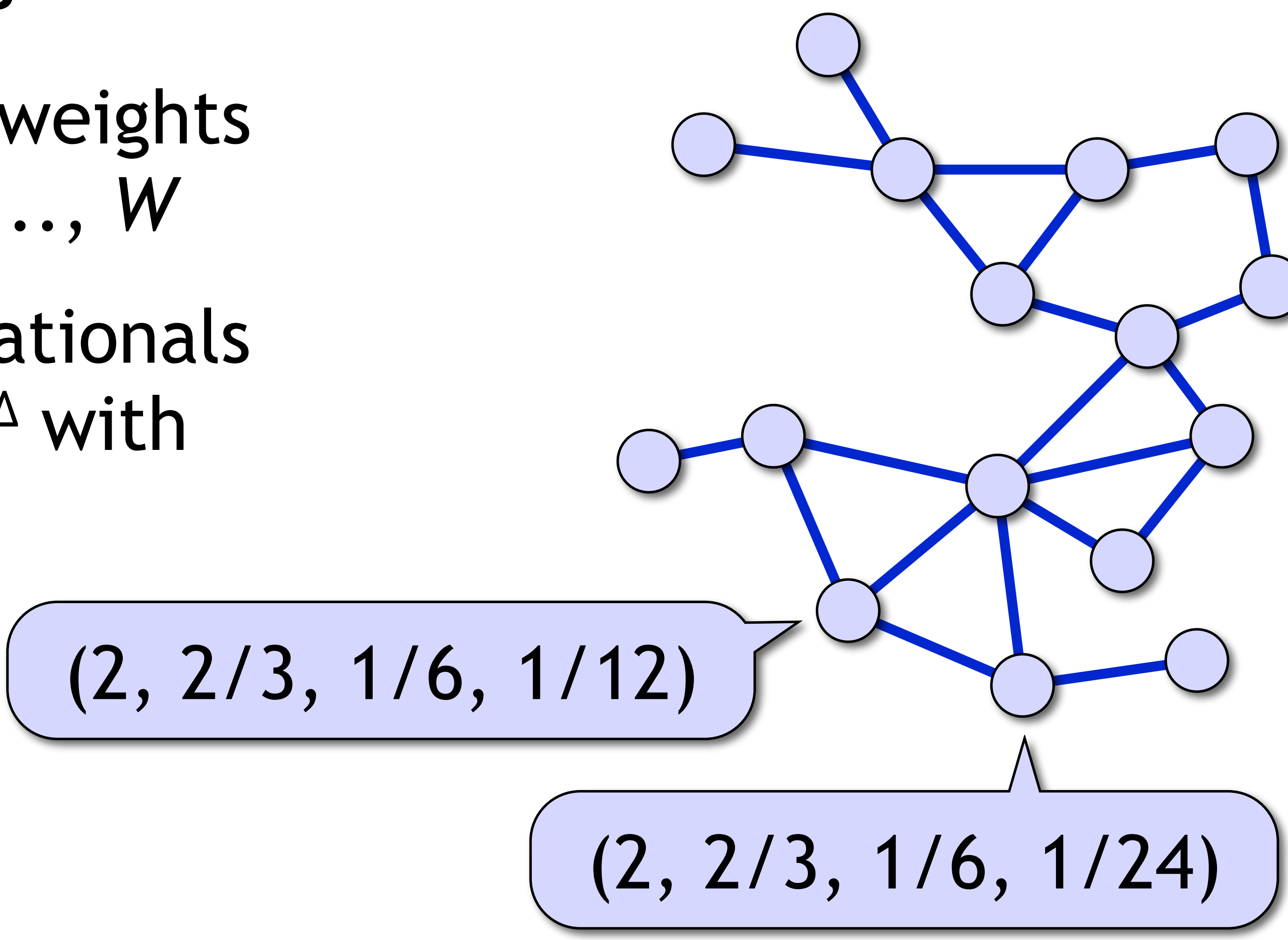
Finding a maximal edge packing: colouring trick

- In Δ rounds all edges are **saturated** or **multicoloured**
  - Saturated edges are good — we're trying to construct a maximal edge packing
  - Why are the multicoloured edges useful?
  - Let's focus on unsaturated nodes and edges



Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ rational numbers
  - Assume that node weights are integers 1, 2, ..., W
  - Then colours are rationals of the form $q/(\Delta!)^\Delta$ with $q \in \{1, 2, \ldots, W\}$



Finding a maximal edge packing: colouring trick

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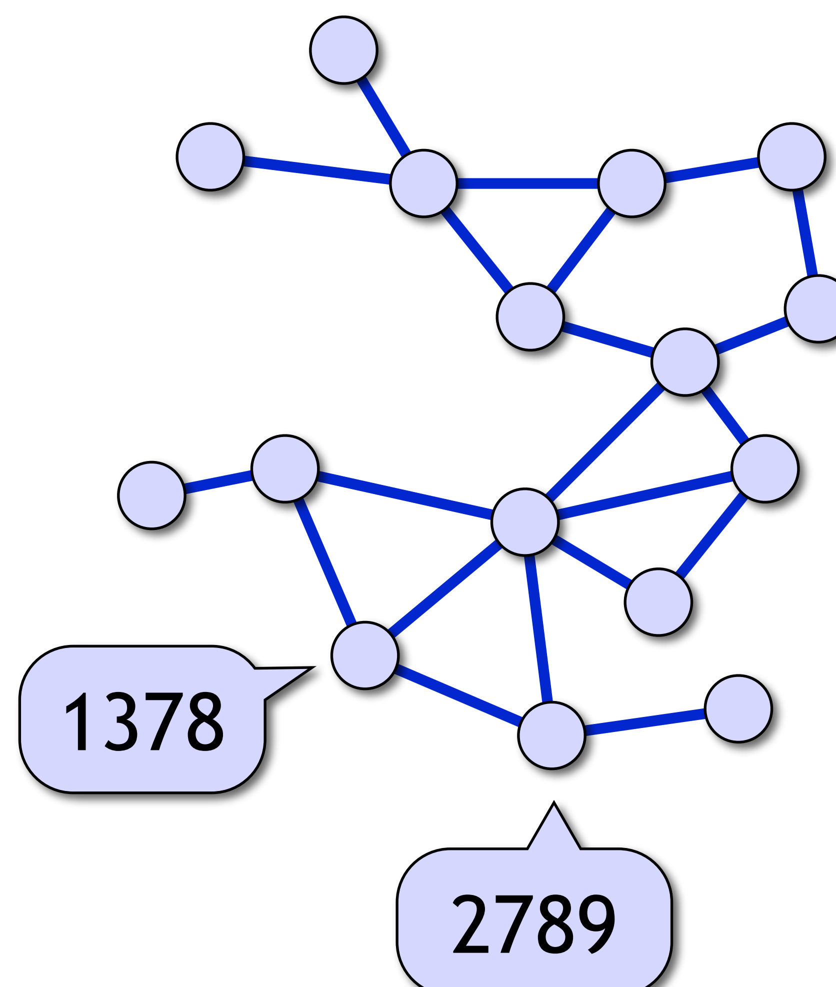

Finding a maximal edge packing: colouring trick

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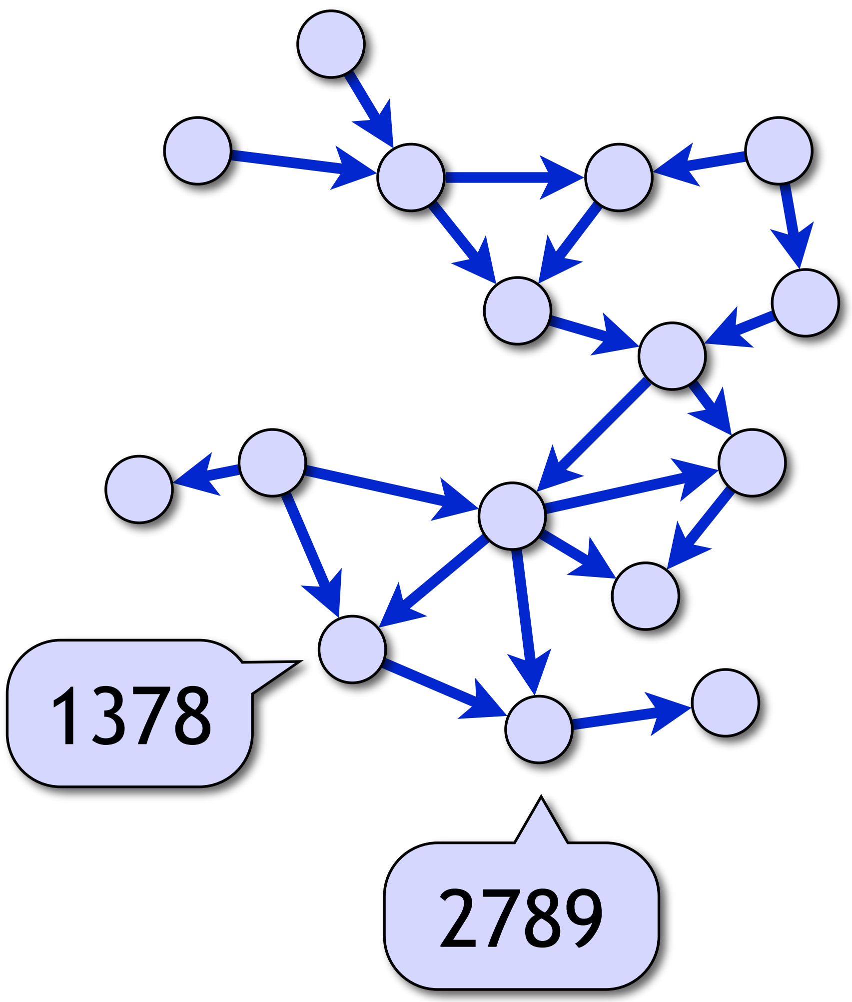
Looks ugly,
but don't worry,
in the end we will
take log* of k

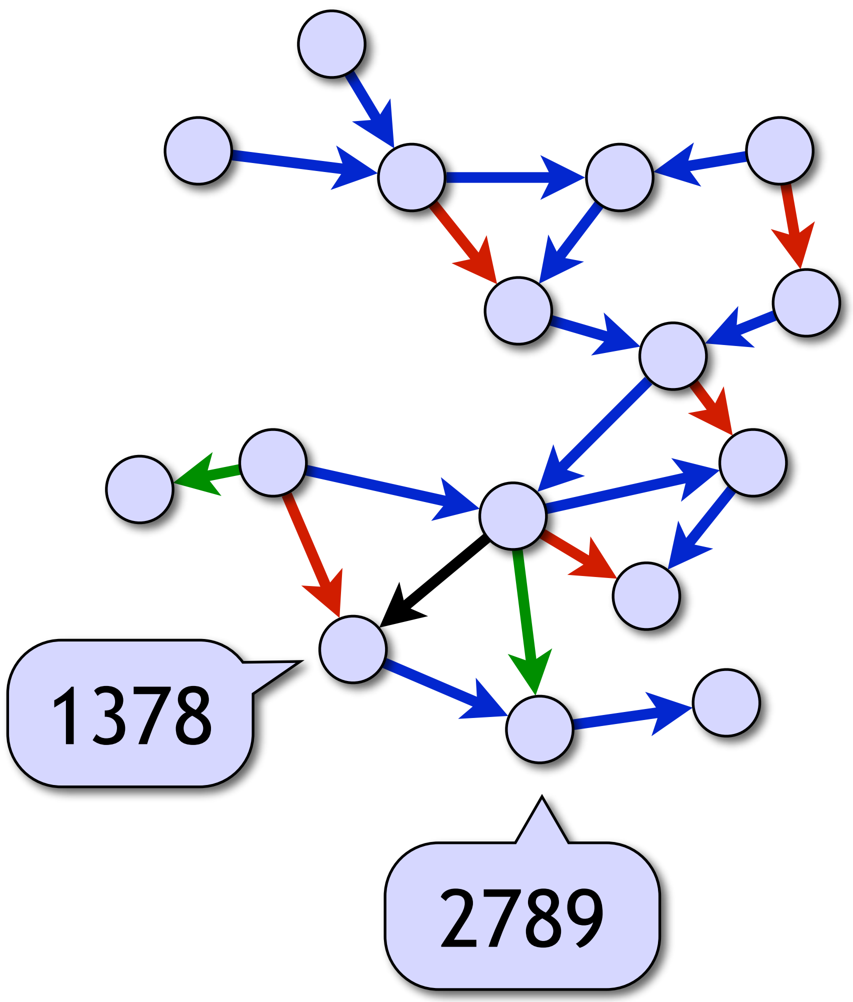
Finding a maximal edge packing: colouring trick

- We have a proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour (acyclic directed graph)



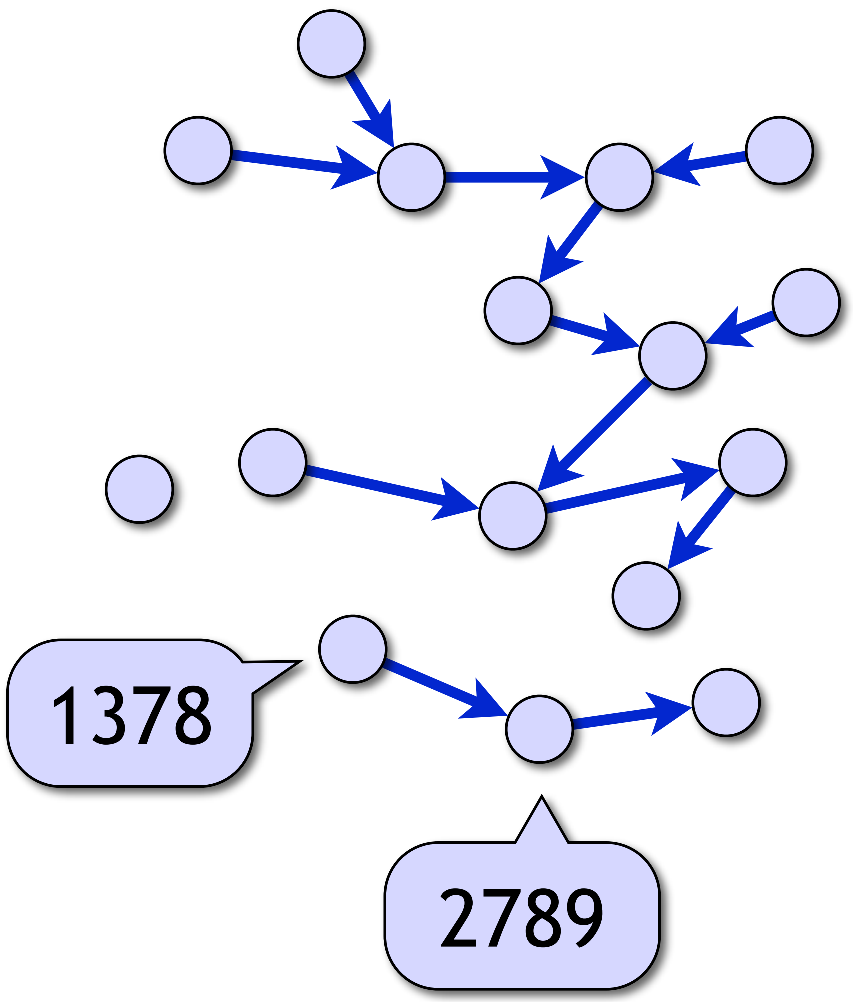
Finding a maximal edge packing: colouring trick

- We have a proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour (acyclic directed graph)
- Partition in Δ forests
  - Each node assigns its outgoing edges to different forests



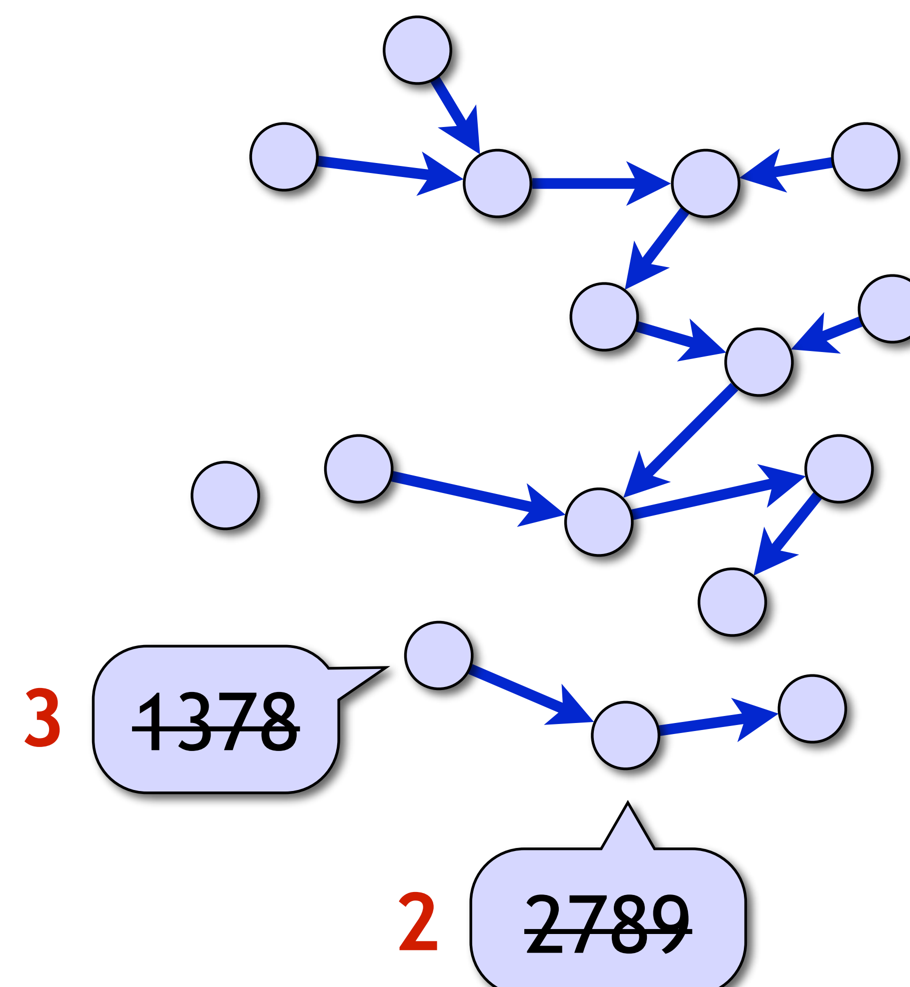
Finding a maximal edge packing: colouring trick

- For each forest in parallel...



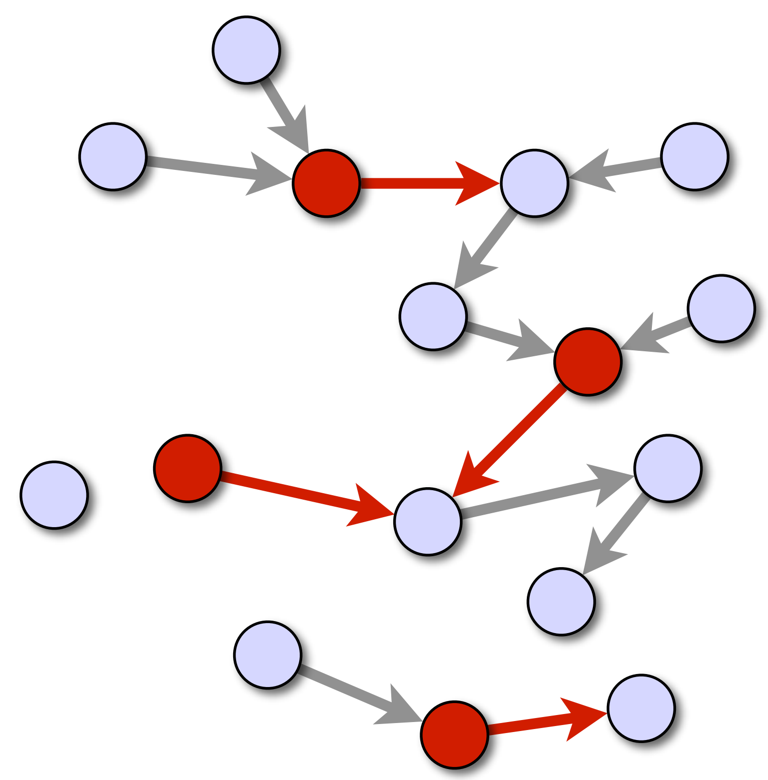
Finding a maximal edge packing: colouring trick

- For each forest in parallel:
  - Use Cole–Vishkin (1986) style colour reduction algorithm
  - Given a k-colouring, finds a **3-colouring** in time O(log* k)
  - Bit manipulation trick: each step replaces a k-colouring with an $O(\log k)$-colouring



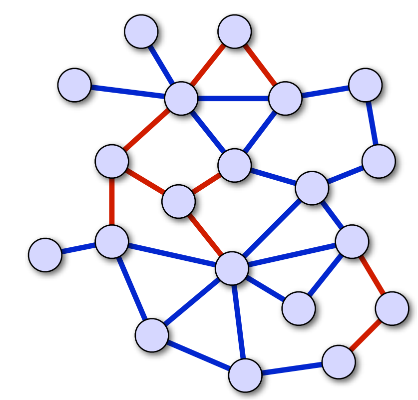
Finding a maximal edge packing: colouring trick

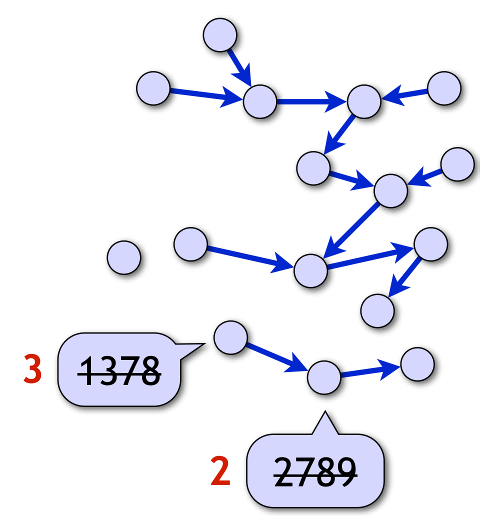
- For each forest and each colour j = 1, 2, 3 in sequence:
  - Saturate all outgoing edges of colour-j nodes
  - Node-disjoint stars, easy to saturate in parallel
- In $O(\Delta)$ rounds we have saturated all edges

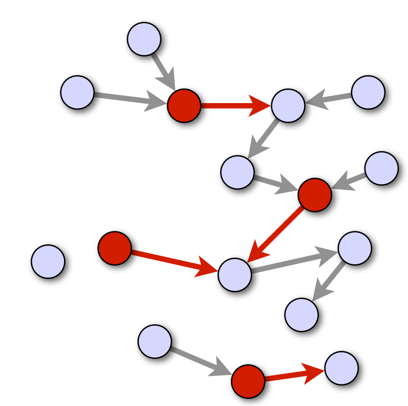


Finding a maximal edge packing: summary

- Total running time:
    - All edges are saturated or multicoloured: $O(\Delta)$
    - Multicoloured forests are 3-coloured: $O(\log^* k)$
    - 3-coloured forests are saturated: $O(\Delta)$
- $O(\Delta + \log^* k) = O(\Delta + \log^* W)$
    - k is huge, but $\log^*$ grows slowly

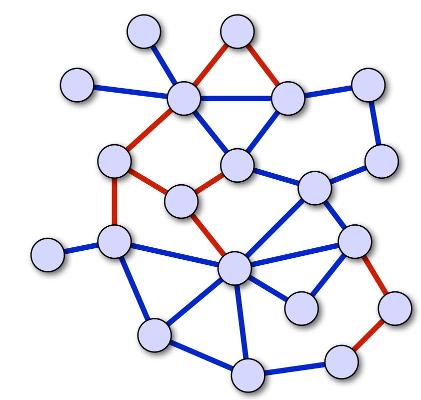


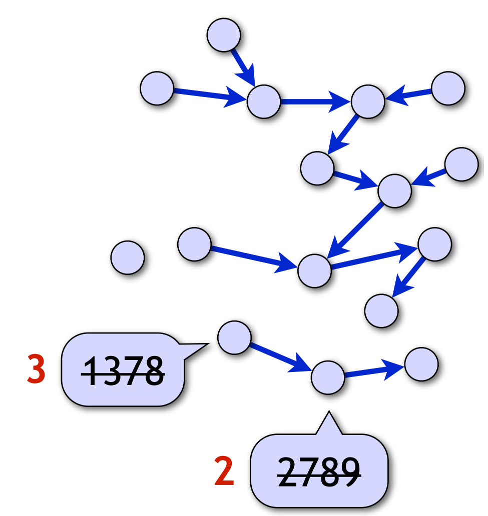


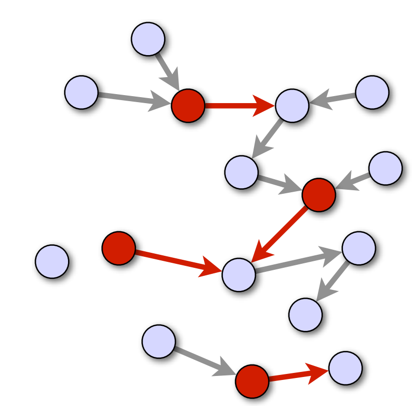


Finding a maximal edge packing: summary

- Maximal edge packing and 2-approximation of vertex cover in time $O(\Delta + \log^* W)$
  - W = maximum node weight
- Unweighted graphs: running time simply $O(\Delta)$, independent of n
- Everything can be implemented in the **port-numbering model**







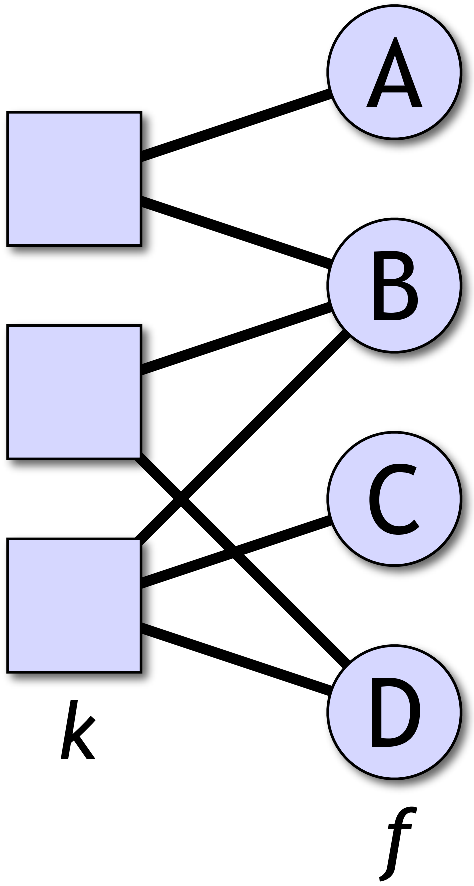
Vertex cover algorithms

- 2-approximation of vertex cover in time $O(\Delta)$ in the **port-numbering model**
  - Insight: consider a more general problem, minimum-***weight*** vertex cover
- 2-approximation of vertex cover in time poly(Δ) in the **broadcast model**?
  - Insight: consider a more general problem, minimum-weight ***set*** cover!

Set cover algorithm

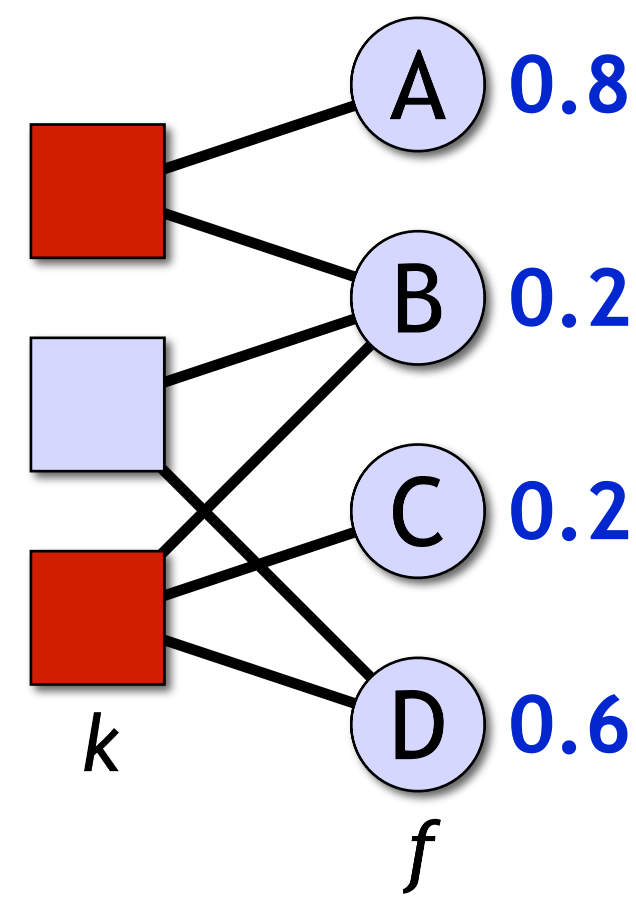
- Set covers in a distributed setting:
  - bipartite graph, “sets” and “elements”
- Degree bounds:
  - element frequency at most *f*
  - set size at most *k*
- Vertex cover:
  - edge ≈ element (*f* = 2)
  - node ≈ set (*k* = Δ)

{A,B}, {B,D},
{B,C,D}



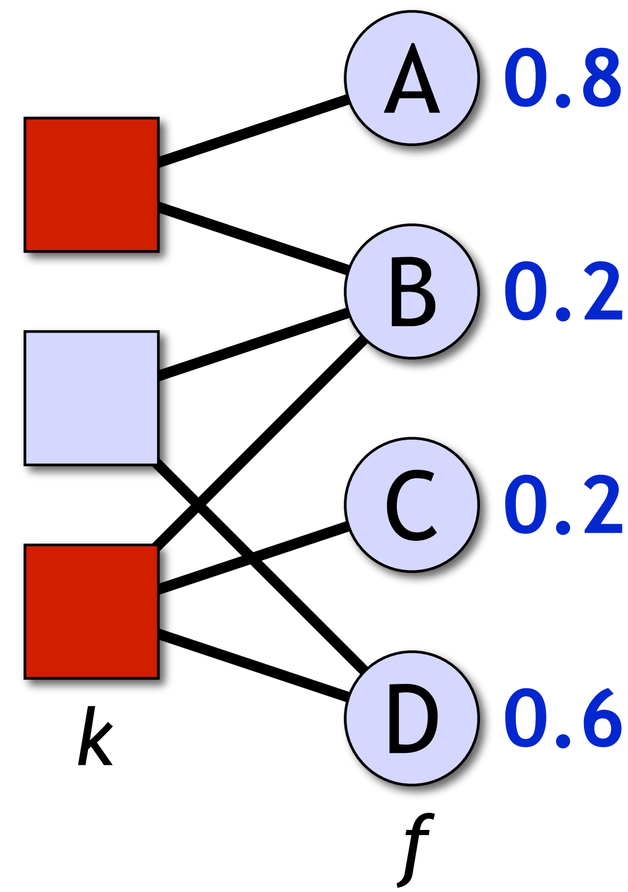
Set cover algorithm

- Similar techniques:
  - Find a **maximal fractional packing**
  - Generalisation of maximal edge packings
  - **Saturated sets**: f-approximation of minimum-weight set cover



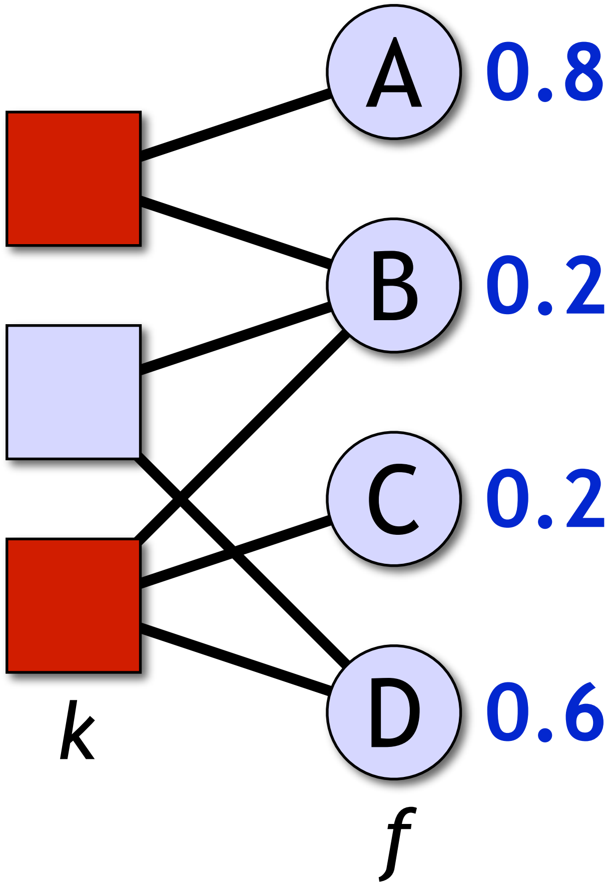
Set cover algorithm

- Similar techniques:
  - Find a **maximal fractional packing**
  - “Greedy but safe” offer/accept rounds
  - Progress guaranteed: something is always saturated *or* multicoloured



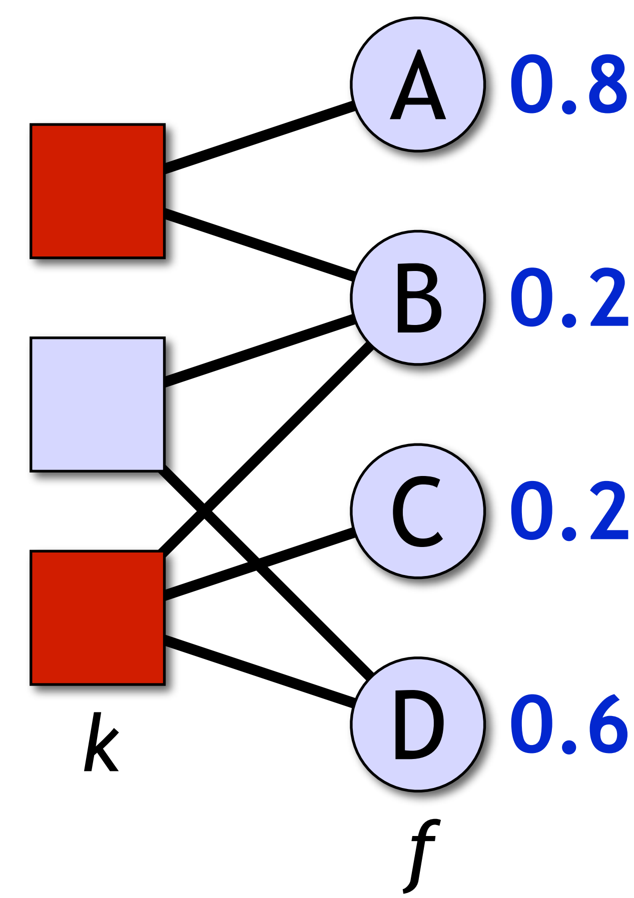
Set cover algorithm

- Dissimilar techniques:
    - Repeated iterations of saturation + colouring phases
    - We don't try to find a proper 3-colouring but a weak 3-colouring
        - Easier in the broadcast model, enough to make some progress
    - Lots of technicalities...



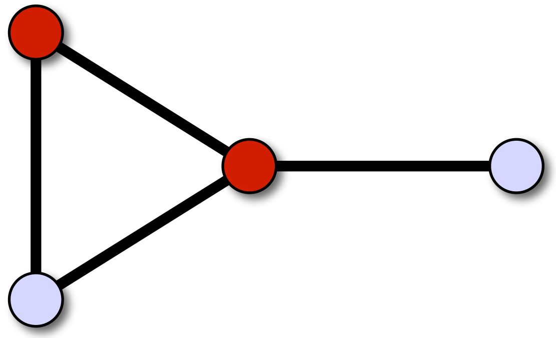
Set cover algorithm

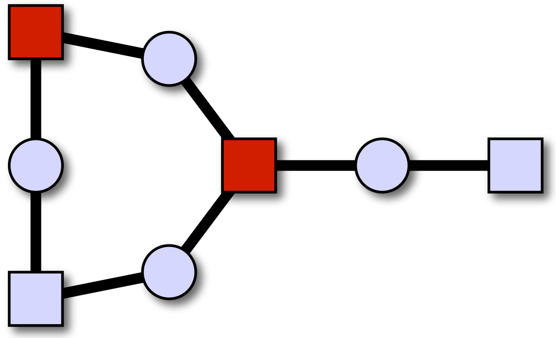
- Maximal fractional packing in $O(f^2k^2 + fk \log^* W)$ rounds, **broadcast model**



Set cover algorithm: application

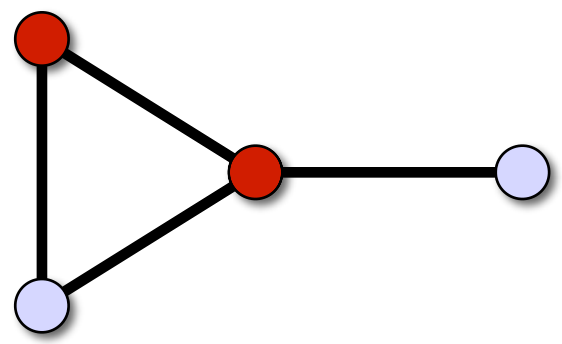
- Use the set cover algorithm to find a **vertex cover**
  - In vertex cover instances, nodes have local state but edges are stateless
  - In set cover instances, both sets and elements have local state
  - Simulation possible, trick: pass around the **full history** of broadcasts, re-compute the states
  - Larger messages, but the same number of rounds

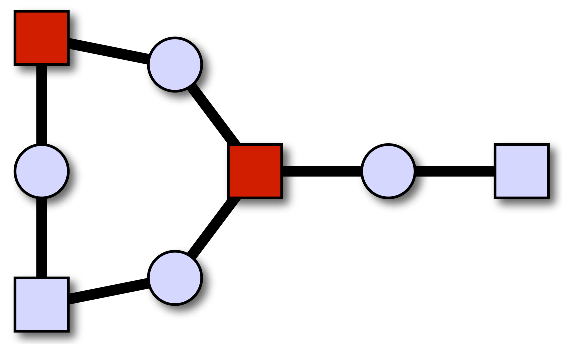




Set cover algorithm: application

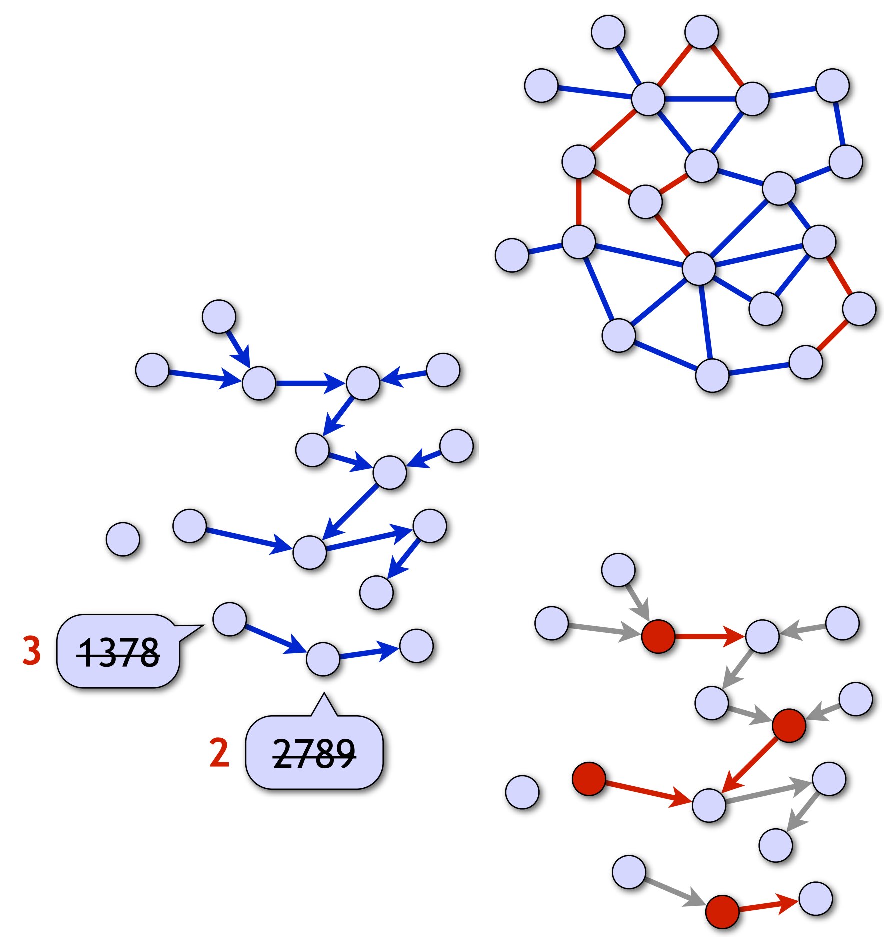
- Use the set cover algorithm to find a vertex cover
- 2-approximation of unweighted vertex cover in $O(\Delta^2)$ rounds, **broadcast model**





Conclusions

- 2-approximation algorithms for vertex cover:
  - Time $O(\Delta)$, **port-numbering model**
  - Time $O(\Delta^2)$, **broadcast model**
- Research questions:
  - Can you do it faster,
    in any model?
  - What else can be solved
    in the broadcast model?



Conclusions

- 2-approximation algorithms for vertex cover:
  - Time $O(\Delta)$, **port-numbering model**
  - Time $O(\Delta^2)$, **broadcast model**
- Take-home message:
  - Sometimes more general problems are easier to solve!