Exercise Round 4 (22.11.2012).

Exercise 1. (1.5 order Itô–Taylor)

Simulate trajectories from the following SDE with 1.5 order strong Itô–Taylor series based method

$$\mathrm{d}x = \tanh(x)\ \mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{1}$$

where $\beta(t)$ is a standard Brownian motion, and compare the resulting histogram to the exact solution

$$p(x,t) = \frac{1}{\sqrt{2\pi\, t}} \cosh(x) \exp\left(-\frac{1}{2}\, t\right) \exp\left(-\frac{1}{2t}\, x^2\right).$$

Exercise 2. (Milstein's method)

Consider the following scalar SDE:

$$\begin{aligned} \mathrm{d}x &= -c\, x\ \mathrm{d}t + g\, x\ \mathrm{d}\beta \\ x(0) &= x_0 \end{aligned} \tag{2}$$

where a, g and x_0 are positive constants and $\beta(t)$ is a standard Brownian motion.

A) Check using the Itô formula that the solution to this equation is

$$x(t) = x_0 \exp\left[(-c - g^2/2)\, t + g\, \beta(t)\right] \tag{3}$$

Hint: $\phi(\beta(t), t) = x_0 \exp\left[(-c - g^2/2)\, t + g\, \beta(t)\right]$.

B) Simulate the equation using Milstein's method with parameters $x_0 = 1$, $c = 1/10$, $g = 1/10$, and check that the histogram at $t = 1$ looks the same as obtained by simulating the above exact solution.

Exercise 3. (Gaussian approximation of SDE)

A) Form a Gaussian assumed density approximation to the SDE in Equation (1) on times $t \in [0, 5]$ and compare it to the exact solution. Compute the Gaussian integrals numerically on a uniform grid.

B) Form Gaussian assumed density approximation to the Equation (2) and numerically compare it to the histogram obtained in 2 B).