ON THE L^4 CONVERGENCE OF PARTICLE FILTERS WITH GENERAL IMPORTANCE DISTRIBUTIONS

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ABSTRACT

In this paper we extend the L^4 proof of Hu et al. (2008) from bootstrap type of particle filters to particle filters with general importance distributions. The result essentially shows that with general importance distributions the particle filter converges provided that the importance weights are bounded. By numerical simulations we also show that this condition is often also a practical requirement for a good performance of a particle filter.

***Index Terms*—** Particle filter; convergence; importance distribution; unbounded function

1. INTRODUCTION

Particle filters [1, 2, 3] are powerful methods for approximate Bayesian filtering in state space models of the form

$$x_t \sim f(x_t \mid x_{t-1}), \qquad y_t \sim g(y_t \mid x_t), \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state of the system, $y_t \in \mathbb{R}^m$ is the measurement, $f(x_t \mid x_{t-1})$ is the transition probability density (w.r.t. Lebesgue measure) modeling the dynamics of the system, and $g(y_t \mid x_t)$ is the conditional probability density of measurements modeling the distribution of measurements.

Particle filters form a weighted set of Monte Carlo samples $\{(x_t^i, w_t^i) \ : \ i = 1, \ldots, N\}$ such that the posterior expectation of a test function $\phi(\cdot)$ can be approximated as

$$\mathbb{E}[\phi(x_t) \mid y_{1:t}] \approx \sum_{i=1}^N w_t^i \, \phi(x_t^i). \tag{2}$$

A particle filter converges if, in a suitable sense, the above approximation becomes exact when $N \to \infty$.

Various types of convergence result for particle filters with general importance distributions, but with bounded test functions can be found in the survey article [4]. Long-term stability results and central limit theorem type of convergence theorems for particle filters (also for unbounded functions), can be found in [5, 6, 7, 8] and references therein. L^4 type of convergence results for the unbounded case have recently been studied in [9, 10], but only in the case of bootstrap type of importance distributions.

In this paper, we extend the proof of Hu et al. (2008) [9] to the case of general importance distributions. The results show that the boundedness of importance weights (along with the model densities) is a sufficient condition for the convergence also in the case of unbounded test functions, which is also a sufficient condition in the bounded case [4]. We also discuss the practical implications of the condition to certain importance distributions proposed in literature.

2. PARTICLE FILTERING

Recall that the Bayesian filter for the state space model in (1) can be written in the abstract form [9]:

$$(\pi_{t|t-1}, \phi) = (\pi_{t-1|t-1}, f \, \phi), \tag{3}$$

$$(\pi_{t|t}, \phi) = \frac{(\pi_{t|t-1}, \phi \, g)}{(\pi_{t|t-1}, g)}, \tag{4}$$

where we have defined

$$(\pi, \phi) = \int \phi(x) \, \pi(\mathrm{d}x), \quad f \, \phi = \int f(x_t \mid x_{t-1}) \, \phi(x_t) \, \mathrm{d}x_t.$$

With this notation, we obtain the bootstrap filter simply by replacing the measures π with their finite-sample approximations and by introducing an additional resampling step. This was the starting point of the analysis in [9].

However, here we wish to analyze the convergence of the more general particle filter which does not correspond to a direct finite-sample approximation of the prediction and update steps above. Instead of sampling from the dynamic model distribution we sample from an importance distribution $q(x_t \mid x_{t-1}, y_t)$ and then compute weights for the samples. For this purpose it is convenient to rewrite the Bayesian filter as a single step

$$(\pi_{t|t}, \phi) = \frac{((\pi_{t-1|t-1}, \rho \, q), \phi)}{((\pi_{t-1|t-1}, \rho \, q), 1)}, \tag{5}$$

where we have defined the importance weights

$$\rho(x_t, x_{t-1}) = \frac{g(y_t \mid x_t) \, f(x_t \mid x_{t-1})}{q(x_t \mid x_{t-1}, y_t)}. \tag{6}$$

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As in [9, 10], to be able to cope with unbounded functions, we need to use a slightly modified version of the standard particle filter in order to guarantee the convergence. The modified particle filter is constructed such that we always have $((\pi^N_{t-1|t-1}, \rho\, q), 1) \geq \gamma_t > 0$, where $\gamma_t > 0$ is a chosen threshold [9]. The modified algorithm is the following.

Algorithm 2.1 (General Modified Particle Filter)**.** *The algorithm is similar to [9], but includes importance distributions.*

1. *Initialize the particles,* $\{x_0^i\}_{i=1}^N \sim \pi_0(\mathrm{d}x_0)$
2. *Draw samples according to* $\bar{x}_t^i \sim \sum_{j=1}^N \alpha_j^i\, q(x_t \mid x_{t-1}^j, y_t)$*, where* α_i^j *are non-negative weights such that* $\sum_{j=1}^N \alpha_j^i = 1$*,* $\sum_{i=1}^N \alpha_j^i = 1$*, and*
$$\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \alpha_j^i\, q(x_t \mid x_{t-1}^j, y_t) = \frac{1}{N} \sum_{j=1}^N q(x_t \mid x_{t-1}^j, y_t).$$
3. *If* $((\pi^N_{t-1|t-1}, \bar{\rho}\, \bar{q}), 1) \geq \gamma_t$*, proceed to step 4 otherwise return to step 2. Note that* $\bar{\rho}$ *and* $\bar{q}$ *are the values evaluated at* $\bar{x}_t^i$*.*
4. *Rename* $\tilde{x}_t^i = \bar{x}_t^i$*, and compute and normalize the weights* $w_t^i = \rho(\tilde{x}_t^i, x_{t-1}^i)$*,* $\tilde{w}_t^i = w_t^i / \sum_{j=1}^N w_t^j$*.*
5. *Resample,* $x_t^i \sim \tilde{\pi}^N_{t|t}(\mathrm{d}x_t) = \sum_{i=1}^N \tilde{w}_t^i\, \delta_{\tilde{x}_t^i}(\mathrm{d}x_t)$
6. *Set* $t = t + 1$ *and repeat from step 2).*

3. CONVERGENCE WITH GENERAL IMPORTANCE DISTRIBUTION

To prove the convergence of the particle filter, we need to impose the following conditions (cf. [9]).

- H0: For any given $y_{1:s}$, we have $((\pi_{s-1|s-1}, \rho\, q), 1) > 0$, where $s = 1, \ldots, t$.
- H1: The dynamic model f, measurement model g, and the importance weights $\rho(x_t, x_{t-1})$ are bounded. That is, there exist constants C_f, C_g, and C_ρ such that $\|f\| \leq C_f$, $\|g\| \leq C_g$, and $\|\rho\| \leq C_\rho$, where the first norm is an operator norm induced by the supremum norm, and the second two are supremum norms of the functions.
- H2: The function $\phi(\cdot)$ satisfies $\sup_{x_s} |\phi(x_s)|^4 g(y_s \mid x_s) < C(y_{1:s})$.

The main convergence theorem is the following.

Theorem 3.1. *Consider the general modified particle filter algorithm and suppose that the conditions H0, H1, and H2 above hold. Then we have the following.*

1. *For a sufficiently large* N*, the algorithm will not run into an infinite loop on steps 2-3.*
2. *Let* $L_t^4(g)$ *be the class of functions satisfying H2. For any* $\phi \in L_t^4(g)$*, there exists a constant* $C_{t|t}$*, independent of* N *such that*
$$\mathbb{E}\left[\left|(\pi^N_{t|t}, \phi) - (\pi_{t|t}, \phi)\right|^4\right] \leq C_{t|t} \frac{||\phi||^4_{t,4}}{N^2}, \quad (7)$$

where $||\phi||^4_{t,4}$ is defined as [9]

$$||\phi||_{t,4} = \max\left\{1, (\pi_{s|s}, |\phi|^4)^{1/4},\ s = 0, 1, \ldots, t\right\}. \quad (8)$$

Proof. The proofs for *initialization and resampling steps* are the same as in [9]. Thus, here, we only prove the convergence of the (combined) prediction and update steps. That is, we prove the convergence of the following:

$$(\pi^N_{t|t}, \phi) - (\pi_{t|t}, \phi) = \frac{(\hat{\pi}^N_{t|t}, \phi)}{(\hat{\pi}^N_{t|t}, 1)} - \frac{(\hat{\pi}_{t|t}, \phi)}{(\hat{\pi}_{t|t}, 1)}, \quad (9)$$

where $\hat{\pi}^N_{t|t} = (\pi^N_{t-1|t-1}, \rho\, q^N)$ and $\hat{\pi}_{t|t} = (\pi_{t-1|t-1}, \rho\, q)$. It is now enough to study the bounded for the following terms:

$$\mathbb{E}\left[\left|(\pi^N_{t|t}, \phi) - (\pi_{t|t}, \phi)\right|^4\right] \quad \text{and} \quad \mathbb{E}[(\pi^N_{t|t}, |\phi|^4)]. \quad (10)$$

At $t = 0$, we have the initialization step for which the proof can be found in [9]. Next, we assume that there exist constants $C_{t-1|t-1}$ and $M_{t-1|t-1}$ such that the following is true:

$$\mathbb{E}\left[\left|(\pi^N_{t-1|t-1}, \phi) - (\pi_{t-1|t-1}, \phi)\right|^4\right] \leq C_{t-1|t-1} \frac{||\phi||^4_{t-1,4}}{N^2}, \quad (11)$$

and

$$\mathbb{E}\left[\left|(\pi^N_{t-1|t-1}, |\phi|^4)\right|\right] \leq M_{t-1|t-1} ||\phi||^4_{t-1,4}. \quad (12)$$

To study the boundedness of (10), we start by studying the numerator terms in (9). We first derive the bound for $\mathbb{E}[|(\hat{\pi}^N_{t|t}, \phi) - (\hat{\pi}_{t|t}, \phi)|^4]$ and then for $\mathbb{E}[(\hat{\pi}^N_{t|t}, |\phi|^4)]$.

Let $\mathcal{F}_{t-1}$ be the σ-algebra generated by x_{t-1}^i. Then we can write $(\hat{\pi}^N_{t|t}, \phi) - (\hat{\pi}_{t|t}, \phi) = \Pi_1 + \Pi_2 + \Pi_3$, where

$$\Pi_1 = (\hat{\pi}^N_{t|t}, \phi) - \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\phi(\tilde{x}_t^i)\, \rho(\tilde{x}_t^i, x_{t-1}) \mid \mathcal{F}_{t-1}], \quad (13)$$

$$\Pi_2 = \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\phi(\tilde{x}_t^i)\, \rho(\tilde{x}_t^i, x_{t-1}) \mid \mathcal{F}_{t-1}] - \frac{1}{N} \sum_{i=1}^N (\pi^{N,\alpha_i}_{t-1|t-1}, f\, \phi\, g), \quad (14)$$

$$\Pi_3 = \frac{1}{N} \sum_{i=1}^N (\pi^{N,\alpha_i}_{t-1|t-1}, f\, \phi\, g) - (\hat{\pi}_{t|t}, \phi). \quad (15)$$

We treat the terms of Π_1, Π_2 and Π_3 separately and, in each case, we assume that $\|\rho\| \leq C_\rho$, and that $\|f\|$ and $\|g\|$ are bounded by some constants. Let $\bar{x}_t^i \sim (\pi_{t-1|t-1}^{N,\alpha_i}, q)$, then

$$\mathbb{E}[\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}^i) \mid \mathcal{F}_{t-1}] = (\pi_{t-1|t-1}^{N,\alpha_i}, f\,\phi\,g),$$
$$(\pi_{t-1|t-1}^{N}, f\,\phi\,g) = \frac{1}{N}\sum_{i=1}^{N}(\pi_{t-1|t-1}^{N,\alpha_i}, f\,\phi\,g). \tag{16}$$

The probability of the threshold γ_t corresponds to event A_t defined as $A_t = \{(\pi_{t-1|t-1}^{N}, f\,g) \geq \gamma_t\}$, where, by (16), we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\rho(\bar{x}_t^i, x_{t-1}^i) \mid \mathcal{F}_{t-1}\right] = (\pi_{t-1|t-1}^{N,\alpha_i}, f\,g). \tag{17}$$

Suppose $\|f\|$ and $\|g\|$ are bounded, and H0 holds. Then, using Markov's inequality and (11), it implies that

$$P\left[\frac{1}{N}\sum_{i=1}^{N}\rho(\bar{x}_s^i, x_{s-1}^i) < \gamma_t \Big| \mathcal{F}_{t-1}\right] \leq \frac{C_{t-1|t-1}\|f\|^4\,\|g\|^4}{N^2\Big|\gamma_t - (\pi_{t|t-1}, g)\Big|^4} = \frac{\tilde{C}_{\gamma_t}}{N^2} = \epsilon. \tag{18}$$

To bound (13), we use Lemmas 7.1, 7.2, 7.3, and 7.5 from [9], (16), and (12), which leads to

$$\begin{aligned}
&\mathbb{E}[|\Pi_1|^4 \mid \mathcal{F}_{t-1}] \\
&\leq \frac{2^4}{N^4}\left[\sum_{i=1}^{N}\frac{\mathbb{E}\left[|\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}^i)|^4|\mathcal{F}_{t-1}\right]}{1-\epsilon}\right] \\
&+ \frac{2^4}{N^4}\left[\left(\sum_{i=1}^{N}\frac{\mathbb{E}\left[|\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}^i)|^2|\mathcal{F}_{t-1}\right]}{1-\epsilon}\right)^2\right] \\
&\leq \frac{2^4}{(1-\epsilon)^2}\left[C_\rho^3\frac{\left(\pi_{t-1|t-1}^N, f\,|\phi|^4\,g\right)}{N^2}\right] \\
&+ \frac{2^4}{(1-\epsilon)^2}\left[C_\rho^2\frac{\left(\pi_{t-1|t-1}^N, f\,|\phi|^2\,g\right)}{N^2}\right] \\
&\leq \frac{2^5\tilde{C}_\rho}{(1-\epsilon)^2}\left[\frac{M_{t-1|t-1}\|\phi\|_{t-1,4}^4}{N^2}\right] = \tilde{C}_{\Pi_1}\frac{\|\phi\|_{t-1,4}^4}{N^2}.
\end{aligned} \tag{19}$$

For the bound of (14), we use Lemmas 7.3 and 7.5, from [9], Eqs. (12), (16), and (18), as well as Jensen's inequality, which leads to

$$\begin{aligned}
&\mathbb{E}\left[\Big|\Pi_2\Big|^4 \mid \mathcal{F}_{t-1}\right] \\
&\leq \frac{2^4\epsilon^2C_\rho^2}{(1-\epsilon)^4N^2}\mathbb{E}\left[\mathbb{E}\left[(N\,(\pi_{t-1|t-1}^N, f\,|\phi|^2\,g))^2\right]\right] \\
&\leq \frac{2^4\epsilon^2}{(1-\epsilon)^4N^2}C_\rho^2\|f\|^2\,\|g\|^2\left(\pi_{t-1|t-1}^N, |\phi|^4\right) \\
&\leq \tilde{C}_{\Pi_2}\frac{\|\phi\|_{t-1,4}^4}{N^2}.
\end{aligned} \tag{20}$$

For the bound of (15), we use (11), which gives:

$$\begin{aligned}
&\mathbb{E}\left[\Big|\Pi_3\Big|^4 \mid \mathcal{F}_{t-1}\right] \\
&\leq ||f||^4||g||^4\mathbb{E}\left[\Big|(\pi_{t-1|t-1}^N, \phi) - (\pi_{t-1|t-1}, \phi)\Big|^4\right] \\
&\leq \tilde{C}_{t-1|t-1}\frac{\|f\|^4\|g\|^4\|\phi\|_{t-1,4}^4}{N^2} = \tilde{C}_{\Pi_3}\frac{\|\phi\|_{t-1,4}^4}{N^2}.
\end{aligned} \tag{21}$$

By combining Eqs. (19), (20), and (21) via Minkowski's inequality, we get

$$\begin{aligned}
&\mathbb{E}\left[\Big|(\hat{\pi}_{t|t}^N, \phi) - (\hat{\pi}_{t|t}, \phi)\Big|^4\right]^{\frac{1}{4}} \\
&\leq \left(\tilde{C}_{\Pi_1}^{1/4} + \tilde{C}_{\Pi_2}^{1/4} + \tilde{C}_{\Pi_3}^{1/4}\right)\frac{\|\phi\|_{t-1,4}}{N^{1/2}} = \hat{C}_{t|t}^{1/4}\frac{\|\phi\|_{t-1,4}}{N^{1/2}},
\end{aligned}$$

which implies

$$\mathbb{E}\left[\Big|(\hat{\pi}_{t|t}^N, \phi) - (\hat{\pi}_{t|t}, \phi)\Big|^4\right] \leq \hat{C}_{t|t}\frac{\|\phi\|_{t-1,4}^4}{N^2}. \tag{22}$$

The bound for $\mathbb{E}[(\hat{\pi}_{t|t}^N, |\phi|^4)]$ can be derived using the same technique as above, which leads to

$$\mathbb{E}\left[\Big|(\hat{\pi}_{t|t}^N, |\phi|^4)\Big|\right] \leq M_{t|t}||\phi||_{t-1,4}^4. \tag{23}$$

Note that if we set $\phi = 1$ in (22) and (23), we get similar bounds for the difference of the denominators.

We finally study the boundedness of (10). For $\mathbb{E}[|(\pi_{t|t}^N, \phi) - (\pi_{t|t}, \phi)|^4]$, we use (22) and (23) with $\phi = 1$ along with Minkowski's inequality, to get

$$\begin{aligned}
&\mathbb{E}\left[\Big|(\pi_{t|t}^N, \phi) - (\pi_{t|t}, \phi)\Big|^4\right]^{\frac{1}{4}} \leq \left|\frac{(\hat{\pi}_{t|t}^N, \phi)\hat{C}_{t|t}^{1/4}}{\gamma_t(\hat{\pi}_{t|t}, 1)}\right|\left(\frac{1}{N^2}\right)^{\frac{1}{4}} \\
&+ \left|\frac{\hat{C}_{t|t}^{1/4}}{(\hat{\pi}_{t|t}, 1)}\right|\left(\frac{\|\phi\|_{t-1,4}^4}{N^2}\right)^{\frac{1}{4}} \leq \frac{1}{N^{1/2}}C_{t|t}^{\frac{1}{4}}\|\phi\|_{t-1,4},
\end{aligned}$$

which thus gives

$$\mathbb{E}\left[\Big|(\pi_{t|t}^N, \phi) - (\pi_{t|t}, \phi)\Big|^4\right] \leq C_{t|t}\frac{\|\phi\|_{t-1,4}^4}{N^2}.$$

For $\mathbb{E}[(\pi_{t|t}^N, |\phi|^4)]$, we similarly get

$$\mathbb{E}\left[(\pi_{t|t}^N, |\phi|^4) - (\pi_{t|t}, |\phi|^4)\right] \leq \bar{M}_{t|t} \|\phi\|_{t-1,4}^4, \tag{24}$$

which thus completes the proof. □

4. PRACTICAL IMPLICATIONS

Our result states that provided that $||f||$, $||g||$, and $||\rho||$ are bounded, we can ensure the convergence. The boundedness of f and g is indeed quite natural, but let's take a closer look at the boundedness of the term ρ, which we defined in (6), on some commonly used importance distributions.

- The *optimal importance distribution* [2] $q(x_t \mid x_{t-1}, y_t) = p(x_t \mid x_{t-1}, y_t)$ leads to $\rho(x_t, x_{t-1}) = \int g(y_t \mid x_t)\, f(x_t \mid x_{t-1})\, \mathrm{d}x_t$ which is guaranteed to be bounded provided that f and g are bounded.
- In the *bootstrap filter* [1, 9] we select $q(x_t \mid x_{t-1}, y_t) = f(x_t \mid x_{t-1})$, which gives $\rho(x_t \mid x_{t-1}) = g(y_t \mid x_t)$ and thus is ensured to be bounded.
- Using *non-linear Kalman filters* to approximate the optimal importance distribution [2, 11, 12] gives $q(x_t \mid x_{t-1}, y_t) = \mathcal{N}(x_t \mid m_t, P_t)$ were, m_t and P_t are mean and covariance computed by the Kalman filter. We can now assure convergence only if the ratio of the optimal importance distribution and its approximation $p(x_t \mid x_{t-1}, y_t)/\mathcal{N}(x_t \mid m_t, P_t)$ is bounded. This requires that the tails of $p(x_t \mid x_{t-1}, y_t)$ are not heavier than the tails of the Gaussian distribution and that the covariance P_t is bounded from below.
- We can also use a multivariate Student's t-distribution with the parameters m_t and P_t above instead of the Gaussian distribution [13]. If we choose the degrees of freedom in the Student's t-distribution to be low enough, then ρ can be assured to be bounded.

Example 4.1 (Linear Gaussian state space model)**.** *Consider the one-dimensional Gaussian random walk model*

$$x_t = x_{t-1} + q_{t-1}, \qquad q_{t-1} \sim \mathcal{N}(0, Q), \tag{25}$$
$$y_t = x_t + r_t, \qquad r_t \sim \mathcal{N}(0, R). \tag{26}$$

The optimal importance distribution is now Gaussian $\mathcal{N}(x_t \mid m_t, P_t)$ with $m_t = x_{t-1} + Q/(Q+R)\ [y_t - x_{t-1}]$, $P = Q - Q^2/(Q+R)$. If we replace the importance distribution with $\mathcal{N}(x_t \mid m_t, c\, P_t)$ where $c < 1$, then the weights become unbounded and thus the particle filter is not guaranteed to converge.

Figure 1 illustrates the effect of the value c to the estimates of the fourth order central moment of the filtering distribution at $t = 24$ with varying number of particles with the parameter values $Q = 1$, $R = 1/2$, and $x_0 = 0$. It can be seen

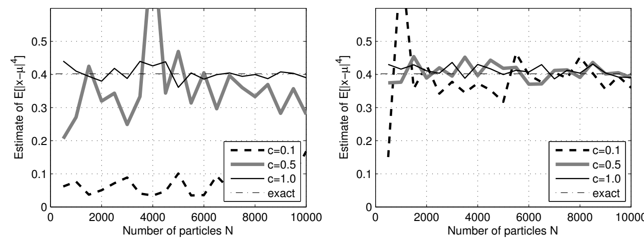

Fig. 1. Illustration of the effect of scaling the importance distributions with c in linear Gaussian example. Left: Gaussian distribution. Right: Student's t-distribution with $\nu = 3$ degrees of freedom. The scaling of variance significantly affects the performance of the particle filter with a Gaussian importance distribution whereas the effect to a Student's t-distribution based particle filter is smaller.

that scaling of the variance in Gaussian importance distribution affects the convergence of the fourth moment estimate whereas with the Student's t-distribution the effect is smaller.

Example 4.2 (Non-linear state space model)**.** *A typically used example of a non-linear model is the following system (see, e.g., [2]):*

$$x_t = \frac{1}{2}\, x_{t-1} + 25\, \frac{x_{t-1}}{1 + (x_{t-1})^2} + 8\, \cos(1.2t) + q_{t-1}, \tag{27}$$
$$y_t = \frac{x_t^2}{20} + r_t, \tag{28}$$

where $q_{t-1} \sim \mathcal{N}(0, 10)$ and $r_t \sim \mathcal{N}(0, 1)$. It is now easy to show that the ratio of the optimal importance distribution and any Gaussian distribution will be uniformly bounded. Thus using a non-linear Kalman filter based Gaussian importance distribution leads to a converging particle filter provided that we do not allow the Gaussian distribution to become singular.

5. CONCLUSION

In this paper, we extended the proof of Hu et al. (2008) [9] to the case of general importance distributions. Our proof shows the L^4 convergence of the particle filter estimates for a general class of unbounded functions provided that the importance weights are bounded. This also implies the probability-one convergence of the estimates [9]. We have analyzed the conditions set by the proof on importance distributions proposed in literature and tested them numerically.

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