Exercise Round 2

The deadline of this exercise round is **November 11, 2014**. The solutions will be gone through during the exercise session in room F255, F-building starting at 14:15.

The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1 (Usage of the Itô formula)

(a) Compute the Itô differential of

$$\phi(\beta) = t + \exp(\beta),$$

where $\beta(t)$ is a Brownian motion with diffusion constant q.

(b) Compute the Itô differential of

$$\phi(x) = x^2,$$

where x solves the scalar SDE

$$\mathrm{d}x = f(x)\,\mathrm{d}t + \sigma\,\mathrm{d}\beta,$$

σ is a constant, and $\beta(t)$ is a standard Brownian motion ($q = 1$).

(c) Compute the Itô differential of

$$\phi(\mathbf{x}) = \mathbf{x}^\mathsf{T}\,\mathbf{x},$$

where

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathbf{x}\,\mathrm{d}t + \mathrm{d}\boldsymbol{\beta},$$

where $\mathbf{F}$ is a constant matrix and the joint diffusion matrix of $\boldsymbol{\beta}$ is $\mathbf{Q}$.

Exercise 2 (Stochastic differential equations)

(a) Check that

$$x(t) = \exp(\beta(t))$$

solves the SDE

$$\mathrm{d}x = \frac{1}{2}\,x\,\mathrm{d}t + x\,\mathrm{d}\beta,$$

where $\beta(t)$ is a standard Brownian motion ($q = 1$).

(b) Solve the following SDE by changing the variable to $y = \ln x$:

$$\mathrm{d}x = -c\, x \; \mathrm{d}\beta,$$

where $c > 0$ is a constant, and $\beta(t)$ is a standard Brownian motion.

(c) Convert the following Stratonovich SDE equation into the equivalent Itô SDE:

$$\begin{aligned} \mathrm{d}x_1 &= -x_2 \circ \mathrm{d}\beta, \\ \mathrm{d}x_2 &= x_1 \circ \mathrm{d}\beta, \end{aligned}$$

where $\beta(t)$ is a scalar Brownian motion.

Exercise 3. (Mean and variance of differential equations)

Derive the mean and covariance equations for the scalar SDE

$$\mathrm{d}x = f(x) \; \mathrm{d}t + \sigma(x) \; \mathrm{d}\beta, \tag{1}$$

where β has the diffusion coefficient q, as follows:

(a) Conclude from the definition of the Itô integral that

$$\mathrm{E}\left[\int_u^v \sigma(x(t)) \; \mathrm{d}\beta(t)\right] = 0$$

for any u and v.

(b) Take expectations from both sides of the SDE (1) and formally divide by $\mathrm{d}t$ to get the differential equation for the mean $m(t)$.

(c) Apply the Itô formula to $\phi(x,t) = (x - m(t))^2$ and take the expectation of the resulting equation to derive the differential equation for the variance.

(d) Write down the mean and covariance differential equations for the scalar SDE

$$\mathrm{d}x = -\lambda\, x \; \mathrm{d}t + \mathrm{d}\beta,$$

where $\lambda > 0$ and solve them with $x(0) = x_0$.