

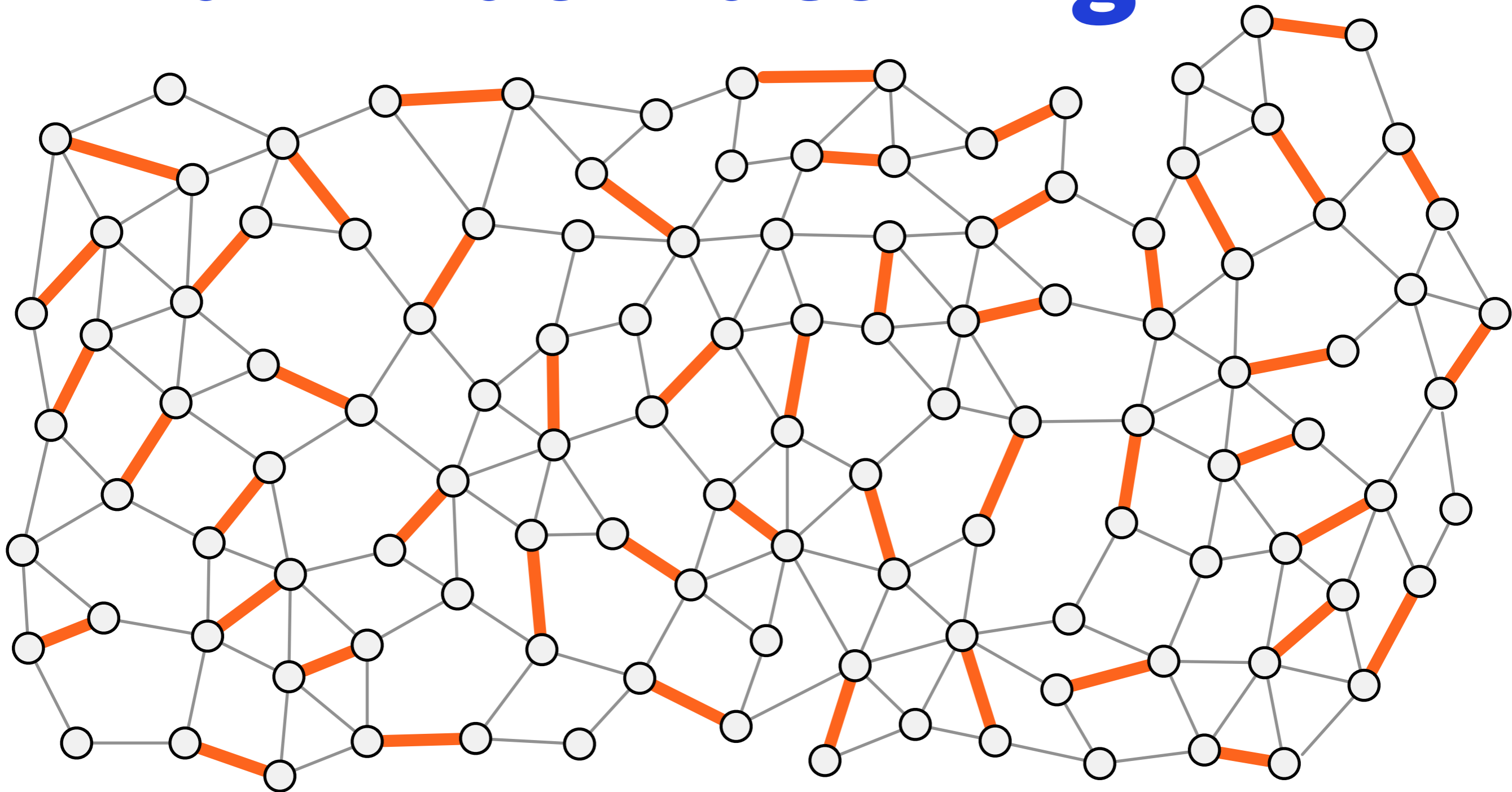
Local coordination and symmetry breaking

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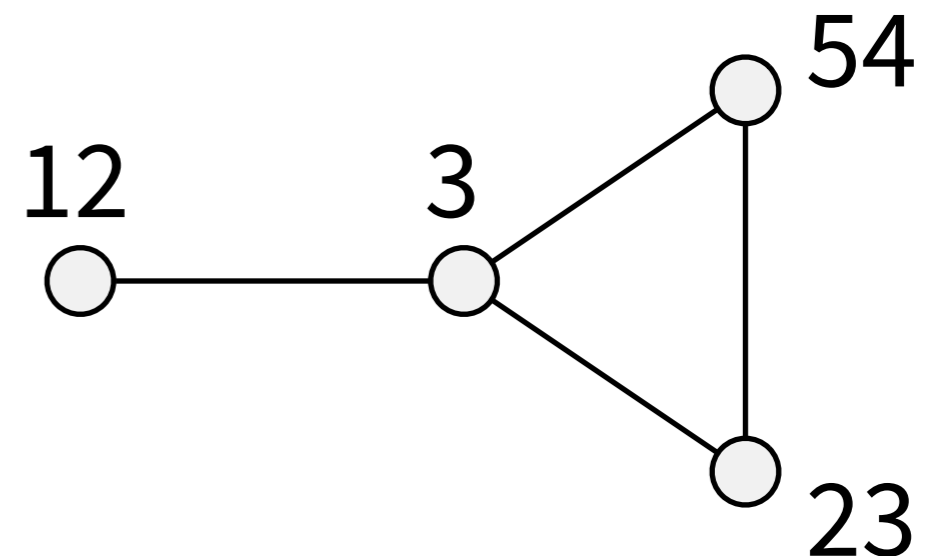
Chalmers, 16 October 2015

Running example: **Maximal matching**



LOCAL model

- **Input:** simple undirected **graph G**
 - communication network
 - nodes labelled with unique $O(\log n)$ -bit identifiers



LOCAL model

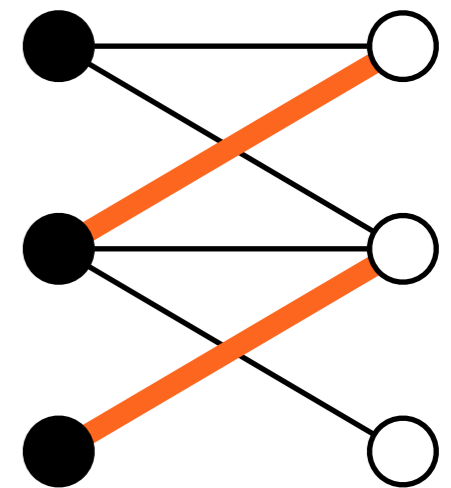
- **Input:** simple undirected graph G
- **Output:** each node v produces a **local output**
 - graph colouring: colour of node v
 - vertex cover: 1 if v is in the cover
 - matching: with whom v is matched

LOCAL model

- **Nodes exchange messages with each other, update local states**
- **Synchronous communication rounds**
- **Arbitrarily large messages**

Maximal matching in 2-coloured graphs

- **Black nodes send proposals to their neighbours, one by one**
- **White nodes accept the first proposal that they get**
- **$O(\Delta)$ communication rounds in graphs of maximum degree Δ**



LOCAL model

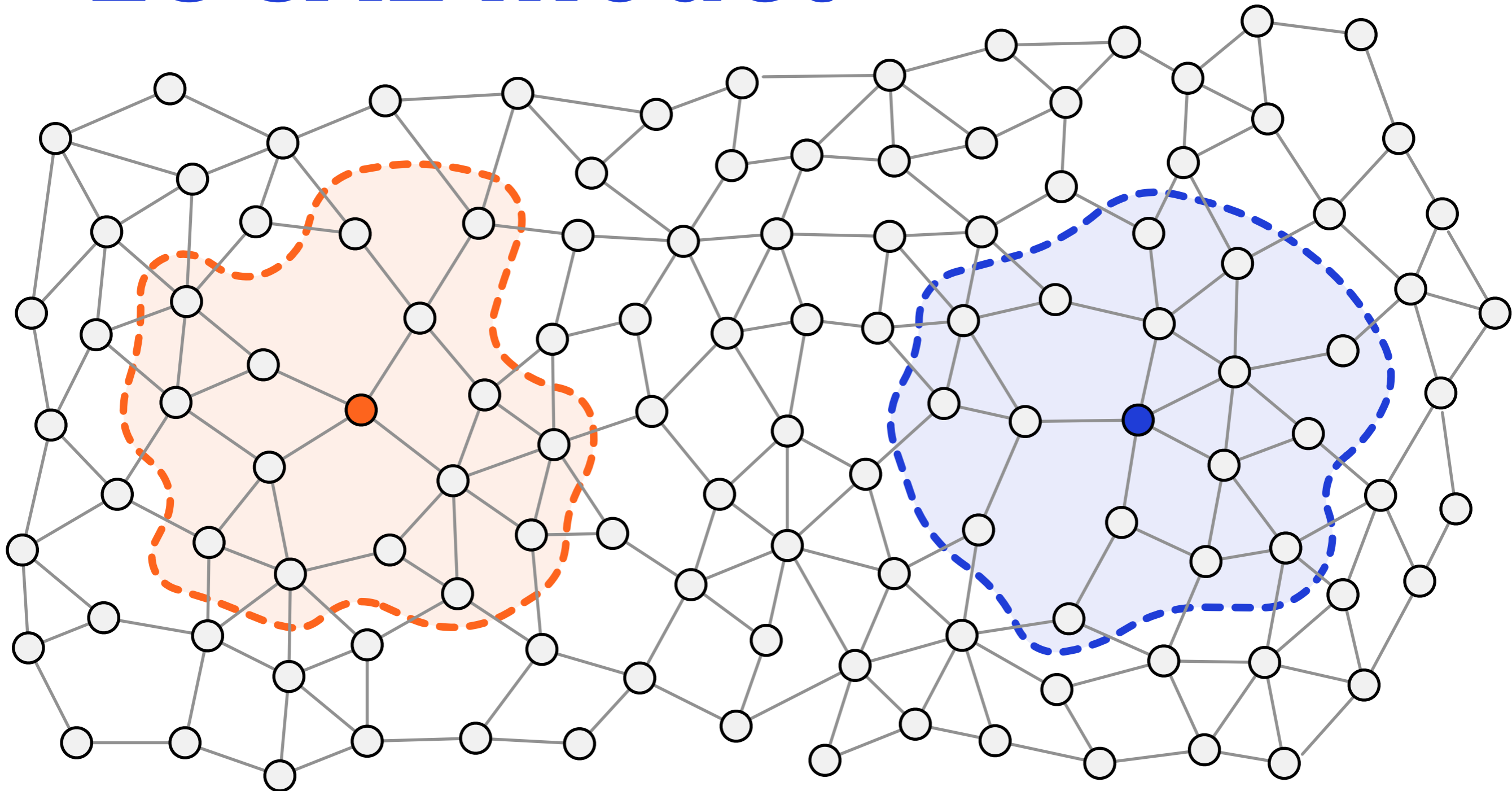
- **Time = number of communication rounds**
 - until all nodes stop and produce their **local outputs**

LOCAL model

- **Time = number of communication rounds**
- **Time = distance:**
 - in t communication rounds, all nodes can learn everything in their **radius- t neighbourhoods**

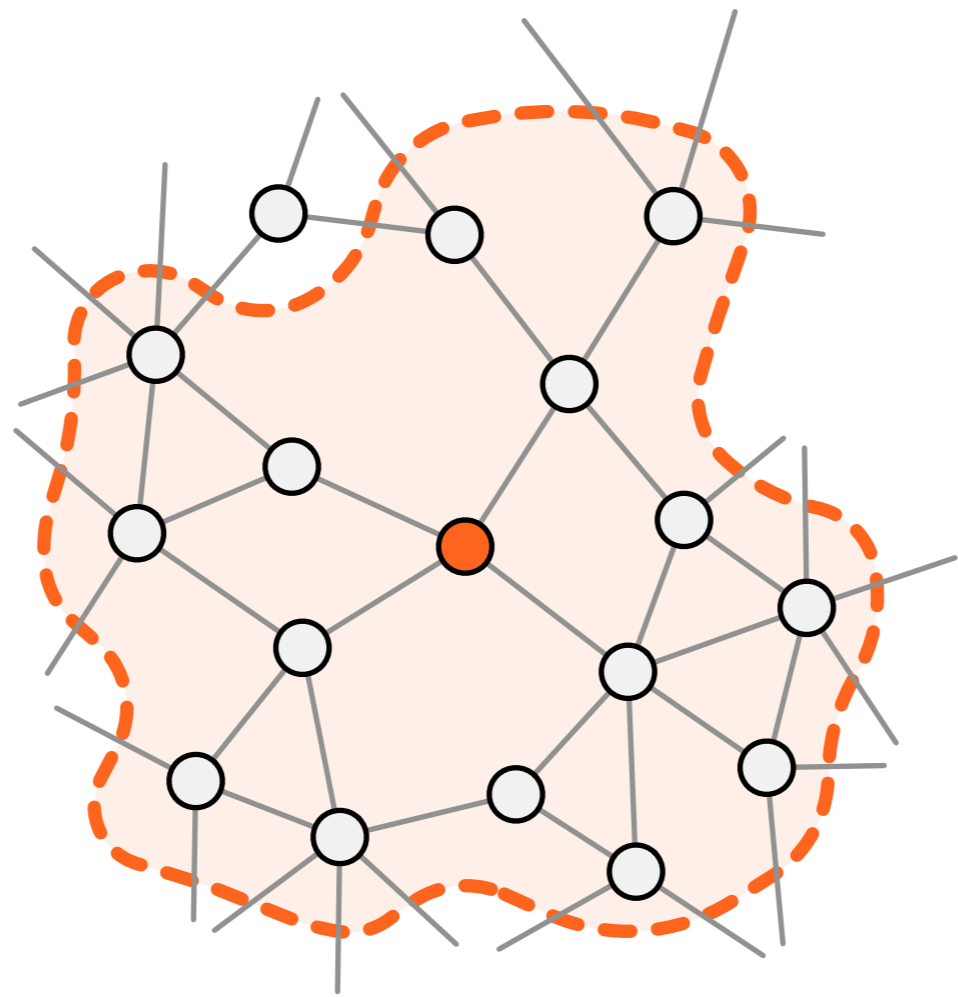
time $t = 2$

LOCAL model



LOCAL model

A:



\mapsto 1











LOCAL model

- **Everything trivial in time $\text{diam}(G)$**
 - all nodes see whole G ,
can compute any function of G
- **What can be solved much faster?**











Distributed time complexity

- n = number of nodes
- Δ = maximum degree
 - $\Delta < n$
- Time complexity $t = t(n, \Delta)$

Landscape











	$O(1)$	$\log^* n$	$\log n$	n
Δ				
$\log \Delta$				
$\log^* \Delta$				
$O(1)$				

Landscape

	$O(1)$	$\log^* n$	$\log n$	n
Δ				
$\log \Delta$				
$\log^* \Delta$				
$O(1)$				











All problems

Landscape

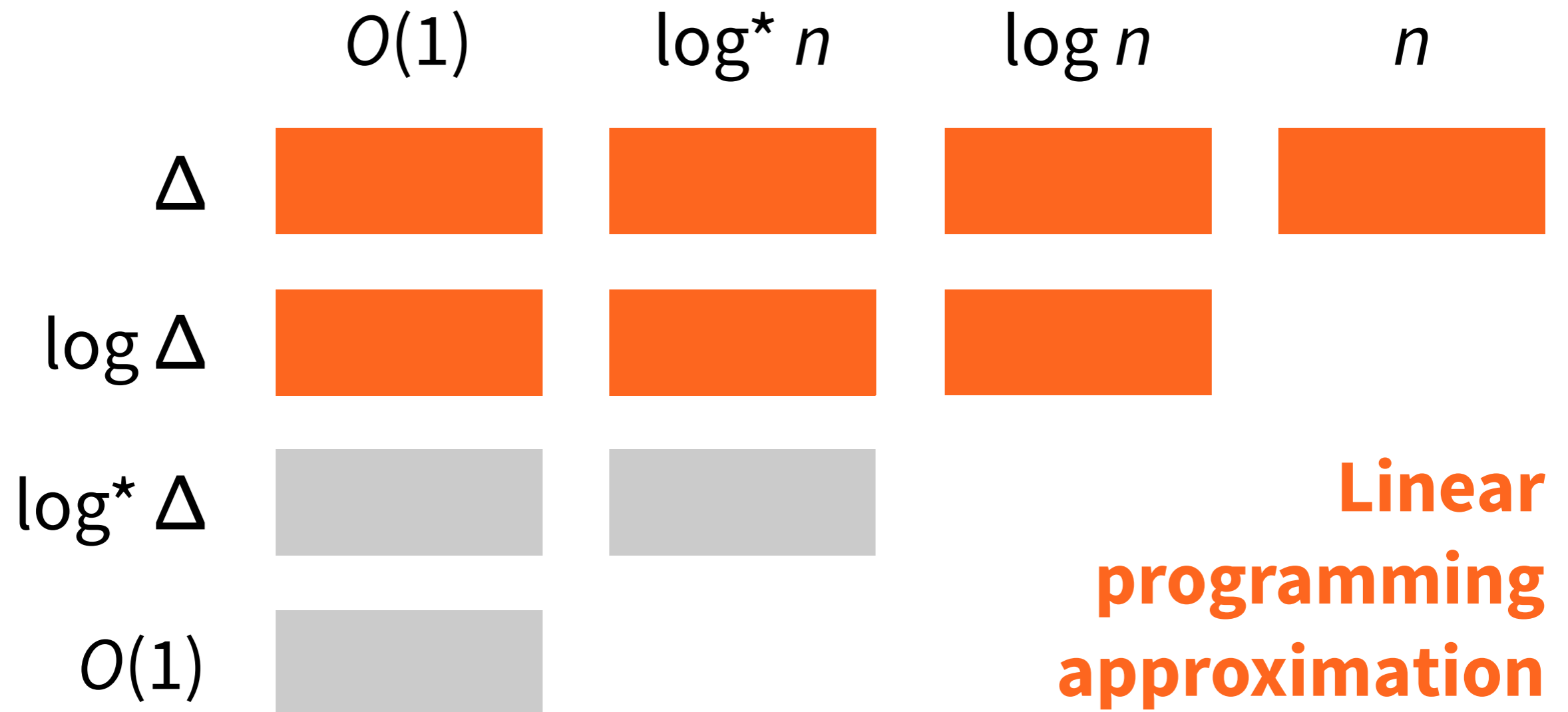
	$O(1)$	$\log^* n$	$\log n$	n
Δ				
$\log \Delta$				
$\log^* \Delta$				
$O(1)$				

Maximal matching











Landscape

	$O(1)$	$\log^* n$	$\log n$	n
Δ				
$\log \Delta$				
$\log^* \Delta$				Bipartite maximal matching
$O(1)$				

Landscape

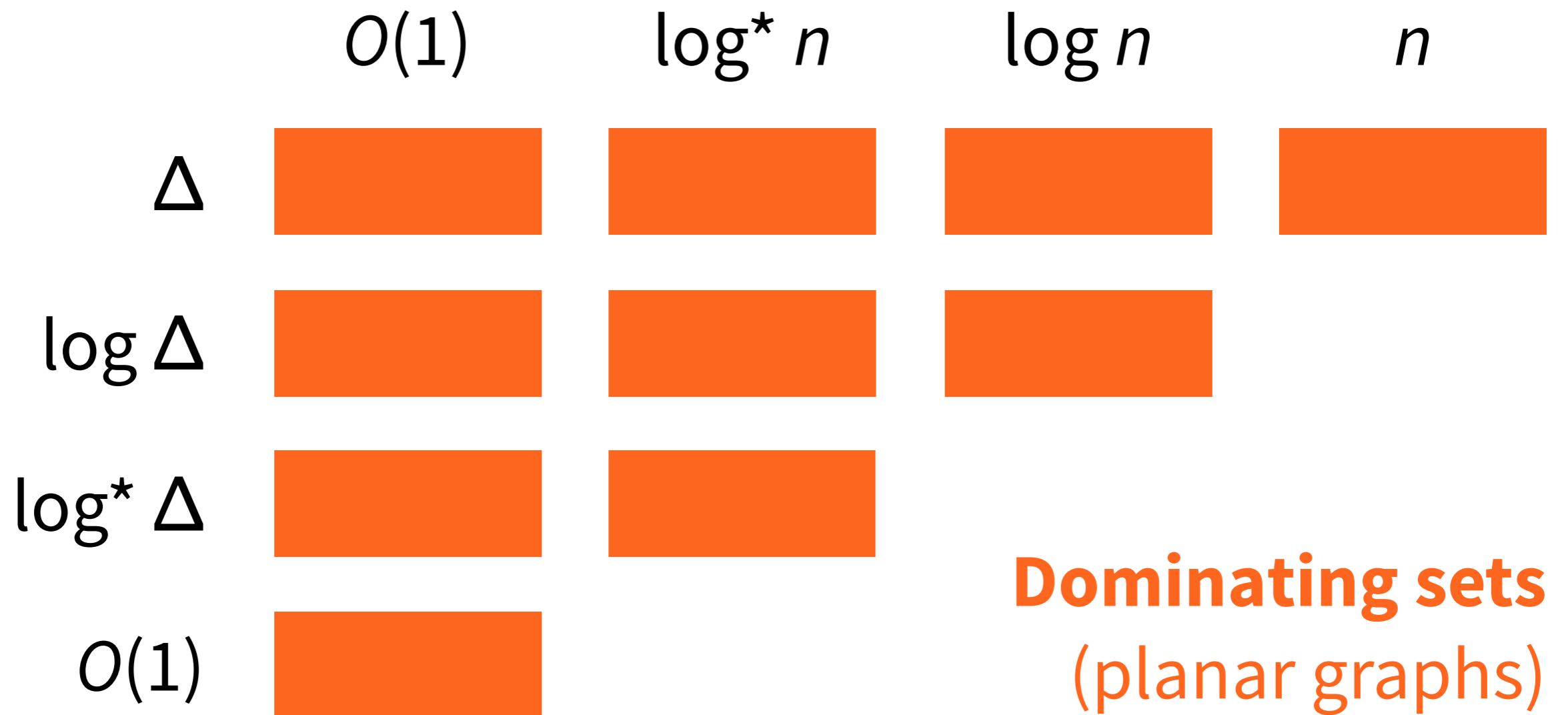


Landscape

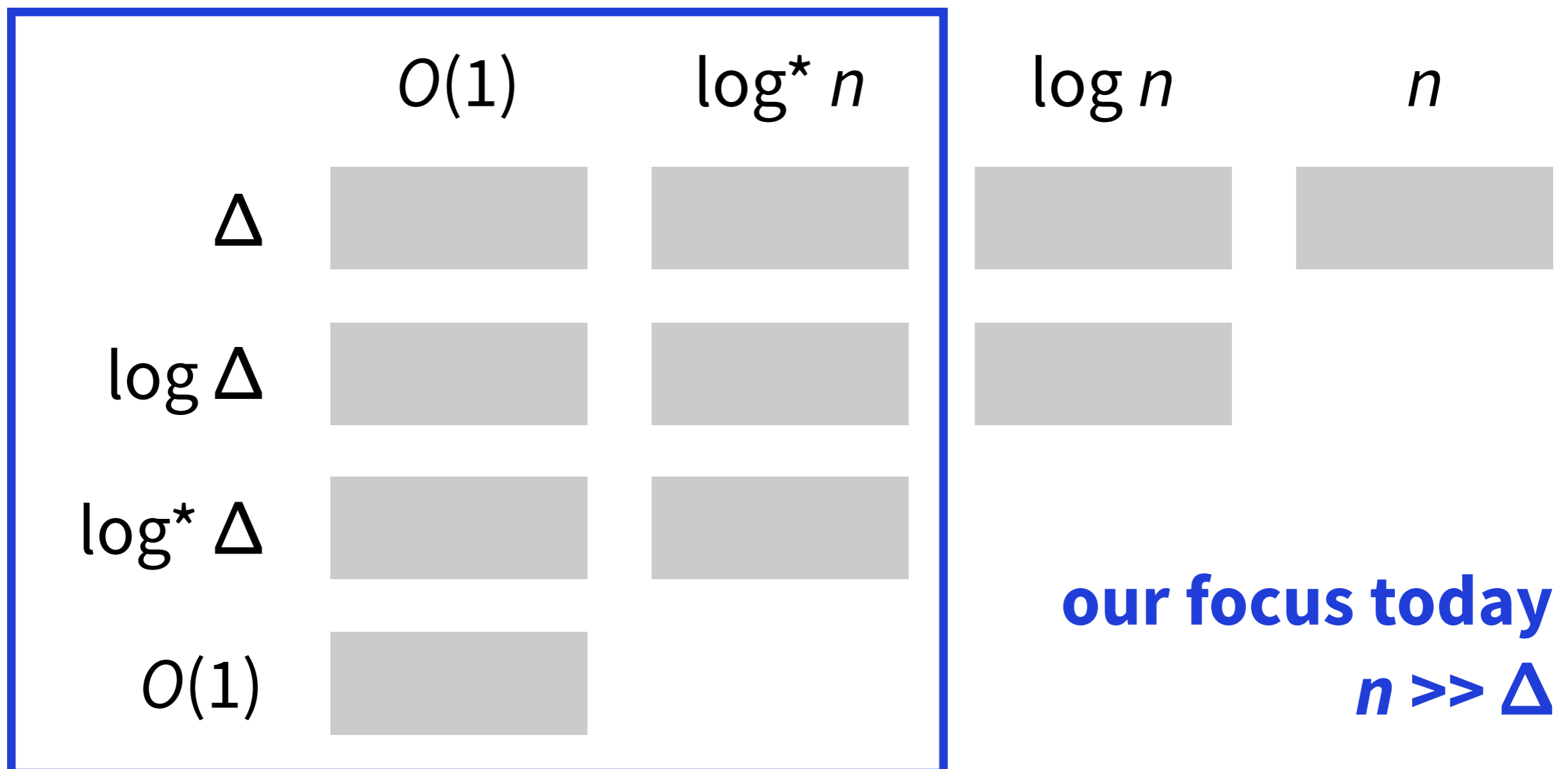
	$O(1)$	$\log^* n$	$\log n$	n
Δ				
$\log \Delta$				
$\log^* \Delta$				
$O(1)$				

Weak colouring
(odd-degree graphs)

Landscape



Landscape



Typical state of the art

	$O(1)$	$\log^* n$	
Δ	no	yes	positive: $O(\log^* n)$
$\log \Delta$			tight bounds
$\log^* \Delta$			as a function of n
$O(1)$			negative: $o(\log^* n)$

Typical state of the art

	$O(1)$	$\log^* n$	
Δ	yes		positive: $O(\Delta)$
$\log \Delta$???		exponential gap as a function of Δ
$\log^* \Delta$	no		
$O(1)$			negative: $o(\log \Delta)$

Typical state of the art

	$O(1)$	$\log^* n$	
Δ	yes		positive: $O(\Delta)$
$\log \Delta$????		exponential gap as a function of Δ — or much worse
$\log^* \Delta$			
$O(1)$			negative: nothing

**fairly well
understood**



$O(1)$

$\log^* n$

Δ



$\log \Delta$



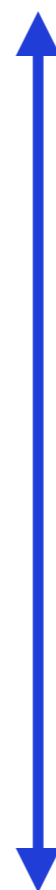
$\log^* \Delta$



$O(1)$



**poorly
understood**



Example:

LP approximation

- **$O(\log \Delta)$: possible**
 - Kuhn et al. (2004, 2006)
- **$o(\log \Delta)$: not possible**
 - Kuhn et al. (2004, 2006)

Example:

Maximal matching

- $O(\Delta + \log^* n)$: possible
 - Panconesi & Rizzi (2001)
- $O(\Delta) + o(\log^* n)$: not possible
 - Linial (1992)
- $o(\Delta) + O(\log^* n)$: unknown

Example: **Bipartite maximal matching**

- **$O(\Delta)$: trivial**
 - Hańćkowiak et al. (1998)
- **$o(\Delta)$: unknown**

Example: **Bipartite maximal matching**

- **$O(\Delta)$: trivial for Δ -regular graphs**
 - Hańćkowiak et al. (1998)
- **$O(1)$: unknown for Δ -regular graphs**

Example:

Semi-matching

- **$O(\Delta^5)$: possible**
 - Czygrinow et al. (2012)
- **$o(\Delta)$: unknown**

Example:

Weak colouring

- **$O(\log^* \Delta)$: possible** (in odd-degree graphs)
 - Naor & Stockmeyer (1995)
- **$o(\log^* \Delta)$: unknown**

**fairly well
understood**



$O(1)$

$\log^* n$

Δ



$\log \Delta$



$\log^* \Delta$



$O(1)$



**poorly
understood**



Orthogonal challenges?

- **n : “symmetry breaking”**
 - fairly well understood
 - Cole & Vishkin (1986), Linial (1992), Ramsey theory ...
- **Δ : “local coordination”**
 - poorly understood

*“symmetry
breaking”*



$O(1)$

$\log^* n$

Δ



$\log \Delta$



$\log^* \Delta$



$O(1)$



*“local
coordination”*



Orthogonal challenges

- **Example: maximal matching, $O(\Delta + \log^* n)$**
- **Restricted versions:**
 - pure symmetry breaking, $O(\log^* n)$
 - pure local coordination, $O(\Delta)$

Orthogonal challenges

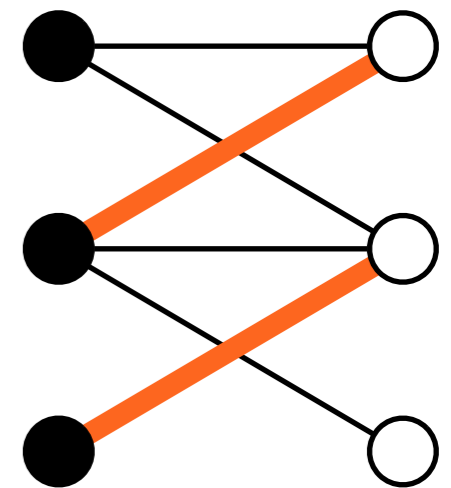
- **Example: maximal matching, $O(\Delta + \log^* n)$**
- **Pure symmetry breaking:**
 - input = cycle
 - no need for local coordination
 - $O(\log^* n)$ is possible and tight

Orthogonal challenges

- **Example: maximal matching, $O(\Delta + \log^* n)$**
- **Pure local coordination:**
 - input = 2-coloured graph
 - no need for symmetry breaking
 - $O(\Delta)$ is possible — is it tight?

Maximal matching in 2-coloured graphs

- **Trivial algorithm:**
 - black nodes send proposals to their neighbours, one by one
 - white nodes accept the first proposal that they get
- **“Coordination” \approx one by one traversal**



Maximal matching in 2-coloured graphs

- **General case:**

- upper bound: $O(\Delta)$
- lower bound: $\Omega(\log \Delta)$ — Kuhn et al.

- **Regular graphs:**

- upper bound: $O(\Delta)$
- lower bound: nothing!

Linear-in- Δ bounds

- **Many combinatorial problems seem to require “local coordination”, takes $O(\Delta)$ time?**
- **Lacking: linear-in- Δ lower bounds**
 - known for restricted algorithm classes (Kuhn & Wattenhofer 2006)

Good news

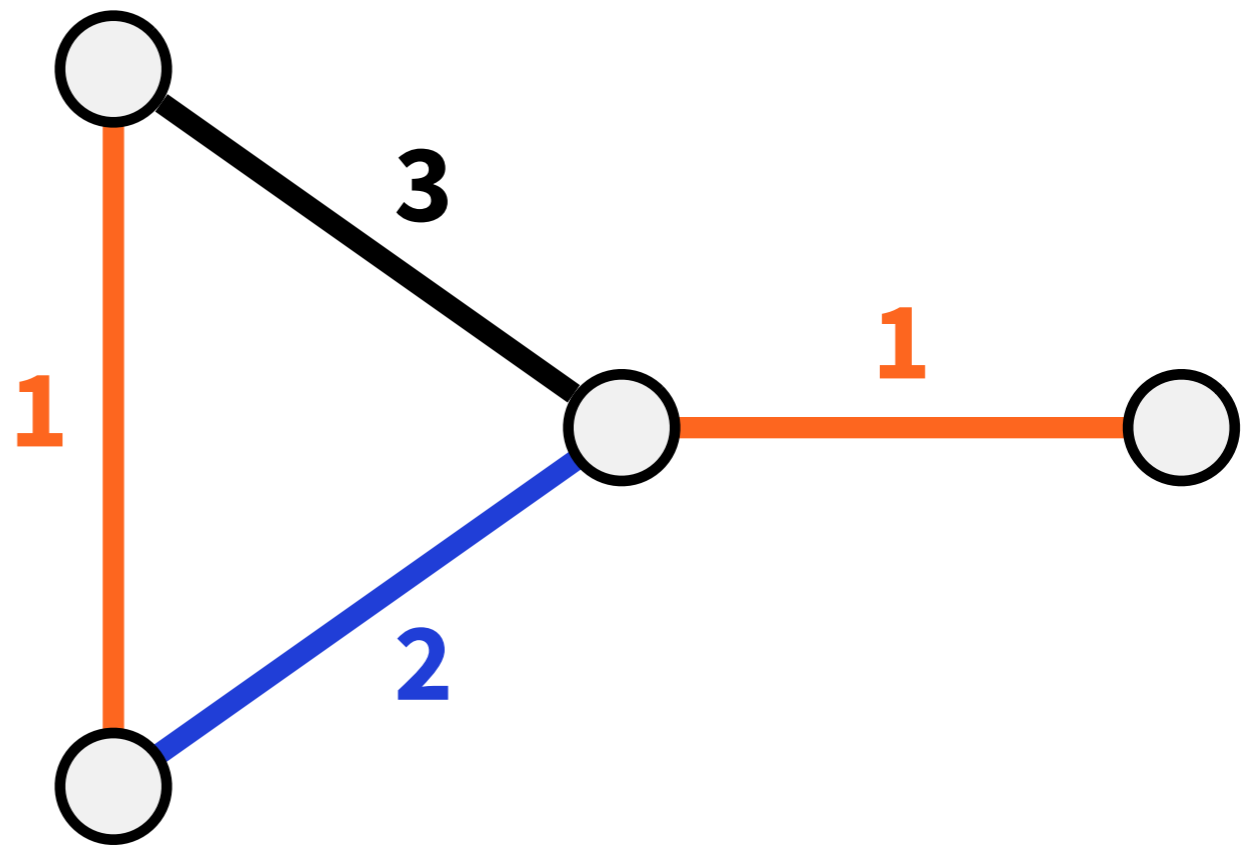
- **We are finally making some progress!**
- **Key problem:** *maximal matching*
- **Start with a “toy model”:**
edge colouring model

EC: edge colouring

No identifiers

No orientations

Edges coloured
with $O(\Delta)$ colours



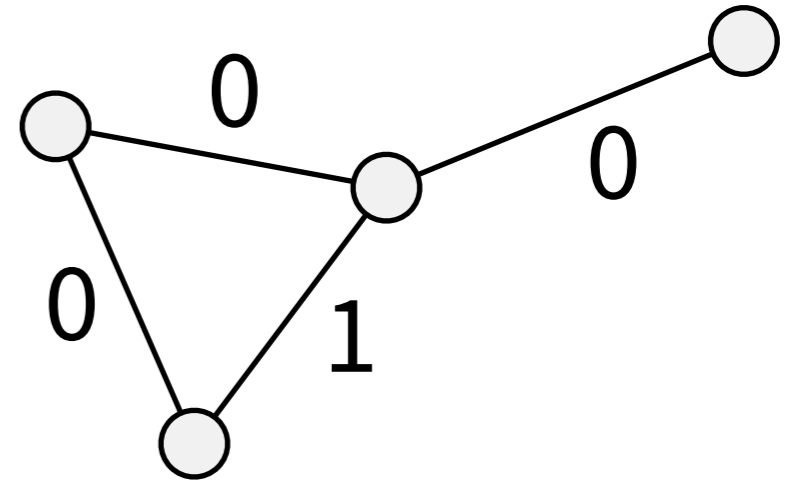
Recent progress

- **Maximal matching in *EC model***
- **$O(\Delta)$: trivial**
 - greedily by colour classes
- **$o(\Delta)$: not possible**
 - PODC 2012

What about the LOCAL model?

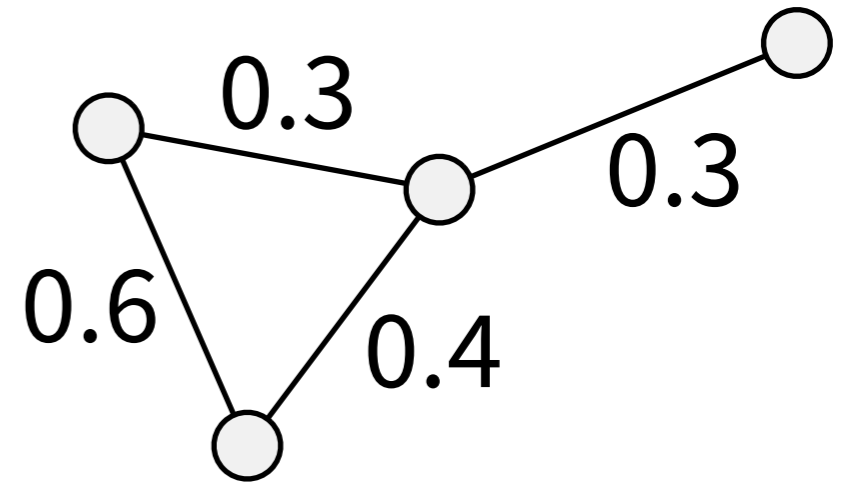
- Not yet there with maximal matchings...
- But we can prove lower bounds for maximal *fractional* matchings!

Matching



- **Edges labelled with integers $\{0, 1\}$**
- **Sum of incident edges at most 1**
- **Maximal matching:**
cannot increase the value of any label

Fractional matching



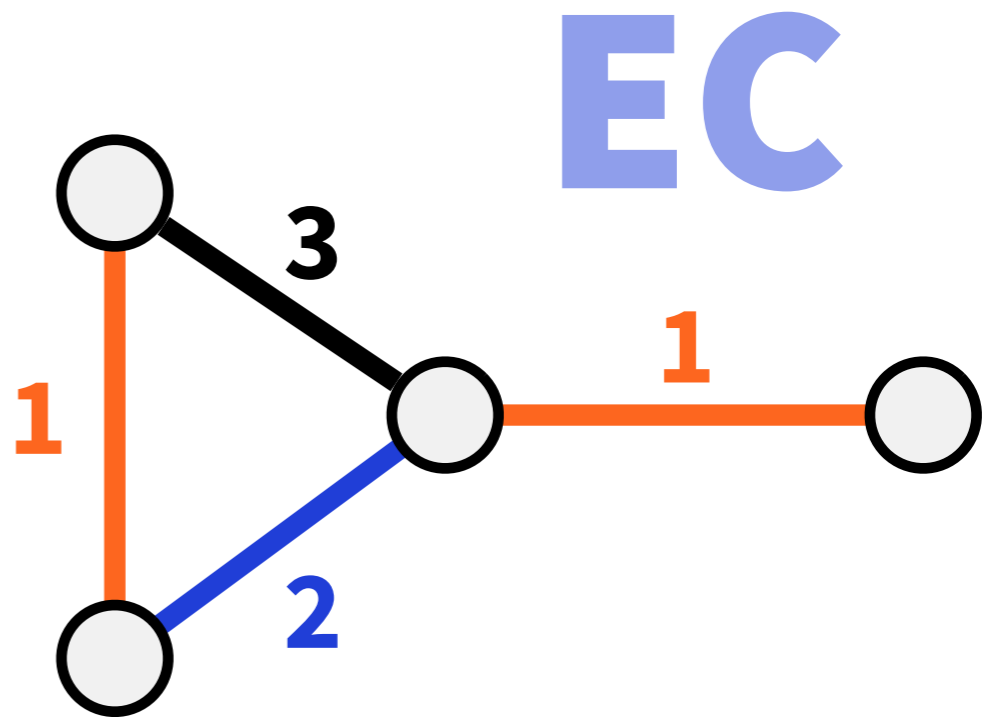
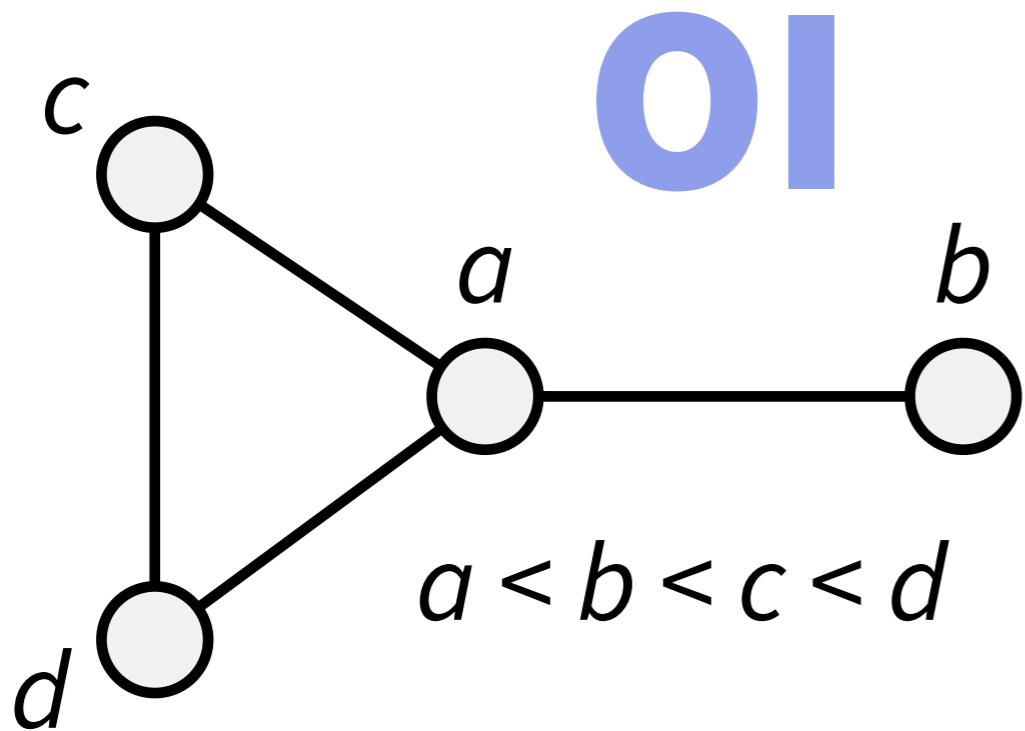
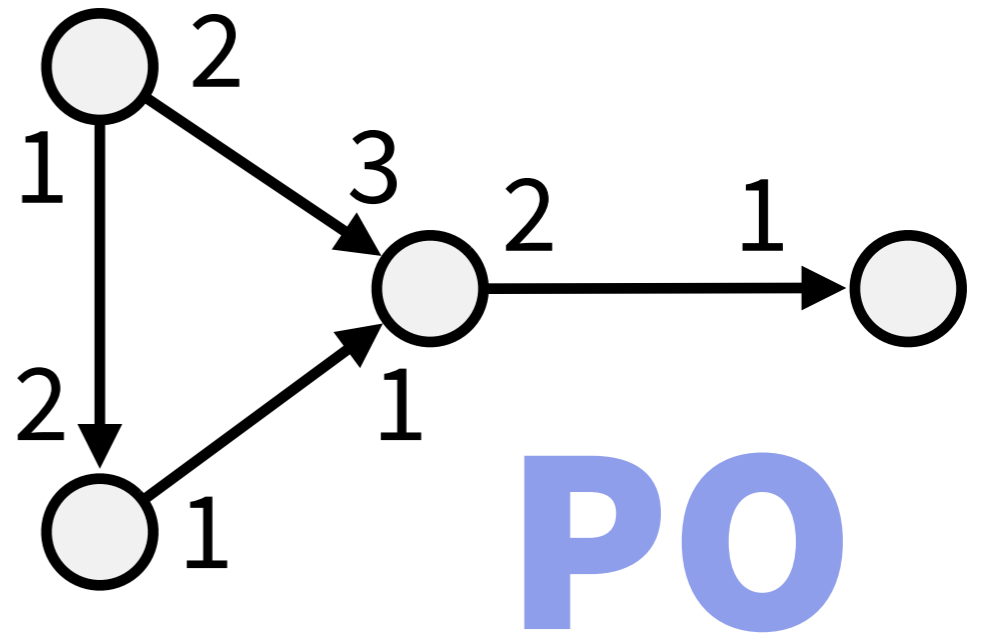
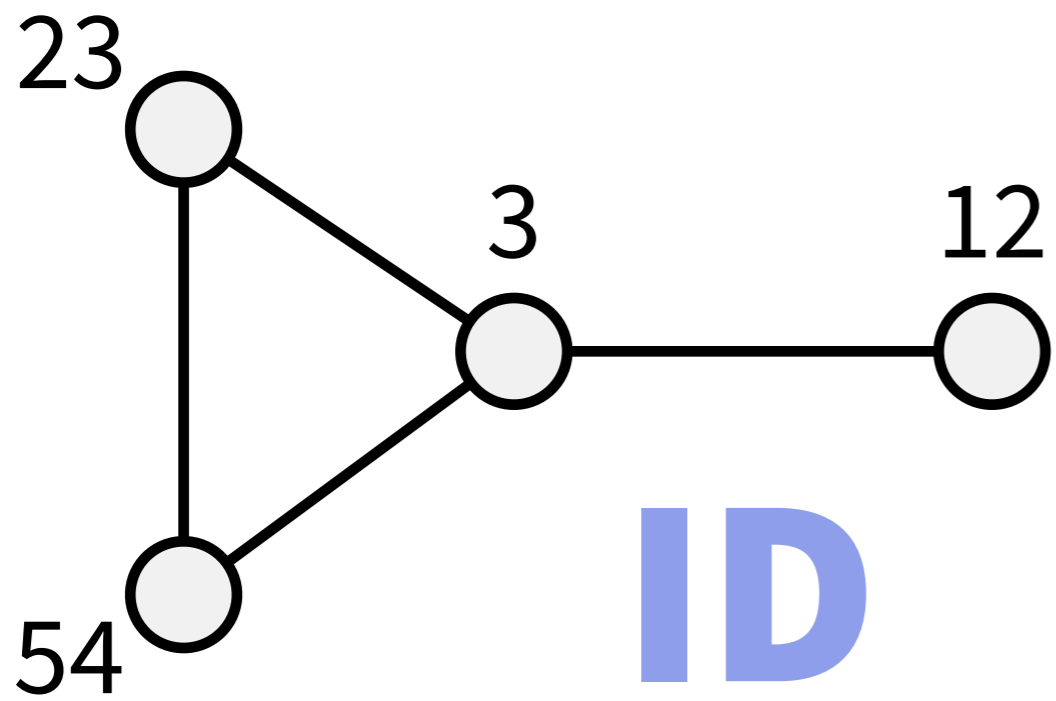
- Edges labelled with real numbers $[0, 1]$
- Sum of incident edges at most 1
- Maximal **fractional** matching:
cannot increase the value of any label

Maximal fractional matching

- **Possible in time $O(\Delta)$**
 - does **not** require symmetry breaking
 - d -regular graph: label all edges with $1/d$
- **Nontrivial part:** graphs that are not regular...

Recent progress

- Maximal *fractional* matching in LOCAL model
- $O(\Delta)$: possible
 - SPAA 2010
- $o(\Delta)$: not possible
 - PODC 2014



State of the art in 2014

- **Problems with $O(\Delta + \log^* n)$ algorithms:**
 - maximal matching
 - maximal independent set
 - vertex colouring with $\Delta+1$ colours
 - edge colouring with $2\Delta-1$ colours ...

State of the art in 2014

- **Problems with $O(\Delta + \log^* n)$ algorithms**
- **Problems with $O(\Delta)$ algorithms:**
 - maximal fractional matching
 - bipartite maximal matching ...

State of the art in 2014

- **Problems with $O(\Delta + \log^* n)$ algorithms**
- **Problems with $O(\Delta)$ algorithms**
- **Some linear-in- Δ lower bounds:**
 - maximal matchings, EC model
 - maximal fractional matchings, LOCAL model

State of the art in 2014

- **All these problems characterised as follows:**
 - any partial solution can be **completed**
 - but completion may be **unique**
- “**Completable but tight**” **problems**
 - greedy algorithm works,
but it may be constrained

State of the art in 2014

- Conjecture: “*completable but tight*” problems cannot be solved in time $o(\Delta) + O(\log^* n)$

State of the art in 2015

- **Conjecture: “*completable but tight*” problems cannot be solved in time $o(\Delta) + O(\log^* n)$**
- ***Wrong!***

State of the art in 2015

- **Barenboim (PODC 2015):**
 - *vertex colouring* with $\Delta+1$ colours
 - can be solved in time $o(\Delta) + O(\log^* n)$

We have a separation!

- **Barenboim (PODC 2015):**
 - *edge colouring* with $2\Delta-1$ colours
 - possible in time $o(\Delta)$ in **EC model**
- **PODC 2012:**
 - *maximal matching*
 - not possible in time $o(\Delta)$ in **EC model**

Next steps?

- **Separation for maximal independent set and $(\Delta+1)$ -vertex colouring in weak models**
- **Model:** anonymous vertex-coloured graphs
- **Lower bound:** just take line graphs
- **Upper bound:** adapt Barenboim's idea ??

Next steps?

- What is the **new conjecture**?
- Which problems require linear-in- Δ rounds?
- $(\Delta+1)$ -colouring: *not*
- Greedy colouring: *perhaps??*
 - lower bounds: e.g. Gavaille et al. (2009)

Next steps?

- **Linear-in- Δ lower bound for bipartite maximal matching**
- **Good:** pure local coordination, no symmetry-breaking needed
- **Needed:** extend known techniques so that they tolerate 2-coloured inputs

Next steps?

- **Poorly understood:** **optimisation problems**
- **Example:** minimum vertex cover (VC)
vs. maximal fractional matchings (MFM)
- **Good:** MFM \rightarrow 2-approximation of VC
- **Needed:** 2-approximation of VC \rightarrow MFM ???

Next steps?

- **Reductions**, conditional lower bounds!
 - hardness, completeness?
- Problems that are *at least as hard as* bipartite maximal matching
- Problems that are *at most as hard as* bipartite maximal matching

Summary

- **Distributed time complexity, LOCAL model**
- $O(\log^* n)$: “**symmetry breaking**”, OK
- $O(\Delta)$: “**local coordination**”, poorly understood
- **Next step: *bipartite* maximal matching**