Instructions. There are three questions, each of them worth 20 points. For the minimum passing grade of 1/5, you will need approximately 30 points, and for the highest grade of 5/5, you will need approximately 50 points. You can answer in English, Finnish, or Swedish.

Question 1. Define the following terms and concepts (10×1 points):

(a) Vertex cover.
(b) Minimal vertex cover.
(c) Minimum vertex cover.
(d) 2-approximation of a minimum vertex cover.
(e) Matching.
(f) Regular graph.
(g) Port-numbered network.
(h) Simple port-numbered network.
(i) Function $\log^* x$.
(j) Monochromatic subset (in the context of Ramsey's theorem).

Solve (2×5 points):

(k) Let $G = (V, E)$ be a graph, and let $M \subseteq E$ be a **maximal matching** in G. Define $C = \bigcup M$, that is, C consists of all nodes that are incident to an edge in matching M. Prove that C is a **2-approximation of a minimum vertex cover** in graph G.

(l) Let $G = (V, E)$ be a **regular graph**. Define $C = V$, that is, C is the set of all nodes. Prove that C is a **2-approximation of a minimum vertex cover** in graph G.

Question 2. Let $\mathscr{F}_\mathrm{P}$ be the family of **path graphs**; that is, $G \in \mathscr{F}_\mathrm{P}$ if G is a connected acyclic simple undirected graph and each node of G has a degree at most 2.

Design a deterministic distributed algorithm A that finds a **2-approximation of a minimum vertex cover** on graph family $\mathscr{F}_\mathrm{P}$, in the port-numbering model. Present your algorithm in a formally precise manner, using the *state machine formalism*. You will have to define

- States_A, the set of states,
- Msg_A, the set of possible messages,
- $\mathrm{init}_{A,d}\colon \mathrm{Input}_A \to \mathrm{States}_A$, the function that initialises the state machine,
- $\mathrm{send}_{A,d}\colon \mathrm{States}_A \to \mathrm{Msg}_A^d$, the function that constructs outgoing messages, and
- $\mathrm{receive}_{A,d}\colon \mathrm{States}_A \times \mathrm{Msg}_A^d \to \mathrm{States}_A$, the function that processes incoming messages.

Note that the set of local inputs is $\mathrm{Input}_A = \{0\}$, and the set of local outputs (stopping states) is $\mathrm{Output}_A = \{0, 1\}$. Prove that your algorithm is correct, and analyse its running time.

Question 3. Let $\mathscr{F}_\mathrm{C}$ be the family of **cycle graphs**; that is, $G \in \mathscr{F}_\mathrm{C}$ if G is a connected 2-regular simple undirected graph. Let Π_0 be the distributed graph problem of finding a **minimal vertex cover**, and let Π_1 be the distributed graph problem of finding a **minimum vertex cover**.

Prove that there is no deterministic distributed algorithm that solves problem Π_1 on graph family $\mathscr{F}_\mathrm{C}$ given Π_0. That is, given a cycle graph and a minimal vertex cover, it is not possible to find a minimum vertex cover. You can use the following result that is familiar from the course material.

> *Theorem.* Assume that A is a distributed algorithm, $X = \mathrm{Input}_A$ is a set of local inputs, $N = (V, P, p)$ and $N' = (V', P', p')$ are port-numbered networks, $f\colon V \to X$ and $f'\colon V' \to X$ are arbitrary functions, and $\phi\colon V \to V'$ is a covering map from (N, f) to (N', f'). Let $x_0, x_1, \ldots$ be the execution of A on (N, f), and let $x'_0, x'_1, \ldots$ be the execution of A on (N', f'). Then for each $t = 0, 1, \ldots$ and each $v \in V$ we have $x_t(v) = x'_t(\phi(v))$.