Exercise Round 3.

Exercise 1. (Kalman Filter with Non-Zero Mean Noises)

Derive the Kalman filter equations for the following linear-Gaussian filtering model with non-zero-mean noises:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1} \\ \mathbf{y}_k &= \mathbf{H}\,\mathbf{x}_k + \mathbf{r}_k, \end{aligned} \tag{1}$$

where $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{m}_q, \mathbf{Q})$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{m}_r, \mathbf{R})$.

Exercise 2. (Filtering of Gaussian Random Walk)

A) Implement the Kalman filter for the Gaussian random walk model (without EKF/UKF toolbox) with Matlab. Draw realizations of state and measurement sequences and apply the filter to it. Plot the evolution of the filtering distribution.

B) Select a finite interval in the state space, say, $x \in [-10, 10]$ and discretize it evenly to N subintervals (e.g. $N = 1000$). Using a suitable numerical approximation to the integrals in the Baysian filtering equations, implement a finite grid approximation the optimal filter for the Gaussian random walk. Verify that the result is practically the same as of the Kalman filter above.

C) Derive the stationary Kalman filter for random walk model, that is, compute the limiting Kalman filter gain when $k \to \infty$. Plot the frequency response of the resulting time-invariant filter. Which type of digital filter is it?

Exercise 3. (Kalman Filter for Noisy Resonator)

Consider the following dynamic model:

$$\mathbf{x}_k = \begin{pmatrix} \cos\omega & \frac{\sin(\omega)}{\omega} \\ -\omega\,\sin(\omega) & \cos(\omega) \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$
$$y_k = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + v_k$$

where $\mathbf{x}_k \in \mathbb{R}^2$ is the state, y_k is the measurement, $v_k \sim N(0, 0.1)$ is a white Gaussian measurement noise, and $\mathbf{q}_k \sim N(\mathbf{0}, \mathbf{Q})$, where

$$\mathbf{Q} = \begin{pmatrix} \frac{q_c\,\omega - q_c\,\cos(\omega)\,\sin(\omega)}{2\omega^3} & \frac{q_c\,\sin^2(\omega)}{2\omega^2} \\ \frac{q_c\,\sin^2(\omega)}{2\omega^2} & \frac{q_c\,\omega + q_c\,\cos(\omega)\,\sin(\omega)}{2\omega} \end{pmatrix} \tag{2}$$

The angular velocity is $\omega = 1/2$ and the spectral density is $q_c = 0.01$. The model is a dicretized version of noisy resonator model with angular velocity ω.

In the file `kf_ex.m` there is simulation of the dynamic model together with a base line solution, where the measurement is directly used as the estimate of the state component x_1 and the second component x_2 is computed as a weighted average of the measurement differences.

A) Implement the Kalman filter for the model and compare its performance (in RMSE sense) to the base line solution. Plot figures of the solutions and report the RMSE values of for both the methods.

B) Compute (numerically) the stationary Kalman filter corresponding to the model. Test this stationary filter against the base line and Kalman filter solutions. Plot the results and report the RMSE values for the solutions. What is the practical difference in the stationary and non-stationary Kalman filter solutions?