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Distributed Algorithms 2020

https://jukkasuomela.fi/da2020/ · October 29, 2023

Chapter 8

Local Neighborhoods

Covering maps can be used to argue that a given problem cannot be solved *at all* with deterministic PN algorithms. Now we will study the concept of *locality*, which can be used to argue that a given problem cannot be solved *fast*, in any model of distributed computing.

8.1 Definitions

Let $N = (V, P, p)$ and $N' = (V', P', p')$ be simple port-numbered networks, with the underlying graphs $G = (V, E)$ and $G' = (V', E')$. Fix the local inputs $f : V \to Y$ and $f' : V' \to Y$, a pair of nodes $v \in V$ and $v' \in V'$, and a radius $r \in \mathbb{N}$. Define the radius-r neighborhoods

$$U = \operatorname{ball}_G(v, r), \quad U' = \operatorname{ball}_{G'}(v', r).$$

We say that (N, f, v) and (N', f', v') have *isomorphic radius-r neighborhoods* if there is a bijection $\psi : U \to U'$ with $\psi(v) = v'$ such that

(a) ψ preserves degrees: $\deg_N(v) = \deg_{N'}(\psi(v))$ for all $v \in U$.

(b) ψ preserves connections and port numbers: $p(u, i) = (v, j)$ if and only if $p'(\psi(u), i) = (\psi(v), j)$ for all $u, v \in U$.

(c) ψ preserves local inputs: $f(v) = f'(\psi(v))$ for all $v \in U$.

The function ψ is called an *r-neighborhood isomorphism from* (N, f, v) *to* (N', f', v'). See Figure 8.1 for an example.

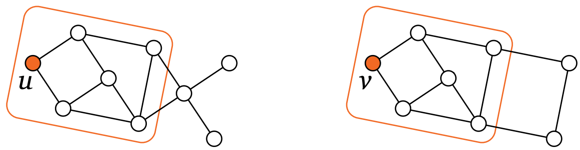


Figure 8.1: Nodes u and v have isomorphic radius-2 neighborhoods, provided that we choose the port numbers appropriately. Therefore in any algorithm A the state of u equals the state of v at time $t = 0, 1, 2$. However, at time $t = 3, 4, \dots$ this does not necessarily hold.

8.2 Local Neighborhoods and Executions

Theorem 8.1. *Assume that*

(a) A *is a deterministic* PN *algorithm with* $X = \mathrm{Input}_A$,

(b) $N = (V, P, p)$ *and* $N' = (V', P', p')$ *are simple port-numbered networks,*

(c) $f : V \to X$ *and* $f' : V' \to X$ *are arbitrary functions,*

(d) $v \in V$ *and* $v' \in V'$,

(e) (N, f, v) *and* (N', f', v') *have isomorphic radius-*r *neighborhoods.*

Let

(f) $x_0, x_1, \dots$ *be the execution of* A *on* (N, f), *and*

(g) $x'_0, x'_1, \dots$ *be the execution of* A *on* (N', f').

Then for each $t = 0, 1, \dots, r$ *we have* $x_t(v) = x'_t(v')$.

Proof. Let G and G' be the underlying graphs of N and N', respectively. We will prove the following stronger claim by induction: for each $t = 0, 1, \dots, r$, we have $x_t(u) = x'_t(\psi(u))$ for all $u \in \mathrm{ball}_G(v, r - t)$.

To prove the base case $t = 0$, let $u \in \mathrm{ball}_G(v, r)$, $d = \deg_N(u)$, and $u' = \psi(u)$; we have

$$x'_0(u') = \mathrm{init}_{A,d}(f'(u')) = \mathrm{init}_{A,d}(f(u)) = x_0(u).$$

For the inductive step, assume that $t \geq 1$ and

$$u \in \text{ball}_G(v, r-t).$$

Let $u' = \psi(u)$. By inductive assumption, we have

$$x'_{t-1}(u') = x_{t-1}(u).$$

Now consider a port $(u,i) \in P$. Let $(s,j) = p(u,i)$. We have $\{s,u\} \in E$, and therefore

$$\text{dist}_G(s,v) \leq \text{dist}_G(s,u) + \text{dist}_G(u,v) \leq 1 + r - t.$$

Define $s' = \psi(s)$. By inductive assumption we have

$$x'_{t-1}(s') = x_{t-1}(s).$$

The neighborhood isomorphism ψ preserves the port numbers: $(s',j) = p'(u',i)$. Hence all of the following are equal:

(a) the message sent by s to port j on round t,
(b) the message sent by s' to port j on round t,
(c) the message received by u from port i on round t,
(d) the message received by u' from port i on round t.

As the same holds for any port of u, we conclude that

$$x'_t(u') = x_t(u). \qquad \square$$

To apply Theorem 8.1 in the LOCAL model, we need to include unique identifiers in the local inputs f and f'.

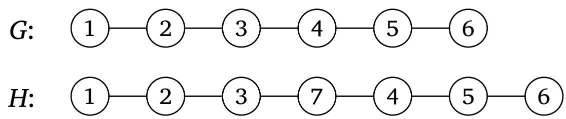
8.3 Example: 2-Coloring Paths

We know from Chapter 1 that one can find a proper 3-coloring of a path very fast, in $O(\log^* n)$ rounds. Now we will show that 2-coloring is much harder; it requires linear time.

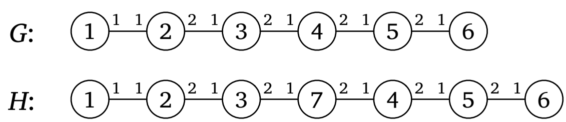
To reach a contradiction, suppose that there is a deterministic distributed algorithm A that finds a proper 2-coloring of any path graph in $o(n)$ rounds in the LOCAL model. Then there has to be a number n_0 such that for any number of nodes $n \geq n_0$, the running time of algorithm A is at most $(n-3)/2$. Pick some integer $k \geq n_0/2$, and consider two paths: path G contains $2k$ nodes, with unique identifiers $1, 2, \ldots, 2k$, and path H contains $2k+1$ nodes, with unique identifiers

$$1, 2, \ldots, k, 2k+1, k+1, k+2, \ldots, 2k.$$

Here is an example for $k = 3$:



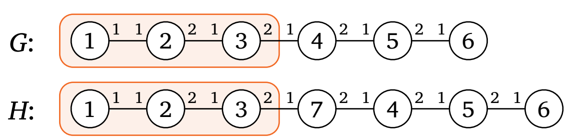
We assign the port numbers so that for all degree-2 nodes port number 1 points towards node 1:



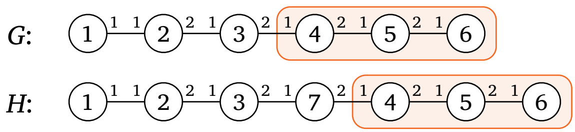
By assumption, the running time of A is at most

$$(n-3)/2 \leq (2k+1-3)/2 = k-1$$

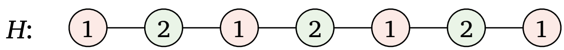
rounds in both cases. Since node 1 has got the same radius-$(k-1)$ neighborhood in G and H, algorithm A will produce the same output for node 1 in both networks:



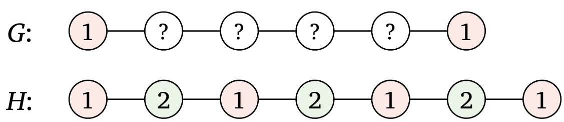
By a similar reasoning, node $2k$ (i.e., the last node of the path) also has to produce the same output in both cases:



However, now we reach a contradiction. In path H, in any proper 2-coloring nodes 1 and $2k$ have the same color—for example, both of them are of color 1, as shown in the following picture:



If algorithm A works correctly, it follows that nodes 1 and $2k$ must produce the same output in path H. However, then it follows that nodes 1 and $2k$ produces the same output also in G, too, but this cannot happen in any proper 2-coloring of G.



We conclude that algorithm A fails to find a proper 2-coloring in at least one of these instances.

8.4 Quiz

Let N a simple port-numbered network with 1000 nodes, such that the underlying graph of N is a cycle. Form another network N' by adding one edge to N. Let A be a LOCAL-model algorithm that runs in 100 rounds. Let f be the output of A in network N, and let f' be the output of A in network N'. At most how many nodes there can be such that their output differs in f and f'?

8.5 Exercises

Exercise 8.1 (edge coloring)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*.

(a) Show that it is possible to find a 2-edge coloring in time $O(n)$ with deterministic PN-algorithms.

(b) Show that it is not possible to find a 2-edge coloring in time $o(n)$ with deterministic PN-algorithms.

(c) Show that it is not possible to find a 2-edge coloring in time $o(n)$ with deterministic LOCAL-algorithms.

Exercise 8.2 (maximal matching)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*.

(a) Show that it is possible to find a maximal matching in time $O(n)$ with deterministic PN-algorithms.

(b) Show that it is not possible to find a maximal matching in time $o(n)$ with deterministic PN-algorithms.

(c) Show that it is possible to find a maximal matching in time $o(n)$ with deterministic LOCAL-algorithms.

Exercise 8.3 (optimization)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*. Can we solve the following problems with deterministic PN-algorithms? If yes, how fast? Can we solve them any faster in the LOCAL model?

(a) Minimum vertex cover.

(b) Minimum dominating set.

(c) Minimum edge dominating set.

Exercise 8.4 (approximation)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*. Can we solve the following problems with deterministic PN-algorithms? If yes, how fast? Can we solve them any faster in the LOCAL model?

(a) 2-approximation of a minimum vertex cover?

(b) 2-approximation of a minimum dominating set?

Exercise 8.5 (auxiliary information). In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*, and we are given a 4-coloring as input. We consider deterministic PN-algorithms.

(a) Show that it is possible to find a 3-coloring in time 1.

(b) Show that it is not possible to find a 3-coloring in time 0.

(c) Show that it is possible to find a 2-coloring in time $O(n)$.

(d) Show that it is not possible to find a 2-coloring in time $o(n)$.

⋆ **Exercise 8.6** (orientations). In this exercise, the graph family $\mathscr{F}$ consists of *cycle graphs*, and we are given some *orientation* as input. The task is to find a *consistent orientation*, i.e., an orientation such that both the indegree and the outdegree of each node is 1.

(a) Show that this problem cannot be solved with any deterministic PN-algorithm.

(b) Show that this problem cannot be solved with any deterministic LOCAL-algorithm in time $o(n)$.

(c) Show that this problem can be solved with a deterministic PN-algorithm if we give n as input to all nodes. How fast? Prove tight upper and lower bounds on the running time.

⋆ **Exercise 8.7** (local indistinguishability). Consider the graphs G_1 and G_2 illustrated in Figure 8.2. Assume that A is a deterministic PN-algorithm with running time 2. Show that A cannot distinguish between nodes v_1 and v_2. That is, there are simple port-numbered networks N_1 and N_2 such that N_i has G_i as the underlying graph, and the output of v_1 in N_1 equals the output of v_2 in N_2.

▷ *hint A*

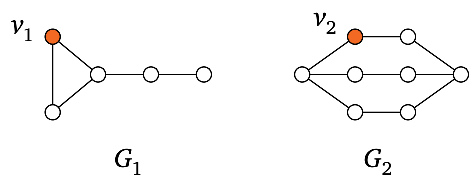


Figure 8.2: Graphs for Exercise 8.7.

8.6 Bibliographic Notes

Local neighborhoods were used to prove negative results in the context of distributed computing by, e.g., Linial [1].

8.7 Bibliography

[1] Nathan Linial. Locality in distributed graph algorithms. *SIAM Journal on Computing*, 21(1):193–201, 1992. doi:10.1137/0221015.

8.8 Hints

A. Argue using both covering maps and local neighborhoods. For $i = 1, 2$, construct a network N_i' and a covering map ϕ_i from N_i' to N_i. Let $v_i' \in \phi_i^{-1}(v_i)$. Show that v_1' and v_2' have isomorphic radius-2 neighborhoods; hence v_1' and v_2' produce the same output. Then use the covering maps to argue that v_1 and v_2 also produce the same outputs. In the construction of N_1', you will need to eliminate the 3-cycle; otherwise v_1' and v_2' cannot have isomorphic neighborhoods.