# Lecture 6: Particle Filtering, Other Approximations, and Continuous-Time Models

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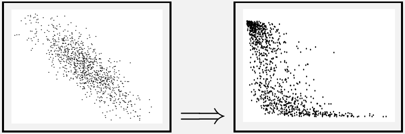
# Particle Filtering: Overview [1/3]

**Demo: Kalman vs. Particle Filtering**:

- Kalman filter animation
- Particle filter animation

# Particle Filtering: Overview [2/3]



- The idea is to form a weighted particle presentation $(\mathbf{x}^{(i)}, w^{(i)})$ of the posterior distribution:

$$p(\mathbf{x}) \approx \sum_i w^{(i)} \, \delta(\mathbf{x} - \mathbf{x}^{(i)}).$$

- Particle filtering = Sequential importance sampling, with additional resampling step.
- Bootstrap filter (also called Condensation) is the simplest particle filter.

# Particle Filtering: Overview [3/3]

- The efficiency of particle filter is determined by the selection of the importance distribution.
- The importance distribution can be formed by using e.g. EKF or UKF.
- Sometimes the optimal importance distribution can be used, and it minimizes the variance of the weights.
- Rao-Blackwellization: Some components of the model are marginalized in closed form $\Rightarrow$ hybrid particle/Kalman filter.

# Bootstrap Filter: Principle

- State density representation is set of samples $\{\mathbf{x}_k^{(i)} : i = 1, \ldots, N\}$.
- Bootstrap filter performs optimal filtering update and prediction steps using Monte Carlo.
- Prediction step performs prediction for each particle separately.
- Equivalent to integrating over the distribution of previous step (as in Kalman Filter).
- Update step is implemented with weighting and resampling.

# Bootstrap Filter: Algorithm

## Bootstrap Filter

1. Generate sample from predictive density of each old sample point $\mathbf{x}_{k-1}^{(i)}$:

$$\tilde{\mathbf{x}}_k^{(i)} \sim p(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)}).$$

2. Evaluate and normalize weights for each new sample point $\tilde{\mathbf{x}}_k^{(i)}$:

$$w_k^{(i)} = p(\mathbf{y}_k \mid \tilde{\mathbf{x}}_k^{(i)}).$$

3. Resample by selecting new samples $\mathbf{x}_k^{(i)}$ from set $\{\tilde{\mathbf{x}}_k^{(i)}\}$ with probabilities proportional to $w_k^{(i)}$.

# Sequential Importance Resampling: Principle

- State density representation is set of weighted samples $\{(\mathbf{x}_k^{(i)}, w_k^{(i)}) \;:\; i = 1, \ldots, N\}$ such that for arbitrary function $\mathbf{g}(\mathbf{x}_k)$, we have

$$\mathrm{E}[\mathbf{g}(\mathbf{x}_k) \mid \mathbf{y}_{1:k}] \approx \sum_i w_k^{(i)} \, \mathbf{g}(\mathbf{x}_k^{(i)}).$$

- On each step, we first draw samples from an importance distribution $\pi(\cdot)$, which is chosen suitably.
- The prediction and update steps are combined and consist of computing new weights from the old ones $w_{k-1}^{(i)} \to w_k^{(i)}$.
- If the "sample diversity" i.e the effective number of different samples is too low, do resampling.

# Sequential Importance Resampling: Algorithm

## Sequential Importance Resampling

1. Draw new point $\mathbf{x}_k^{(i)}$ for each point in the sample set $\{\mathbf{x}_{k-1}^{(i)}, i = 1, \ldots, N\}$ from the importance distribution:

$$\mathbf{x}_k^{(i)} \sim \pi(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_{1:k}), \qquad i = 1, \ldots, N.$$

2. Calculate new weights

$$w_k^{(i)} = w_{k-1}^{(i)} \frac{p(\mathbf{y}_k \mid \mathbf{x}_k^{(i)})\, p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_{1:k})}, \qquad i = 1, \ldots, N.$$

   and normalize them to sum to unity.

3. If the effective number of particles is too low, perform resampling.

# Effective Number of Particles and Resampling

The estimate for the effective number of particles can be computed as:

$$n_{\text{eff}} \approx \frac{1}{\sum_{i=1}^{N} \left(w_k^{(i)}\right)^2},$$

## Resampling

1. Interpret each weight $w_k^{(i)}$ as the probability of obtaining the sample index $i$ in the set $\{\mathbf{x}_k^{(i)} \mid i = 1, \ldots, N\}$.
2. Draw $N$ samples from that discrete distribution and replace the old sample set with this new one.
3. Set all weights to the constant value $w_k^{(i)} = 1/N$.

# Constructing the Importance Distribution

- Use the dynamic model as the importance distribution $\Rightarrow$ Bootstrap filter.
- Use the optimal importance distribution

$$\pi(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{y}_{1:k}) = p(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{y}_{1:k}).$$

- Approximate the optimal importance distribution by UKF $\Rightarrow$ unscented particle filter.
- Instead of UKF also EKF, SLF or any Gaussian filter can be, for example, used.
- Simulate availability of optimal importance distribution $\Rightarrow$ auxiliary SIR (ASIR) filter.

# Rao-Blackwellized Particle Filtering: Principle [1/2]

- Consider a conditionally Gaussian model of the form

$$
\begin{aligned}
\mathbf{s}_k &\sim p(\mathbf{s}_k \,|\, \mathbf{s}_{k-1}) \\
\mathbf{x}_k &= \mathbf{A}(\mathbf{s}_{k-1})\,\mathbf{x}_{k-1} + \mathbf{q}_k, \qquad \mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}) \\
\mathbf{y}_k &= \mathbf{H}(\mathbf{s}_k)\,\mathbf{x}_k + \mathbf{r}_k, \qquad \mathbf{r}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{R})
\end{aligned}
$$

- The model is of the form

$$
\begin{aligned}
p(\mathbf{x}_k, \mathbf{s}_k | \mathbf{x}_{k-1}, \mathbf{s}_{k-1}) &= \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{A}(\mathbf{s}_{k-1})\mathbf{x}_{k-1}, \mathbf{Q})\, p(\mathbf{s}_k \,|\, \mathbf{s}_{k-1}) \\
p(\mathbf{y}_k \,|\, \mathbf{x}_k, \mathbf{s}_k) &= \mathrm{N}(\mathbf{y}_k \,|\, \mathbf{H}(\mathbf{s}_k), \mathbf{R})
\end{aligned}
$$

- The full model is non-linear and non-Gaussian in general.
- But given the values $\mathbf{s}_k$ the model is Gaussian and thus Kalman filter equations can be used.

# Rao-Blackwellized Particle Filtering: Principle [1/2]

- The idea of the Rao-Blackwellized particle filter:
  - Use Monte Carlo sampling to the values $\mathbf{s}_k$
  - Given these values, compute distribution of $\mathbf{x}_k$ with Kalman filter equations.
  - Result is a Mixture Gaussian distribution, where each particle consist of value $\mathbf{s}_k^{(i)}$, associated weight $w_k^{(i)}$ and the mean and covariance conditional to the history $\mathbf{s}_{1:k}^{(i)}$
- The explicit RBPF equations can be found in the lecture notes.
- If the model is almost conditionally Gaussian, it is also possible to use e.g. EKF, SLF or UKF instead of the linear KF.

# Particle Filter: Advantages

- No restrictions in model – can be applied to non-Gaussian models, hierarchical models etc.
- Global approximation.
- Approaches the exact solution, when the number of samples goes to infinity.
- In its basic form, very easy to implement.
- Superset of other filtering methods – Kalman filter is a Rao-Blackwellized particle filter with one particle.

# Particle Filter: Disadvantages

- Computational requirements much higher than of the Kalman filters.
- Problems with nearly noise-free models, especially with accurate dynamic models.
- Good importance distributions and efficient Rao-Blackwellized filters quite tricky to implement.
- Very hard to find programming errors (i.e., to debug).

# Multiple Model Kalman Filtering

- Algorithm for estimating true mode(l) or its parameter from a finite set of alternatives.
- Assume that we are given $N$ possible models/modes, and one of them is true.
- If $s$ is the model or mode index, the problem can be written in form:

$$
\begin{aligned}
P(s = i) &= \pi_0^i \\
\mathbf{x}_k &= \mathbf{A}(s)\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1} \\
\mathbf{y}_k &= \mathbf{H}(s)\,\mathbf{x}_k + \mathbf{r}_k,
\end{aligned}
$$

  where $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}(s))$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{R}(s))$.
- Can be solved in closed form with $s$ parallel Kalman filters.

# Switching Dynamic Linear Models

- Assume that we have $N$ possible models, but the true model is assumed to change in time.
- If the model index $s_k$ is modeled as Markov chain, we have:

$$
\begin{aligned}
P(s_0 = i) &= \pi_0^i \\
P(s_k = i \mid s_{k-1} = j) &= \Pi_{ij}.
\end{aligned}
$$

- Given the model/mode $s_k$ we have linear Gaussian model:

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{A}(s_k)\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1} \\
\mathbf{y}_k &= \mathbf{H}(s_k)\,\mathbf{x}_k + \mathbf{r}_k,
\end{aligned}
$$

- Closed form solution would require running Kalman filters for each possible history $s_{1:k} \Rightarrow N^k$ filters, not feasible.

# Switching Dynamic Linear Models (cont.)

- Retain huge number of hypotheses and prune ones with lowest probabilities $\Rightarrow$ multiple hypothesis tracking (MHT).
- Use a Rao-Blackwellized particle filter (RBPF) or plain particle filter.
- Classical alternatives:
  - 1st order Generalized pseudo-Bayesian (GPB1) filter uses single Gaussian and one-step integration over modes.
  - 2nd order Generalized pseudo-Bayesian (GPB2) filter uses sum (mixture) of $N$ Gaussians and two-step integration.
  - Interacting multiple models (IMM) filter uses sum of $N$ Gaussians, and mixing of Gaussians in prediction and normal multiple model update.

# Variational Kalman Smoother

- Variation Bayesian analysis based framework for estimating the parameters of linear state space models.
- Idea: Fix $\mathbf{Q} = \mathbf{I}$ and assume that the joint distribution of states $\mathbf{x}_1, \ldots, \mathbf{x}_T$ and parameters $\mathbf{A}, \mathbf{H}, \mathbf{R}$ is approximately separable:

$$
\begin{aligned}
& p(\mathbf{x}_1, \ldots, \mathbf{x}_T, \mathbf{A}, \mathbf{H}, \mathbf{R} \mid \mathbf{y}_1, \ldots, \mathbf{y}_T) \\
& \quad \approx p(\mathbf{x}_1, \ldots, \mathbf{x}_T \mid \mathbf{y}_1, \ldots, \mathbf{y}_T) \, p(\mathbf{A}, \mathbf{H}, \mathbf{R} \mid \mathbf{y}_1, \ldots, \mathbf{y}_T).
\end{aligned}
$$

- The resulting EM-algorithm consist of alternating steps of smoothing with fixed parameters and estimation of new parameter values.
- The general equations of the algorithm are quite complicated and assume that all the model parameters are to be estimated.

# Recursive Variational Bayesian Estimation of Noise Variances

- Algorithm for estimating unknown time-varying measurement variances.
- Assume that the joint filtering distribution of state and measurement noise variance is approximately separable:

$$p(\mathbf{x}_k, \sigma_k^2 \mid y_1, \ldots, y_k) \approx p(\mathbf{x}_k \mid y_1, \ldots, y_k)\, p(\sigma_k^2 \mid y_1, \ldots, y_k)$$

- Variational Bayesian analysis leads to algorithm, where the natural representation is

$$\begin{aligned} p(\sigma_k^2 \mid y_1, \ldots, y_k) &= \mathrm{InvGamma}(\sigma_k^2 \mid \alpha_k, \beta_k) \\ p(\mathbf{x}_k \mid y_1, \ldots, y_k) &= \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k). \end{aligned}$$

- The update step consists of a fixed-point iteration for computing new $\alpha_k, \beta_k, \mathbf{m}_k, \mathbf{P}_k$ from the old ones.

# Outlier Rejection and Multiple Target Tracking

- Outlier Rejection / Clutter Modeling:
  - Probabilistic Data Association (PDA)
  - Monte Carlo Data Association (MCDA)
  - Multiple hypothesis tracking (MHT)
- Multiple Target Tracking
  - Multiple hypothesis tracking (MHT)
  - Joint Probabilistic Data Association (JPDA)
  - Rao-Blackwellized Particle Filtering (RBMCDA) for Multiple Target Tracking

# Continuous-Discrete Pendulum Model

- Consider the pendulum model, which was first stated as

$$
\begin{aligned}
d^2\theta/dt^2 &= -g\,\sin(\theta) + w(t) \\
y_k &= \sin(\theta(t_k)) + r_k,
\end{aligned}
$$

  where $w(t)$ is "Gaussian white noise" and $r_k \sim \mathrm{N}(0, \sigma^2)$.
- With state $\mathbf{x} = (\theta, d\theta/dt)$, the model is of the abstract form

$$
\begin{aligned}
d\mathbf{x}/dt &= \mathbf{f}(\mathbf{x}) + \mathbf{w}(t) \\
\mathbf{y}_k &= \mathbf{h}(\mathbf{x}(t_k)) + \mathbf{r}_k
\end{aligned}
$$

  where $\mathbf{w}(t)$ has the covariance (spectral density) $\mathbf{Q}_c$.
- Continuous-time dynamics + discrete-time measurement = Continuous-discrete (-time) filtering model.

# Discretization of Continuous Dynamics [1/4]

- Previously we assumed that the measurements are obtained at times $t_k = 0, \Delta t, 2\Delta t, \ldots$
- The state space model was then Euler-discretized as

$$\mathbf{x}_k = \mathbf{x}_{k-1} + \mathbf{f}(\mathbf{x}_{k-1})\,\Delta t + \mathbf{q}_{k-1}$$
$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k$$

- But what should be the variance of $\mathbf{q}_k$?
- Consistency: The same variance for single step of length $\Delta t$, and 2 steps of length $\Delta t/2$:

$$\mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}_c\,\Delta t)$$

# Discretization of Continuous Dynamics [2/4]

- Now the Extended Kalman fiter (EKF) for this model is
  - Prediction:

$$
\begin{aligned}
\mathbf{m}_k^- &= \mathbf{m}_{k-1} + \mathbf{f}(\mathbf{m}_{k-1})\,\Delta t \\
\mathbf{P}_k^- &= (\mathbf{I} + \mathbf{F}\,\Delta t)\,\mathbf{P}_{k-1}\,(\mathbf{I} + \mathbf{F}\,\Delta t)^T + \mathbf{Q}_c\,\Delta t \\
&= \mathbf{P}_{k-1} + \mathbf{F}\,\mathbf{P}_{k-1}\,\Delta t + \mathbf{P}_{k-1}\,\mathbf{F}^T\,\Delta t \\
&\quad + \mathbf{F}\,\mathbf{P}_{k-1}\,\mathbf{F}^T\,\Delta t^2 + \mathbf{Q}_c\,\Delta t
\end{aligned}
$$

  - Update:

$$
\begin{aligned}
\mathbf{S}_k &= \mathbf{H}(\mathbf{m}_k^-)\,\mathbf{P}_k^-\,\mathbf{H}^T(\mathbf{m}_k^-) + \mathbf{R} \\
\mathbf{K}_k &= \mathbf{P}_k^-\,\mathbf{H}^T(\mathbf{m}_k^-)\,\mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\,[\mathbf{y}_k - \mathbf{h}(\mathbf{m}_k^-)] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\,\mathbf{S}_k\,\mathbf{K}_k^T
\end{aligned}
$$

# Discretization of Continuous Dynamics [3/4]

- But what happens if $\Delta t$ is not "small", that is, if we get measurements quite rarely?
  - We can use more Euler steps between measurements.
  - We can perform the EKF prediction on each step.
  - We can even compute the limit of infinite number of steps.
- If we let $\delta t = \Delta t / n$, the prediction becomes:

$$
\begin{aligned}
&\hat{\mathbf{m}}_0 = \mathbf{m}_{k-1}; \quad \hat{\mathbf{P}}_0 = \mathbf{P}_{k-1} \\
&\text{for } i = 1 \ldots n \\
&\qquad \hat{\mathbf{m}}_i = \hat{\mathbf{m}}_{i-1} + \mathbf{f}(\hat{\mathbf{m}}_{i-1})\, \delta t \\
&\qquad \hat{\mathbf{P}}_i = \hat{\mathbf{P}}_{i-1} + \mathbf{F}\, \hat{\mathbf{P}}_{i-1}\, \delta t + \hat{\mathbf{P}}_{i-1}\, \mathbf{F}^T\, \delta t \\
&\qquad\qquad + \mathbf{F}\, \hat{\mathbf{P}}_{i-1}\, \mathbf{F}^T\, \delta t^2 + \mathbf{Q}_c\, \delta t \\
&\text{end} \\
&\mathbf{m}_k^- = \hat{\mathbf{m}}_n; \quad \mathbf{P}_k^- = \hat{\mathbf{P}}_n.
\end{aligned}
$$

# Discretization of Continuous Dynamics [4/4]

- By re-arranging the equations in the for-loop, we get

$$
\begin{aligned}
(\hat{\mathbf{m}}_i - \hat{\mathbf{m}}_{i-1})/\delta t &= \mathbf{f}(\hat{\mathbf{m}}_{i-1}) \\
(\hat{\mathbf{P}}_i - \hat{\mathbf{P}}_{i-1})/\delta t &= \mathbf{F}\,\hat{\mathbf{P}}_{i-1} + \hat{\mathbf{P}}_{i-1}\,\mathbf{F}^T + \mathbf{F}\,\hat{\mathbf{P}}_{i-1}\,\mathbf{F}^T\,\delta t + \mathbf{Q}_c
\end{aligned}
$$

- In the limit $\delta t \to 0$, we get the differential equations

$$
\begin{aligned}
d\hat{\mathbf{m}}/dt &= \mathbf{f}(\hat{\mathbf{m}}(t)) \\
d\hat{\mathbf{P}}/dt &= \mathbf{F}(\hat{\mathbf{m}}(t))\,\hat{\mathbf{P}}(t) + \hat{\mathbf{P}}(t)\,\mathbf{F}^T(\hat{\mathbf{m}}(t)) + \mathbf{Q}_c
\end{aligned}
$$

- The initial conditions are

$$
\begin{aligned}
\hat{\mathbf{m}}(0) &= \mathbf{m}_{k-1} \\
\hat{\mathbf{P}}(0) &= \mathbf{P}_{k-1}
\end{aligned}
$$

- The final prediction is

$$
\begin{aligned}
\mathbf{m}_k^- &= \hat{\mathbf{m}}(\Delta t) \\
\mathbf{P}_k^- &= \hat{\mathbf{P}}(\Delta t)
\end{aligned}
$$

## Continuous-Discrete EKF

- Prediction: between the measurements integrate the following differential equations from $t_{k-1}$ to $t_k$:

$$
\begin{aligned}
d\mathbf{m}/dt &= \mathbf{f}(\mathbf{m}(t)) \\
d\mathbf{P}/dt &= \mathbf{F}(\mathbf{m}(t))\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^T(\mathbf{m}(t)) + \mathbf{Q}_c
\end{aligned}
$$

- Update: at measurements do the EKF update

$$
\begin{aligned}
\mathbf{S}_k &= \mathbf{H}(\mathbf{m}_k^-)\,\mathbf{P}_k^-\,\mathbf{H}^T(\mathbf{m}_k^-) + \mathbf{R} \\
\mathbf{K}_k &= \mathbf{P}_k^-\,\mathbf{H}^T(\mathbf{m}_k^-)\,\mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\,[\mathbf{y}_k - \mathbf{h}(\mathbf{m}_k^-)] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\,\mathbf{S}_k\,\mathbf{K}_k^T,
\end{aligned}
$$

where $\mathbf{m}_k^-$ and $\mathbf{P}_k^-$ are the results of the prediction step.

# Continuous-Discrete SLF, UKF, PF etc.

- The equations

$$\begin{aligned} d\mathbf{m}/dt &= \mathbf{f}(\mathbf{m}(t)) \\ d\mathbf{P}/dt &= \mathbf{F}(\mathbf{m}(t))\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^T(\mathbf{m}(t)) + \mathbf{Q}_c \end{aligned}$$

  actually generate a Gaussian process approximation $\mathbf{x}(t) \sim \mathrm{N}(\mathbf{m}(t), \mathbf{P}(t))$ to the solution of non-linear stochastic differential equation (SDE)

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}) + \mathbf{w}(t)$$

- We could also use statistical linearization or unscented transform and get a bit different limiting differential equations.
- Also possible to generate particle approximations by a continuous-time version of importance sampling (based on Girsanov theorem).

# More general SDE Theory

- The most general SDE model usually considered is of the form

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}) + \mathbf{L}(\mathbf{x})\,\mathbf{w}(t)$$

- Formally, $\mathbf{w}(t)$ is a Gaussian white noise process with zero mean and covariance function

$$\mathrm{E}[\mathbf{w}(t)\,\mathbf{w}^T(t')] = \mathbf{Q}_c\,\delta(t' - t)$$

- The distribution $p(\mathbf{x}(t))$ is non-Gaussian and it is given by the following partial differential equation:

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left(f_i(\mathbf{x})\,p\right) + \frac{1}{2}\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\left([\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^T]_{ij}\,p\right)$$

- Known as Fokker-Planck equation or Kolmogorov forward equation.

# More general SDE Theory (cont.)

- In more rigorous theory, we actually must interpret the SDE as integral equation

$$\mathbf{x}(t) - \mathbf{x}(s) = \int_s^t \mathbf{f}(\mathbf{x})\, dt + \int_s^t \mathbf{L}(\mathbf{x})\, \mathbf{w}(t)\, dt$$

- In Ito's theory of SDE's the second integral is defined as stochastic integral w.r.t. Brownian motion $\beta(t)$:

$$\mathbf{x}(t) - \mathbf{x}(s) = \int_s^t \mathbf{f}(\mathbf{x})\, dt + \int_s^t \mathbf{L}(\mathbf{x})\, d\beta(t)$$

  i.e., formally $\mathbf{w}(t)\, dt = d\beta(t)$ or $\mathbf{w}(t) = d\beta(t)/dt$
- However, Brownian motion is nowhere differentiable!
- Brownian motion is also called as Wiener process.

# More general SDE Theory (cont. II)

- In stochastics, the integral equation is often written as

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x})\,dt + \mathbf{L}(\mathbf{x})\,d\beta(t)$$

- In engineering (control theory, physics) it is customary to formally divide with $dt$ to get

$$d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x}) + \mathbf{L}(\mathbf{x})\,\mathbf{w}(t)$$

- So called Stratonovich's theory is more consistent with this white noise interpretation than Ito's theory.
- In mathematical sense Stratonovich's theory defines the stochastic integral $\int_s^t \mathbf{L}(\mathbf{x})\,d\beta(t)$ a bit differently – also the Fokker-Planck equation is different.

# Cautions About White Noise

- White noise is actually only formally defined as derivative of Brownian motion.
- White noise can only be defined in distributional sense – for this reason non-linear functions of it $g(\mathbf{w}(t))$ are not well-defined.
- For this reason, the following more general type of SDE does not make sense:

$$d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x}, \mathbf{w})$$

- We must also be careful in interpreting the multiplicative term in the equation

$$d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x}) + \mathbf{L}(\mathbf{x})\,\mathbf{w}(t)$$

# Formal Optimal Continuous-Discrete Filter

## Optimal continuous-discrete filter

1. **Prediction step:** Solve the Kolmogorov-forward (Fokker-Planck) partial differential equation.

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial}{\partial x_i} (f_i(\mathbf{x})\, p) + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( [\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^T]_{ij}\, p \right)$$

2. **Update step:** Apply the Bayes' rule.

$$p(\mathbf{x}(t_k) \mid \mathbf{y}_{1:k}) = \frac{p(\mathbf{y}_k \mid \mathbf{x}(t_k))\, p(\mathbf{x}(t_k) \mid \mathbf{y}_{1:k-1})}{\int p(\mathbf{y}_k \mid \mathbf{x}(t_k))\, p(\mathbf{x}(t_k) \mid \mathbf{y}_{1:k-1})\, \mathrm{d}\mathbf{x}(t_k)}$$

# General Continuous-Time Filtering

- We could also model the measurements as a continuous-time process:

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}) + \mathbf{L}(\mathbf{x})\,\mathbf{w}(t)$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x}) + \mathbf{n}(t)$$

- Again, one must be very careful in interpreting the white noise processes $\mathbf{w}(t)$ and $\mathbf{n}(t)$.
- The filtering equations become a stochastic partial differential equation (SPDE) called Kushner-Stratonovich equation.
- The equation for the unnormalized filtering density is called the Zakai equation, which also is a SPDE.
- It is also possible to take the continuous-time limit of the Bayesian smoothing equations (result is a PDE).

# Kalman-Bucy Filter

- If the system is linear

$$
\begin{aligned}
d\mathbf{x}/dt &= \mathbf{F}\,\mathbf{x} + \mathbf{w}(t) \\
\mathbf{y} &= \mathbf{H}\,\mathbf{x} + \mathbf{n}(t)
\end{aligned}
$$

  we get the continuous-time Kalman-Bucy filter:

$$
\begin{aligned}
d\mathbf{m}/dt &= \mathbf{F}\,\mathbf{m} + \mathbf{K}\,(\mathbf{y} - \mathbf{H}\,\mathbf{m}) \\
d\mathbf{P}/dt &= \mathbf{F}\,\mathbf{P} + \mathbf{P}\,\mathbf{F}^T + \mathbf{Q}_c - \mathbf{K}\,\mathbf{R}\,\mathbf{K}^T,
\end{aligned}
$$

  where $\mathbf{K} = \mathbf{P}\,\mathbf{H}^T\,\mathbf{R}^{-1}$.
- The stationary solution to these equations is equivalent to the continuous-time Wiener filter.
- Non-linear extensions (EKF, SLF, UKF, etc.) can be obtained similarly to the discrete-time case.

- Let's return to linear stochastic differential equations:

$$d\mathbf{x}/dt = \mathbf{F}\,\mathbf{x} + \mathbf{w}$$

- Assume that $\mathbf{F}$ is time-independent. For example, in car-tracking model we had a model of this type.
- Given $\mathbf{x}(0)$ we can now actually solve the equation

$$\mathbf{x}(t) = \exp(t\,\mathbf{F})\,\mathbf{x}(0) + \int_0^t \exp((t-s)\,\mathbf{F})\,\mathbf{w}(s)\,ds,$$

where exp(.) is the matrix exponential function:

$$\exp(t\mathbf{F}) = \mathbf{I} + t\mathbf{F} + \frac{1}{2!}t^2\mathbf{F}^2 + \frac{1}{3!}t^3\mathbf{F}^3 + \dots$$

- Note that we are treating $\mathbf{w}(s)$ as an ordinary function, which is not generally justified!

## Solution of LTI SDE (cont.)

- We can also solve the equation on predefined time points $t_1, t_2, \ldots$ as follows:

$$\mathbf{x}(t_k) = \exp((t_k - t_{k-1})\,\mathbf{F})\,\mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \exp((t_k - s)\,\mathbf{F})\,\mathbf{w}(s) ds$$

- The first term is of the form $\mathbf{A}\,\mathbf{x}(t_{k-1})$, where the matrix is a known constant $\mathbf{A} = \exp(\Delta t\,\mathbf{F})$.
- The second term is a zero mean Gaussian random variable and its covariance can be calculated as:

$$\begin{aligned}\mathbf{Q} &= \int_{t_{k-1}}^{t_k} \exp((t_k - s)\,\mathbf{F})\,\mathbf{Q}_c \exp((t_k - s)\,\mathbf{F})^T ds \\ &= \int_{0}^{\Delta t} \exp((\Delta t - s)\,\mathbf{F})\,\mathbf{Q}_c \exp((\Delta t - s)\,\mathbf{F})^T ds\end{aligned}$$

# Solution of LTI SDE (cont. II)

- Thus the continuous-time system is in a sense equivalent to the discrete-time system

$$\mathbf{x}(t_k) = \mathbf{A}\,\mathbf{x}(t_{k-1}) + \mathbf{q}_k$$

where $\mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ and

$$\begin{aligned}\mathbf{A} &= \exp(\Delta t\, A) \\ \mathbf{Q} &= \int_0^{\Delta t} exp((\Delta t - s)\,\mathbf{F})\,\mathbf{Q}_c \exp((\Delta t - s)\,\mathbf{F})^T ds\end{aligned}$$

- An analogous equivalent discretization is also possible with time-varying linear stochastic differential equation models.
- A continuous-discrete Kalman filter can be always implemented as a discrete-time Kalman filter by forming the equivalent discrete-time system.

# Wiener Velocity Model

- For example, consider the Wiener velocity model (= white noise acceleration model):

$$d^2x/dt^2 = w(t),$$

which is equivalent to the state space model

$$d\mathbf{x}/dt = \mathbf{F}\,\mathbf{x} + \mathbf{w}$$

with $\mathbf{F} = (0\ 1;\ 0\ 0), \mathbf{x} = (x, dx/dt), \mathbf{Q}_c = (0\ 0;\ 0\ q)$.

- Then we have

$$\mathbf{A} = \exp(\Delta t\,\mathbf{F}) = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{Q} &= \int_0^{\Delta t} \exp((\Delta t - s)\,\mathbf{F})\,\mathbf{Q}_c\,\exp((\Delta t - s)\,\mathbf{F})^T ds \\
&= \begin{pmatrix} \Delta t^3/3\,q & \Delta t^2/2\,q \\ \Delta t^2/2\,q & \Delta t\,q \end{pmatrix}
\end{aligned}$$

which might look familiar.

# Mean and Covariance Differential Equations

- Note that in the linear (time-invariant) case

$$d\mathbf{x}/dt = \mathbf{F}\,\mathbf{x} + \mathbf{w}$$

we could also write down the differential equations

$$\begin{aligned} d\mathbf{m}/dt &= \mathbf{F}\,\mathbf{m} \\ d\mathbf{P}/dt &= \mathbf{F}\,\mathbf{P} + \mathbf{P}\,\mathbf{F}^T + \mathbf{Q}_c \end{aligned}$$

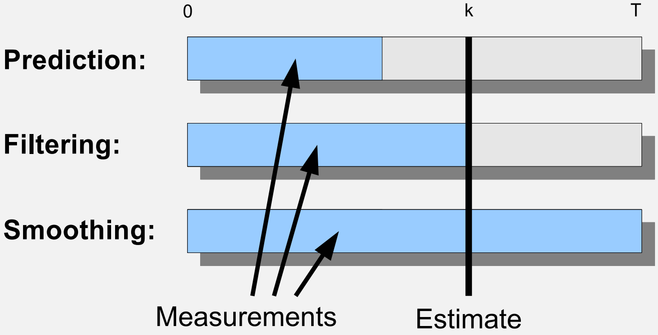
which exactly give the evolution of mean and covariance.

- The solutions of these equations are

$$\begin{aligned} \mathbf{m}(t) &= \exp(t\,\mathbf{F})\,\mathbf{m}_0 \\ \mathbf{P}(t) &= \exp(t\,\mathbf{F})\,\mathbf{P}_0\,\exp(t\,\mathbf{F})^T \\ &\quad + \int_0^t \exp((t-s)\,\mathbf{F})\,\mathbf{Q}_c\,\exp((t-s)\,\mathbf{F})^T\,ds, \end{aligned}$$

which are consistent with the previous results.

# Optimal Smoothing . . .




. . . the topic of next week.

# Optimal Control Theory

- Assume that the physical system can be modeled with differential equation with input **u**

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

- Determine $\mathbf{u}(t)$ such that $\mathbf{x}(t)$ and $\mathbf{u}(t)$ satisfy certain constraints and minimize a cost functional.
- For example, steer a space craft to moon such that the consumed of fuel is minimized.
- If the system is linear and cost function quadratic, we get linear quadratic controller (or regulator).

# Stochastic (Optimal) Control Theory

- Assume that the system model is stochastic:

$$
\begin{aligned}
d\mathbf{x}/dt &= \mathbf{f}(\mathbf{x}, \mathbf{u}) + \mathbf{w}(t) \\
\mathbf{y}_k &= \mathbf{h}(\mathbf{x}(t_k)) + \mathbf{r}_k
\end{aligned}
$$

- Given only the measurements $\mathbf{y}_k$, find $\mathbf{u}(t)$ such that $\mathbf{x}(t)$ and $\mathbf{u}(t)$ satisfy the constraints and minimize a cost function.
- If linear Gaussian, we have
  Linear Quadratic (LQ) controller + Kalman filter = Linear Quadratic Gaussian (LQG) controller
- In general, not simply a combination of optimal filter and deterministic optimal controller.
- Model Predictive Control (MPC) is a well-known approximation algorithm for constrained problems.

# Spatially Distributed Systems

- Infinite dimensional generalization of state space model is the stochastic partial differential equation (SPDE)

$$\frac{\partial \mathbf{x}(t, \mathbf{r})}{\partial t} = \mathscr{F}_r \, \mathbf{x}(t, \mathbf{r}) + \mathscr{L}_r \, \mathbf{w}(t, \mathbf{r}),$$

  where $\mathscr{F}_r$ and $\mathscr{L}_r$ are linear operators (e.g. integro-differential operators) in $\mathbf{r}$-variable and $\mathbf{w}(\cdot)$ is a time-space white noise.

- Practically every SPDE can be converted into this form with respect to any variable (which is relabeled as t).
- For example, stochastic heat equation

$$\frac{\partial x(t, r)}{\partial t} = \frac{\partial^2 x(t, r)}{\partial r^2} + w(t, r).$$

# Spatially Distributed Systems (cont.)

- The solution to the SPDE is analogous to finite-dimensional case:

$$\mathbf{x}(t,\mathbf{r}) = \mathscr{U}_r(t)\,\mathbf{x}(0,\mathbf{r}) + \int_0^t \mathscr{U}_r(t-s)\,\mathscr{L}_r\,\mathbf{w}(s,\mathbf{r})\,ds.$$

- $\mathscr{U}_r(t) = \exp(t\,\mathscr{F}_r)$ is the evolution operator – corresponds to propagator in quantum mechanics.
- Spatio-temporal Gaussian process models can be naturally formulated as linear SPDE's.
- Recursive Bayesian estimation with SPDE models lead to infinite-dimensional Kalman filters and RTS smoothers.
- SPDE's can be approximated with finite models by usage of finite-differences or finite-element methods.

# Summary

- Particle filters use weighted set of samples (particles) for approximating the filtering distributions.
- Sequential importance resampling (SIR) is the general framework and bootstrap filter is a simple special case of it.
- In Rao-Blackwellized particle filters a part of the state is sampled and part is integrated in closed form with Kalman filter.
- Other filtering algorithms than EKF, SLF, UKF and PF are, for example, multiple model Kalman filters and IMM algorithm.
- Specialized filtering algorithms exist also, e.g., for parameter estimation, outlier rejection and multiple target tracking.

# Summary (cont.)

- In continuous-discrete filtering, the dynamic model is a continuous-time process and measurement are obtained at discrete times.
- In continuous-discrete EKF, SLF and UKF the continuous-time non-linear dynamic model is approximated as a Gaussian process.
- In continuous-time filtering, the both the dynamic and measurements models are continuous-time processes.
- The theories of continuous and continuous-discrete filtering are tied to the theory of stochastic differential equations.