High-density approximations for a spatial Poisson noise with long-range dependence

Lasse Leskelä
Helsinki University of Technology

Joint work with I. Kaj, I. Norros, and V. Schmidt

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Outline

1. The model

2. Limit theorems

3. Properties of the limits

Earlier work: *Mikosch, Resnick, Rootzén, and Stegeman (2002); Kaj and Taqqu (2004)*

Poisson grain model

Random subsets of $\mathbb{R}^d$ called *grains*: $x_j + (\rho v_j)^{1/d} C$

- $C \subset \mathbb{R}^d$ such that $|C| = 1$, C is compact and *starlike*:

$$rC \subset C, \quad r \in [0,1]$$

- x_j points of a PRM on $\mathbb{R}^d$ with intensity λ

- Independent grain volumes ρV with $\mathsf{E}\, V = 1$ and $\mathsf{P}(V \in \cdot) = F(\cdot)$

Poisson noise field

$J_{\lambda,\rho}(y) = \#$ grains containing y

$$J_{\lambda,\rho} = \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} 1(y \in x + (\rho v)^{1/d} C)\, N_\lambda(dx, dv),$$

where $N_{\lambda,\rho}(dx, dv)$ is a PRM on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity $\lambda dx\, F(dv)$

Problem: describe $J_{\lambda,\rho}$ as the *mean grain density* $\lambda \to \infty$ and the *mean grain volume* $\rho \to 0$

Note: in 1D, $J_{\lambda,\rho}(y)$ is known as the $M/G/\infty$ process with arrival intensity λ and mean call duration ρ

Poisson noise as a random linear functional

Define for $\phi \in L^1$

$$J_{\lambda,\rho}(\phi) = \int_{\mathbb{R}^d} J_{\lambda,\rho}(y)\,\phi(y)\,dy$$

Then

$$J_{\lambda,\rho}(\phi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \rho v\, m_\phi(x, \rho v)\, N_\lambda(dx, dv),$$

where m_ϕ are the averages

$$m_\phi(x, v) = v^{-1} \int_{x+v^{1/d}C} \phi(y)\,dy$$

Scaling limit for volumes with light tails

Theorem 1. *Assume* $\mathsf{E}\, V^2 < \infty$. *Then, as* $\lambda \to \infty$ *and* $\rho \to 0$,

$$\frac{J_{\lambda,\rho}(\phi) - \mathsf{E}\, J_{\lambda,\rho}(\phi)}{\rho(\lambda\, \mathsf{E}\, V^2)^{1/2}} \xrightarrow{d} W(\phi), \quad \phi \in L^1 \cap L^2,$$

where W *is the Gaussian random linear functional with* $\mathsf{E}\, W(\phi) = 0$ *and*

$$\mathsf{E}\, W(\phi) W(\psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x)\psi(y)\, dx\, dy.$$

W can also be defined as a Gaussian random measure with Lebesgue control measure

Volumes with heavy tails

Assume that for a slowly varying $L(v)$ and $\gamma \in (1, 2)$,

$$P(V > v) = v^{-\gamma} L(v)$$

Then E(# grains covering the origin with volume > 1) equals

$$\lambda\rho \int_{1/\rho}^{\infty} vF(dv) \sim \frac{\lambda\rho^{\gamma} L(1/\rho)}{1 - \gamma^{-1}}, \quad \rho \to 0$$

Three regimes:

Small-grain scaling	$\lambda\,\rho^{\gamma} L(1/\rho) \to 0$
Intermediate scaling	$\lambda\,\rho^{\gamma} L(1/\rho) \to \gamma^{-1}$
Large-grain scaling	$\lambda\,\rho^{\gamma} L(1/\rho) \to \infty$

Limit for heavy tails, small-grain scaling

Theorem 2. *Assume that* $P(V > v) = L(v)v^{-\gamma}$ *with* $\gamma \in (1,2)$ *and* $\lambda\rho^{\gamma}L(1/\rho) \to 0$*. Then as* $\lambda \to \infty$ *and* $\rho \to 0$*,*

$$\frac{J_{\lambda,\rho}(\phi) - \mathsf{E}\, J_{\lambda,\rho}(\phi)}{\rho(1/(1-F))^{\leftarrow}(\gamma\lambda)} \xrightarrow{d} \Lambda_\gamma(\phi), \quad \phi \in L^1 \cap L^2,$$

where Λ_γ *is the independently scattered* γ*-stable random measure on* $\mathbb{R}^d$ *with Lebesgue control measure and unit skewness.*

Note: in 1D: $t \mapsto \Lambda_\gamma(1_{[0,t]})$ is a γ-stable Lévy process

Limit for heavy tails, intermediate scaling

Theorem 3. *Assume that* $P(V > v) = L(v)v^{-\gamma}$ *with* $\gamma \in (1, 2)$ *and* $\lambda\rho^{\gamma}L(1/\rho) \to \gamma^{-1}$. *Then as* $\lambda \to \infty$ *and* $\rho \to 0$,

$$J_{\lambda,\rho}(\phi) - \mathsf{E}\, J_{\lambda,\rho}(\phi) \xrightarrow{d} J^{*}_{\gamma,C}(\phi), \quad \phi \in L^1 \cap L^2,$$

where

$$J^{*}_{\gamma,C}(\phi) = \int_{\mathbb{R}^d} \int_0^{\infty} v m_{\phi}(x, v) \, (N_{\gamma}(dx, dv) - dx\, v^{-\gamma-1} dv),$$

and $N_{\gamma}(dx, dv)$ *is a PRM on* $\mathbb{R}^d \times \mathbb{R}_+$ *with intensity* $dx\, v^{-\gamma-1} dv$.

Limit for heavy tails, large-grain scaling

Theorem 4. *Assume that* $P(V > v) = L(v)v^{-\gamma}$ *with* $\gamma \in (1,2)$ *and* $\lambda\rho^\gamma L(1/\rho) \to \infty$. *Then as* $\lambda \to \infty$ *and* $\rho \to 0$,

$$\frac{J_{\lambda,\rho}(\phi) - \mathsf{E}\, J_{\lambda,\rho}(\phi)}{(\gamma\lambda\rho^\gamma L(1/\rho))^{1/2}} \xrightarrow{d} W_{\gamma,C}(\phi), \quad \phi \in L^1 \cap L^2,$$

where $W_{\gamma,C}$ *is the Gaussian random linear functional on* $L^1 \cap L^2$ *with* $\mathsf{E}\, W_{\gamma,C}(\phi) = 0$ *and*

$$\mathsf{E}\, W_{\gamma,C}(\phi)W_{\gamma,C}(\psi) = \iint \phi(x)K_{\gamma,C}(x-y)\phi(y)\, dx\, dy,$$

where

$$K_{\gamma,C}(x) = \int_0^\infty \left| v^{-1/d}x + C \cap C \right| v^{-\gamma} dv.$$

Properties of the limits

The Gaussian limit $W_{\gamma,C}$ is self-similar with $H = (3-\gamma)/2$:

$$W_{\gamma,C}(\phi \circ a^{-1}) \stackrel{d}{=} a^{Hd} W_{\gamma,C}(\phi)$$

The stable limit Λ_γ is self-similar with $H = 1/\gamma$:

$$\Lambda_\gamma(\phi \circ a^{-1}) \stackrel{d}{=} a^{Hd} \Lambda_\gamma(\phi)$$

In both cases, $H \in (1/2, 1)$

All the limits, $W_{\gamma,C}, J^*_{\gamma,C}, \Lambda_\gamma$ are stationary (=translation invariant)

Gaussian limit in the symmetric case

When C is the closed ball centered at the origin,

$$K_{\gamma,C}(x) = K_{\gamma,C}(e_1)\,|x|^{-(\gamma-1)d}, \quad e_1 = (1, 0, \ldots, 0)$$

Thus,

$$W_{\gamma,C} \stackrel{d}{=} K_{\gamma,C}(e_1)^{1/2} W_H, \quad H = (3-\gamma)/2,$$

where W_H is the Gaussian random linear functional on $L^1 \cap L^2$ with $\mathsf{E}\, W_H(\phi) = 0$ and

$$\mathsf{E}\, W_H(\phi) W_H(\psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\phi(x)\psi(y)}{|x-y|^{(2-2H)d}}\, dx\, dy.$$

The covariance kernel is called the *Riesz potential*

White noise representation of W_H

For $H \in (1/2, 1)$,

$$W_H(\phi) \stackrel{d}{=} c_{H,d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\phi(y)dy}{|x-y|^{(3/2-H)d}} W(dx),$$

where W is the Gaussian white noise on L^2

W_H in one-dimensional parameter space

When $d = 1$,

$$\mathsf{E}\, W_H(\phi)^2 = \frac{1}{2H-1}\,\mathsf{E}\left(\int_0^\infty \phi\, dB_H\right)^2,$$

where B_H is fBm with Hurst parameter H

Thus

$$W_H : \phi \mapsto c_H \int \phi\, dB_H$$

with $W_H(1_{[0,t]}) = c_H B_H(t)$

This way W_H extends the fractional Gaussian noise $\frac{d}{dt}B_H$ to multidimensional parameter space

Proofs of Theorems 1–4

The proofs are based on

(i) Fourier transform of the Poisson noise

(ii) Extended Potter's bounds

(iii) Hardy–Littlewood maximal theorem

Fourier transform of the Poisson noise

When N is a PRM with intensity η and

$$\int (|\phi| \wedge \phi^2) d\eta < \infty,$$

then the stochastic integral $\int \phi(dN - d\eta)$ exists, and

$$\mathsf{E}\, e^{i \int \phi(dN - d\eta)} = e^{\int \Psi(\phi) d\eta}$$

with

$$\Psi(v) = e^{iv} - 1 - iv.$$

Note: linearity $\Longrightarrow$ no need to consider $\sum_{j=1}^{n} \theta_j \phi_j$

Potter's bounds

Lemma 1. *Let* $V \geq 0$ *be a random variable such that* $P(V > v) = L(v)v^{-\gamma}$ *for some* $\gamma > 0$*. Then for any* $\epsilon \in (0, \gamma)$ *there exist positive numbers* c_ϵ *and* ρ_ϵ *such that for all* $\rho \in (0, \rho_\epsilon)$*,*

$$\frac{L(v/\rho)}{L(1/\rho)} \leq c_\epsilon \left(v^{-\epsilon} \vee v^{+\epsilon} \right) \quad \forall v > 0.$$

Hardy–Littlewood maximal theorem

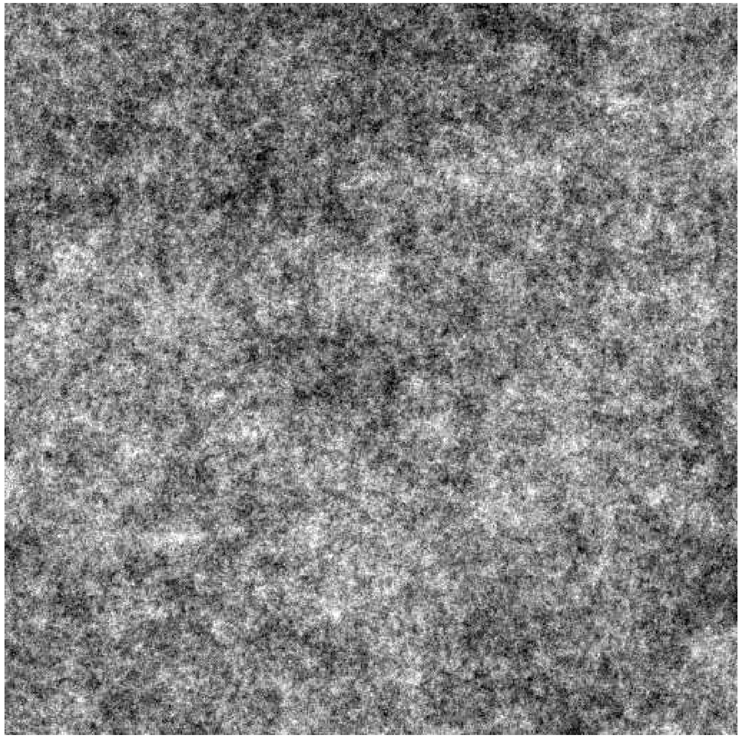
Let ϕ_* be the *Hardy–Littlewood maximal function* of ϕ,

$$\phi_*(x) = \sup_{v>0} \; v^{-1} \int_{x+v^{1/d}C} |\phi(y)| \, dy$$

For all $p > 1$,

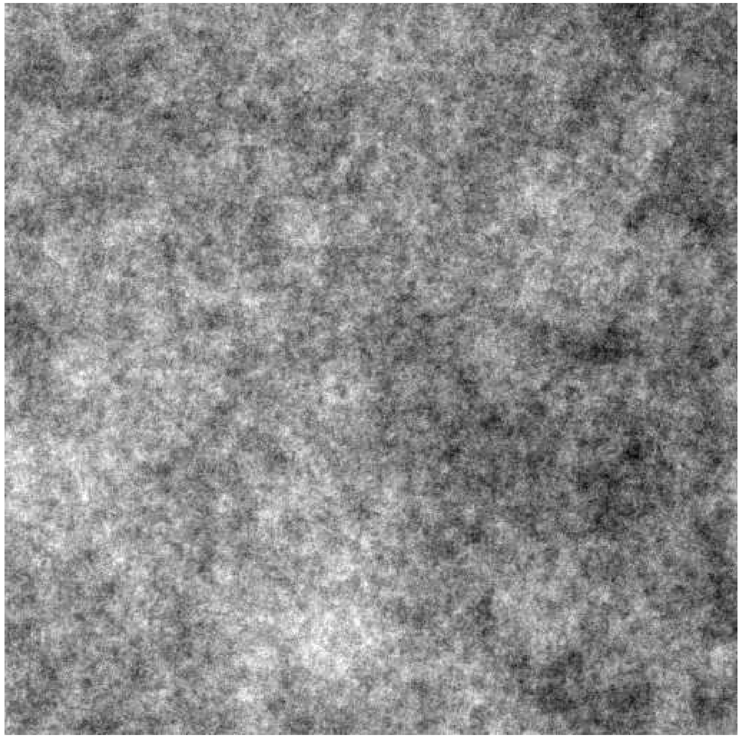
$$\phi \in L^p \implies \phi_* \in L^p$$

2D fractional Gaussian noise with $H = 0.90$



Picture taken by Penttinen and Virtamo (2004)

2D fractional Gaussian noise with $H = 0.97$



Picture taken by Penttinen and Virtamo (2004)

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