Joint work with Alkida Balliu, Sebastian Brandt,
Juho Hirvonen, Dennis Olivetti, Mikaël Rabie

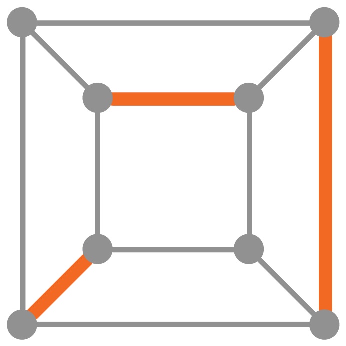
Best paper award at FOCS 2019

Lower bounds for maximal matchings and maximal independent sets

Jukka Suomela · Aalto University · Finland

Two classical graph problems

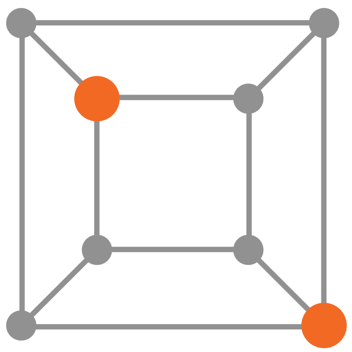
Maximal matching



matching = set of non-adjacent edges

maximal = not a strict subset of another matching

Maximal independent set

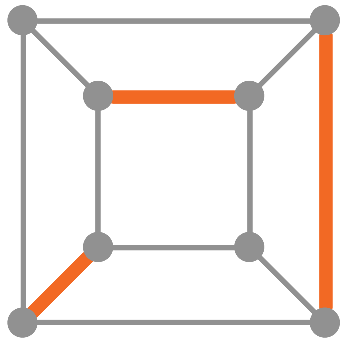


independent set = set of non-adjacent nodes

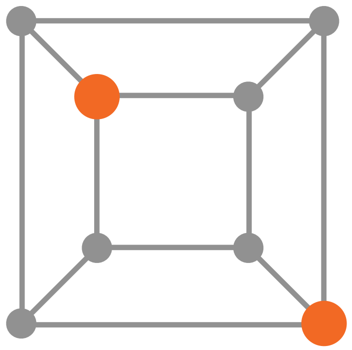
maximal = not a strict subset of another independent set

Two classical graph problems

Maximal matching

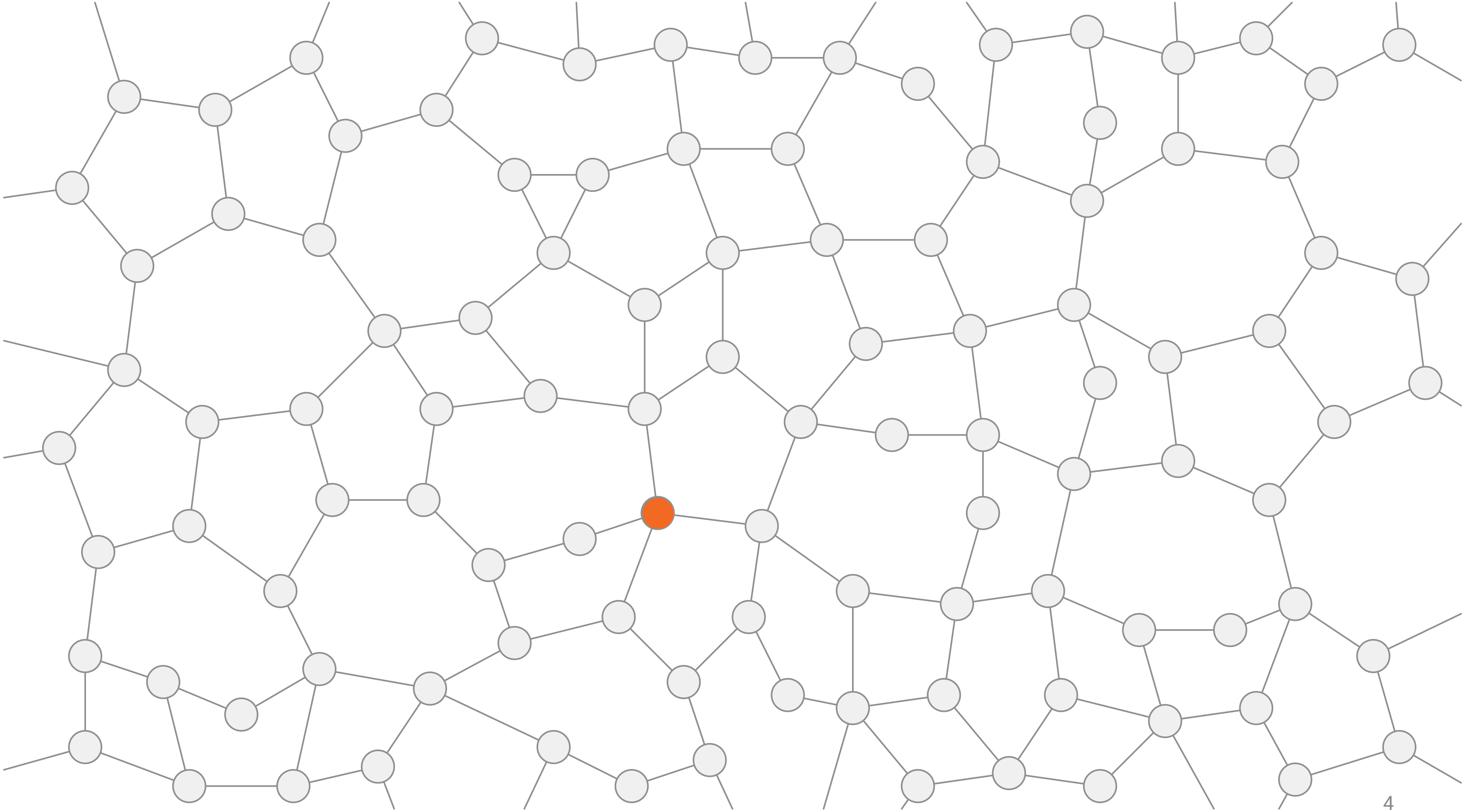


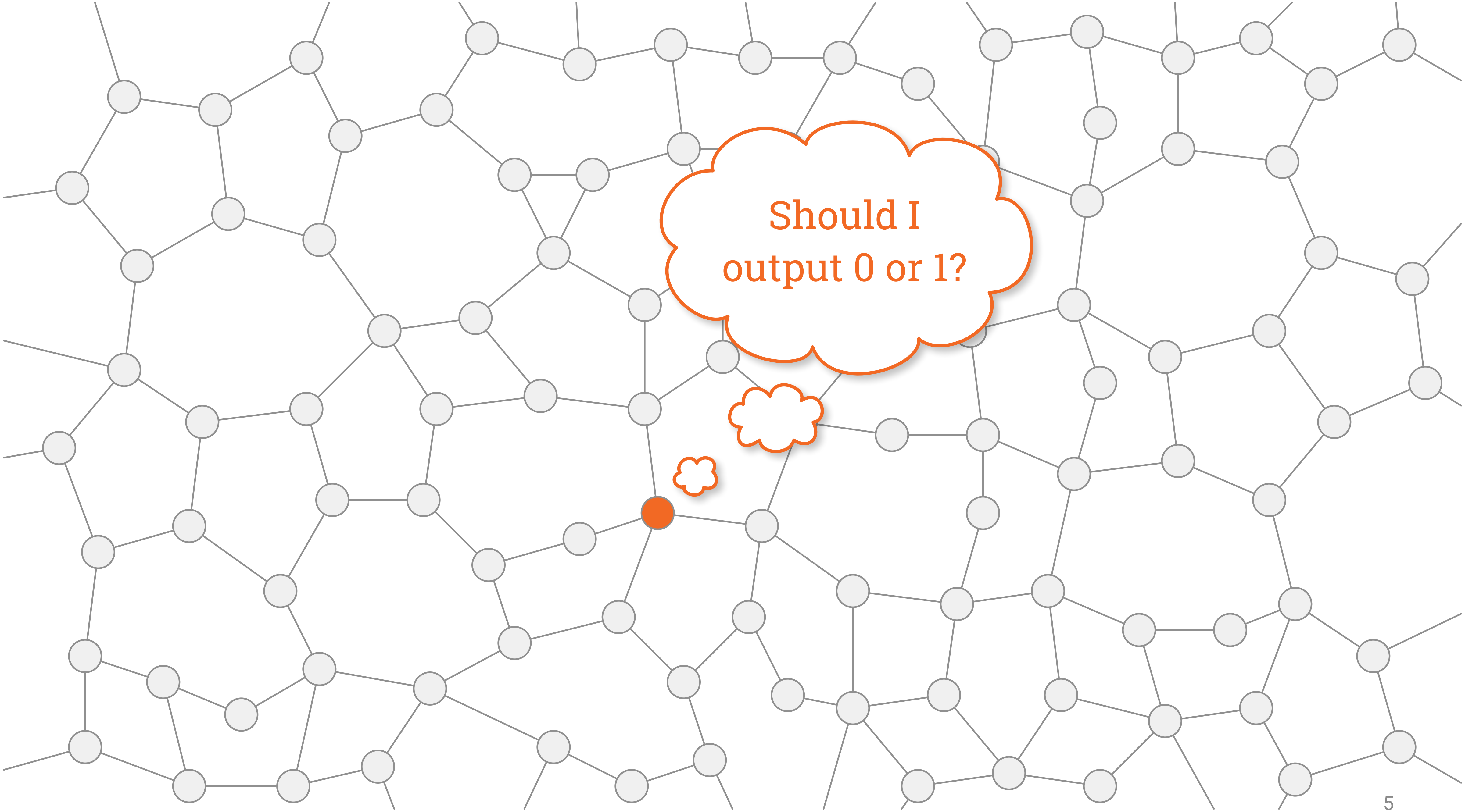
Maximal independent set

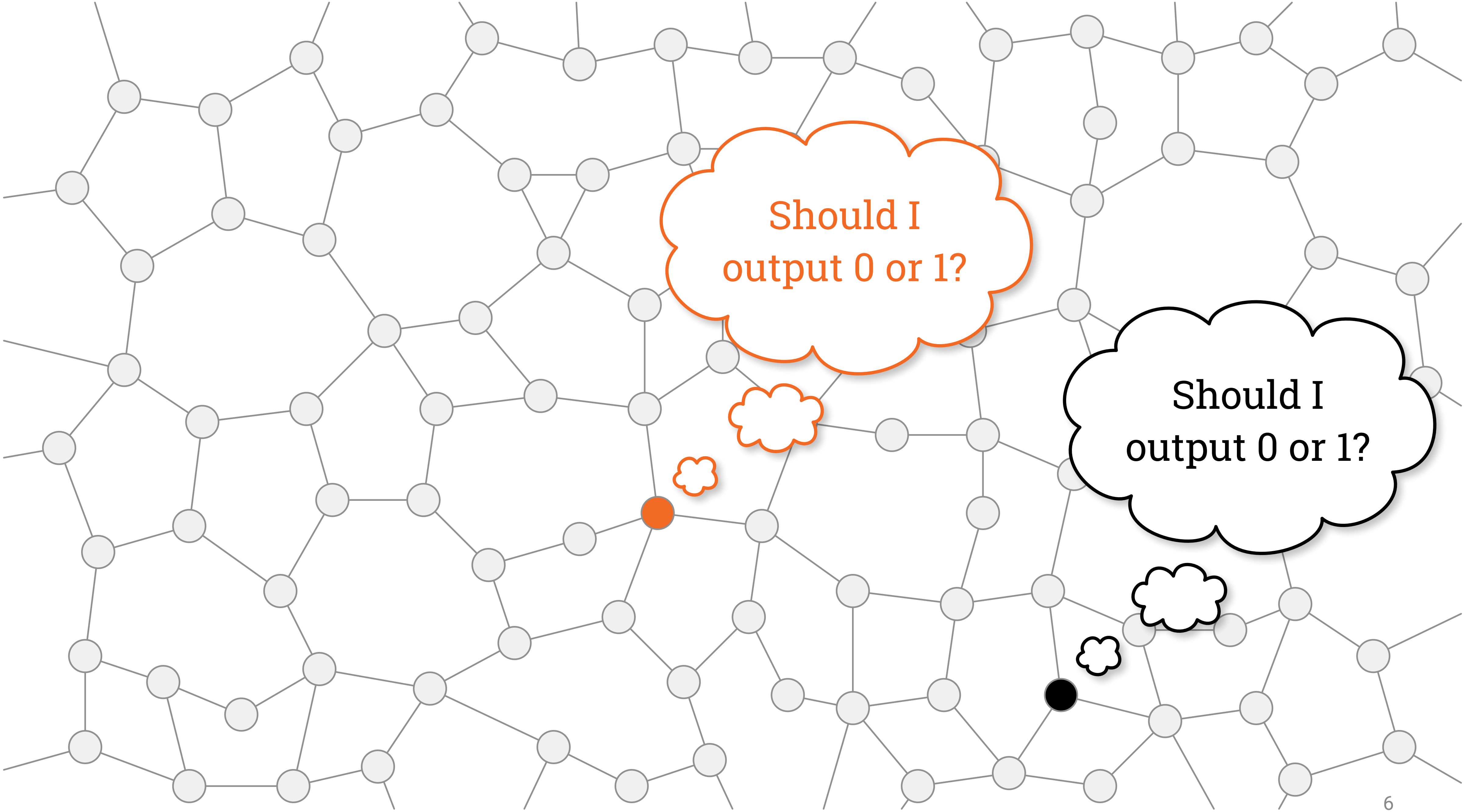


Trivial linear-time centralized, sequential algorithm:
add edges/nodes until stuck

How easy is it to solve these problems in a distributed setting?




Should I
output 0 or 1?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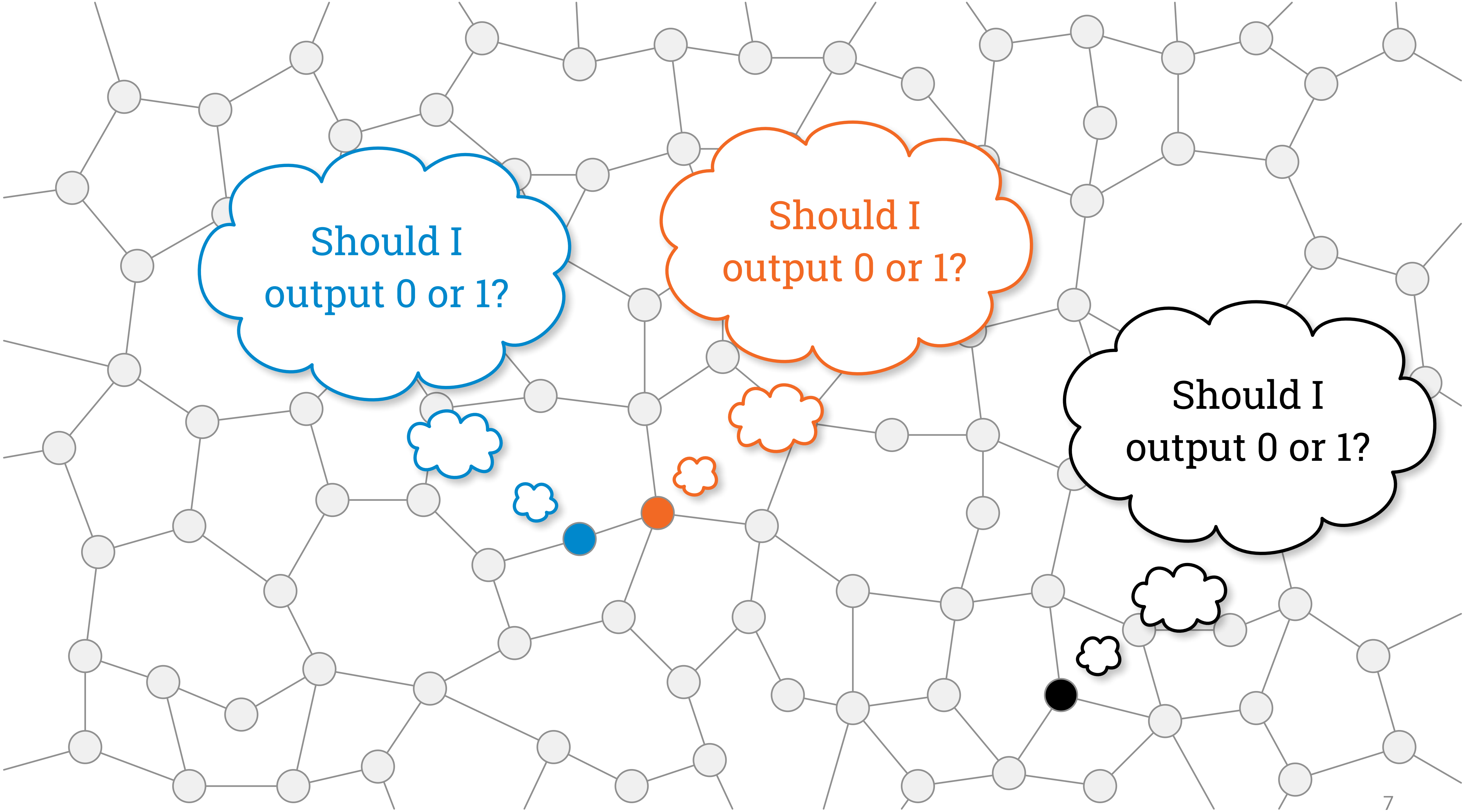
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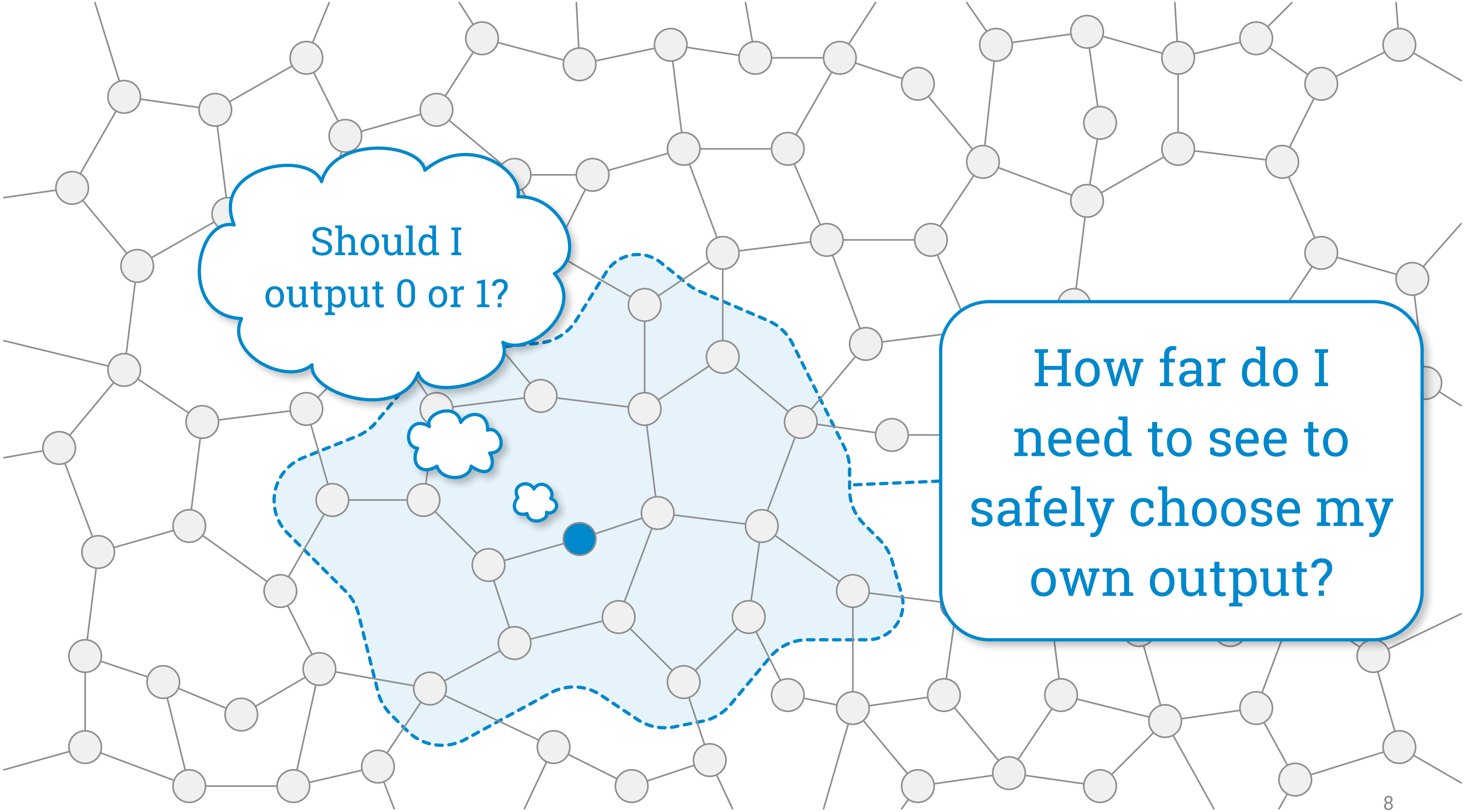
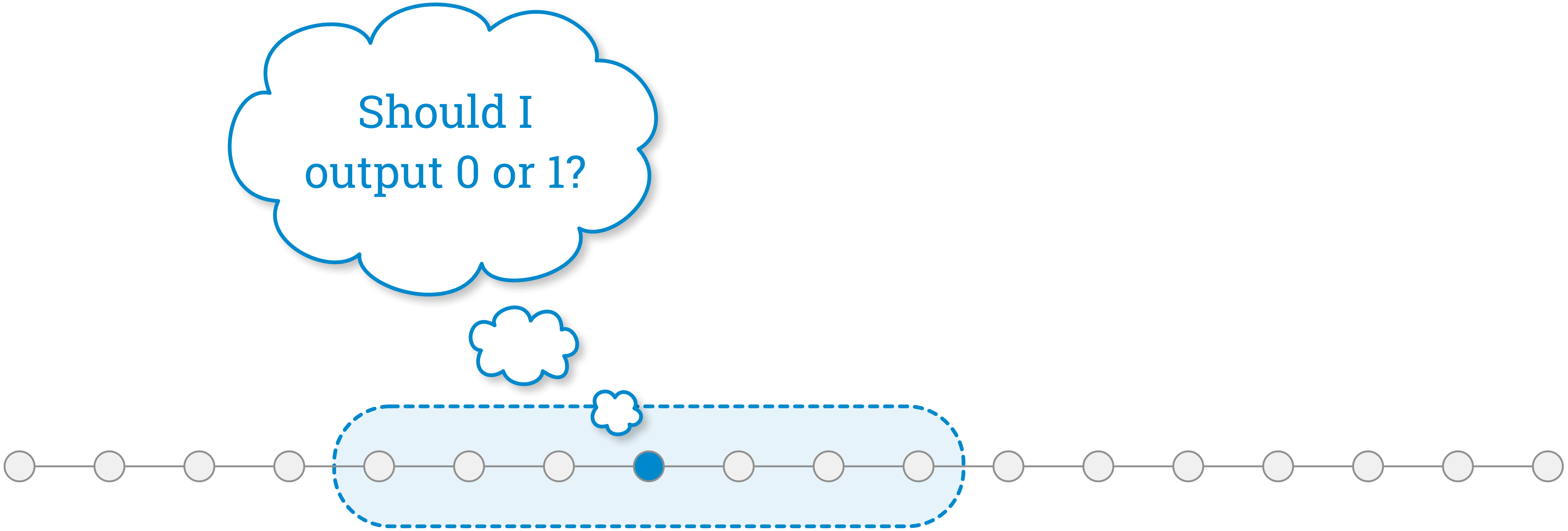
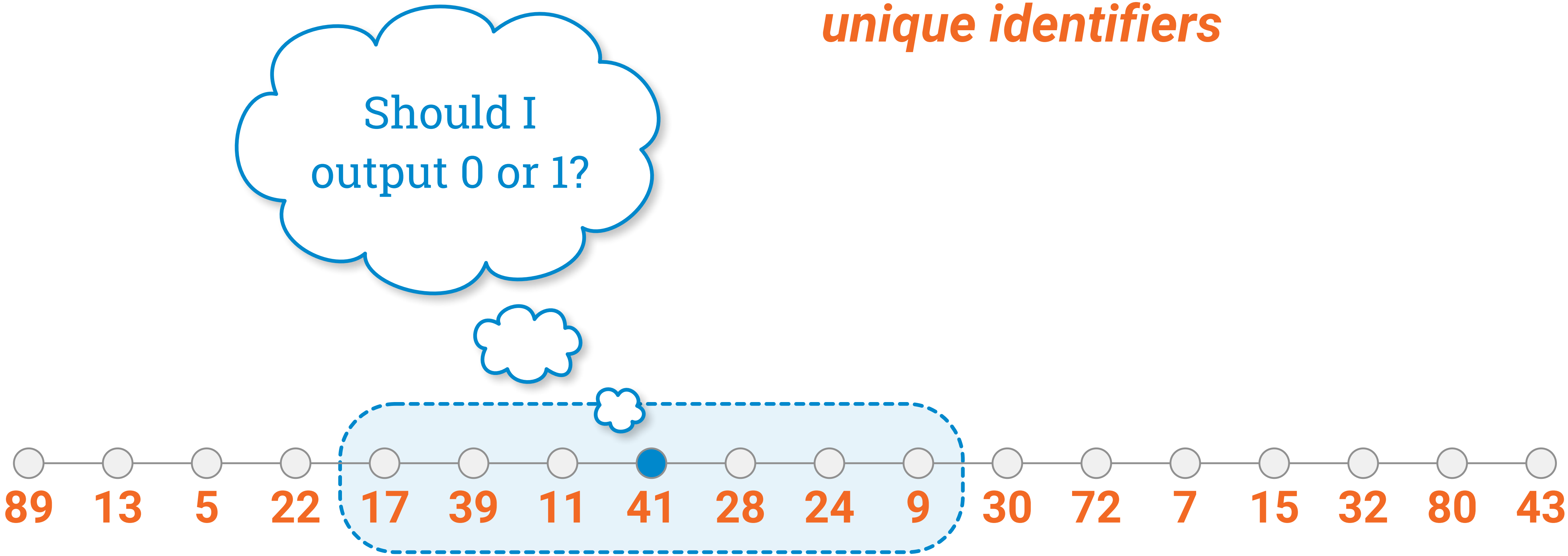

Should I output 0 or 1?
How far do I need to see to safely choose my own output?

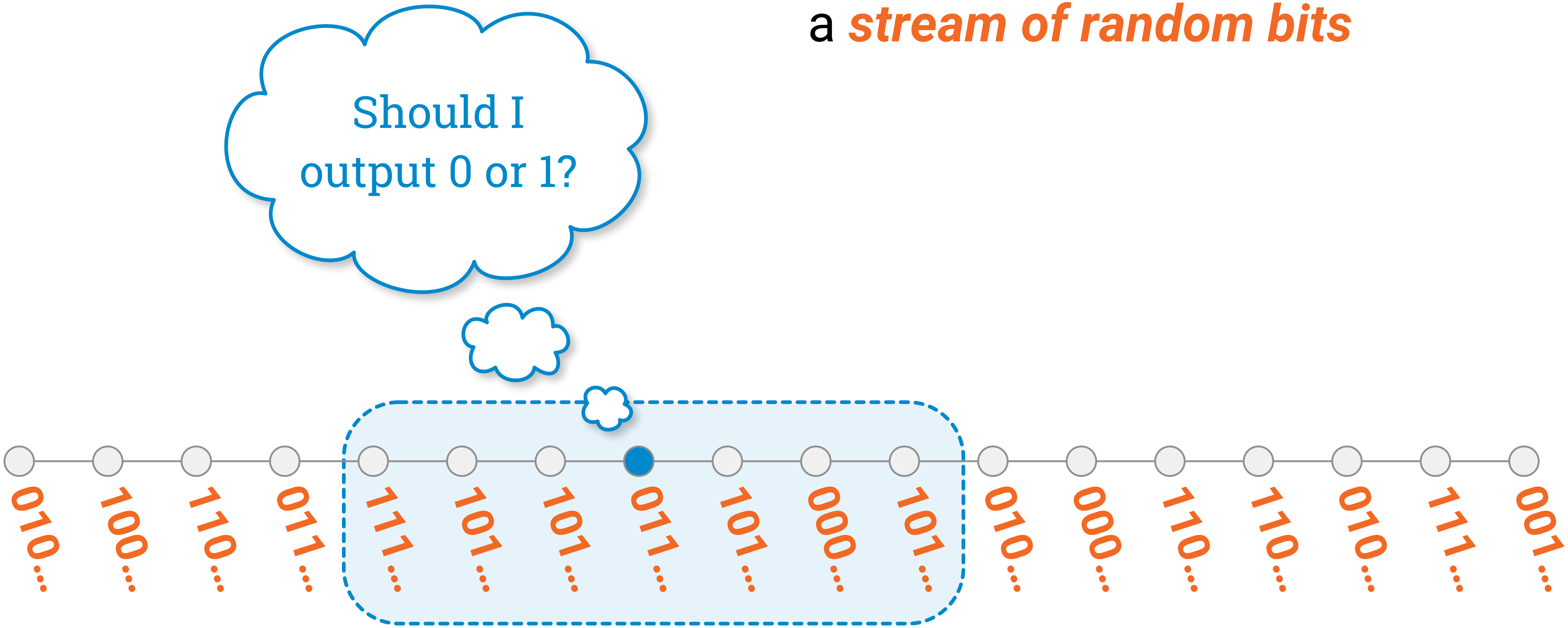
But what if everyone sees the same local neighborhood?



Deterministic distributed algorithms: we assume that nodes are labeled with ***unique identifiers***



Randomized distributed algorithms: we assume that nodes are labeled with a ***stream of random bits***



How far do you need to see?

- More formally: time complexity in the **LOCAL model** of distributed computing
- Two equivalent perspectives:
  - ***how far*** does a node need to see to pick its own part of the solution?
  - ***how many communication rounds*** are needed in a message-passing system until all nodes can stop and announce their own outputs?
- Worst-case setting:
  - worst-case input graph
  - worst-case assignment of unique identifiers

n = number of nodes

Δ = maximum degree

Old news:

$O(\Delta + \log^* n)$

Our result:

this is tight!

This project started around 2011...

State of the art in the early 2010s

- **Four key problems** that have been studied actively since the 1980s
- Trivial to solve with centralized sequential algorithms

maximal independent set	maximal matching
$(\Delta+1)$-vertex coloring	$(2\Delta-1)$-edge coloring

- All of these are ***"symmetry-breaking"*** problems
  - adjacent nodes/edges in the middle of a regular graph need to produce different outputs

State of the art in the early 2010s

- ***Lower bounds:***
  - Linial (1987, 1992), Naor (1991), Kuhn, Moscibroda, Wattenhofer (2004)

maximal independent set	maximal matching
$(\Delta+1)$-vertex coloring	$(2\Delta-1)$-edge coloring

- ***Upper bounds:***
  - Cole & Vishkin (1986), Luby (1985, 1986), Alon, Babai, Itai (1986), Israeli & Itai (1986), Panconesi & Srinivasan (1996), Hanckowiak, Karonski, Panconesi (1998, 2001), Panconesi & Rizzi (2001) …

State of the art in the early 2010s

- Algorithms for solving each of these problems in $O(\Delta + \log^* n)$ rounds
- Is this the best one can do, and why?

maximal independent set	maximal matching
$(\Delta+1)$-vertex coloring	$(2\Delta-1)$-edge coloring

- Well-known that $O(\Delta) + o(\log^* n)$ is not possible
  - holds both for deterministic and randomized algorithms
- What about $o(\Delta) + O(\log^* n)$???

How to make sense of O(Δ + log* n)?

- Why $O(\Delta + \log^* n)$?
  - $O(\log^* n)$: "**symmetry-breaking**", adjacent nodes do different things
  - $O(\Delta)$: ???
- How to study dependency on n in isolation?
  - just look at bounded-degree graphs, let $\Delta = O(1)$
  - well-understood thanks to Linial (1987, 1992), Naor (1991)
- How to study dependency on Δ in isolation?
  - you can't set $n = O(1)$ and see what happens…
  - but maybe we can eliminate "**symmetry-breaking**" concerns?

Eliminate “O(log* n)” part

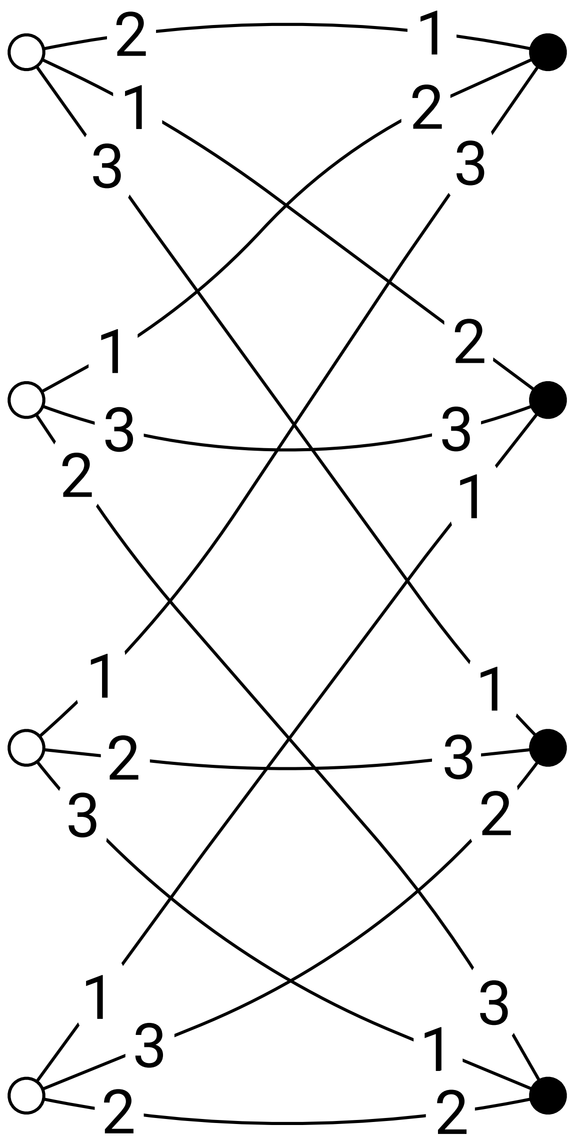
- Could we find simple special cases of these problems that would be solvable in $O(\Delta)$ time, independently of n?
- Yes! Examples:
  - maximal matching: $\boldsymbol{O(\Delta + \log^* n)}$
  - maximal ***fractional*** matching: $\boldsymbol{O(\Delta)}$
  - maximal matching in ***bipartite*** graphs: $\boldsymbol{O(\Delta)}$
  - maximal matching in ***edge-colored*** graphs: $\boldsymbol{O(\Delta)}$
- Could we first prove a lower bound for one of these?
  - and if so, would it help to understand the general case?

Let’s look at this in more detail…

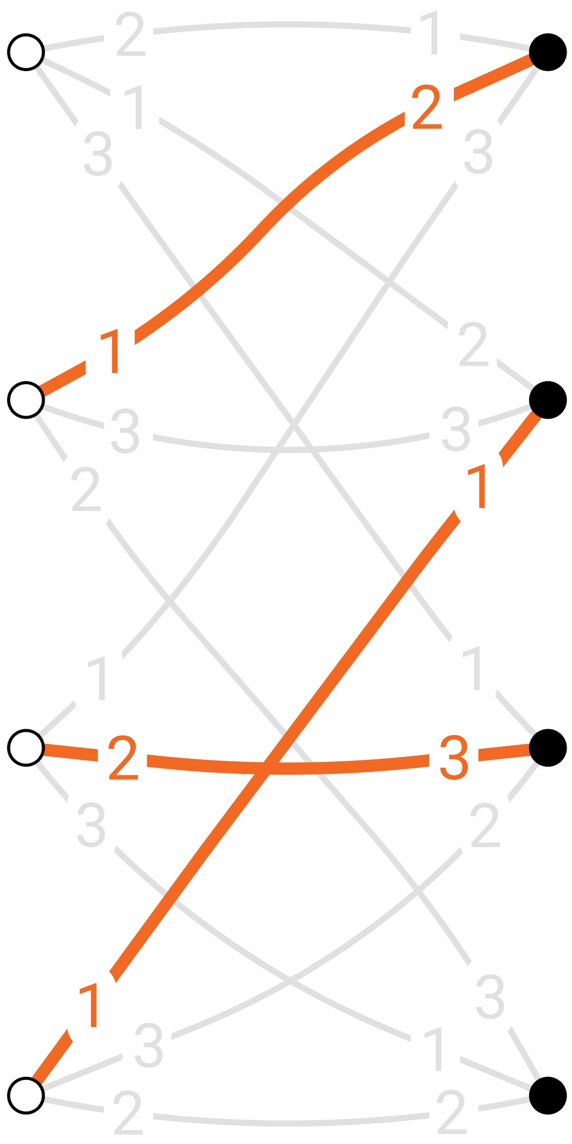
computer network with port numbering

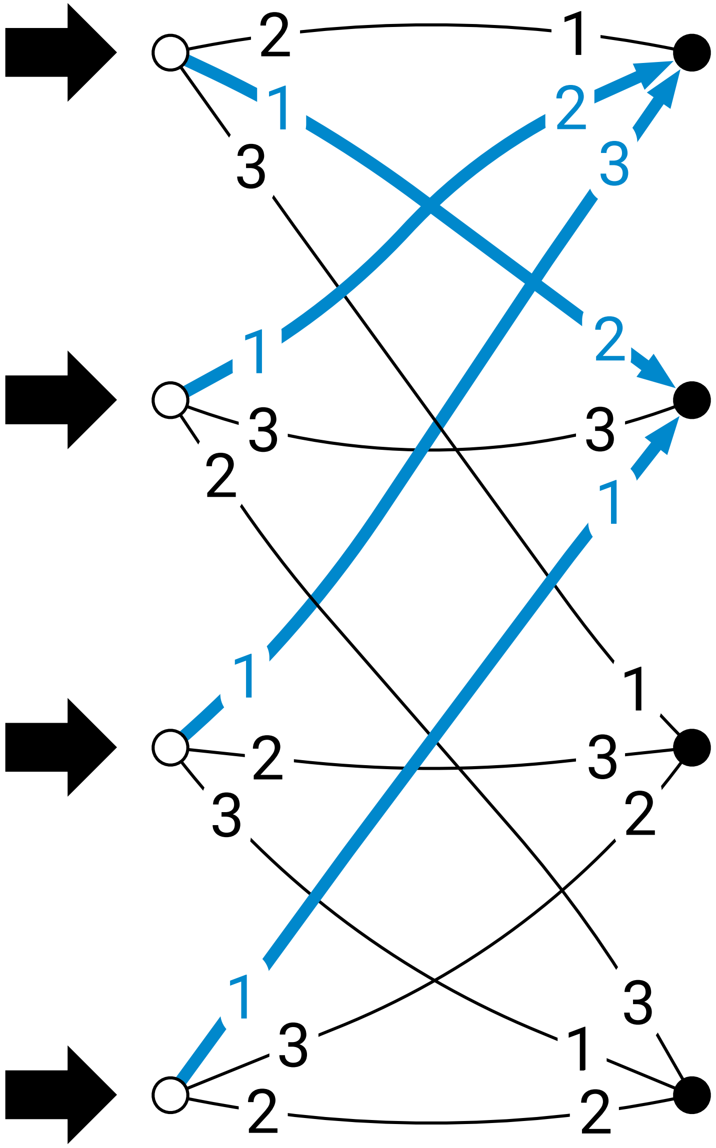
bipartite, 2-colored graph

Δ-regular (here Δ = 3)



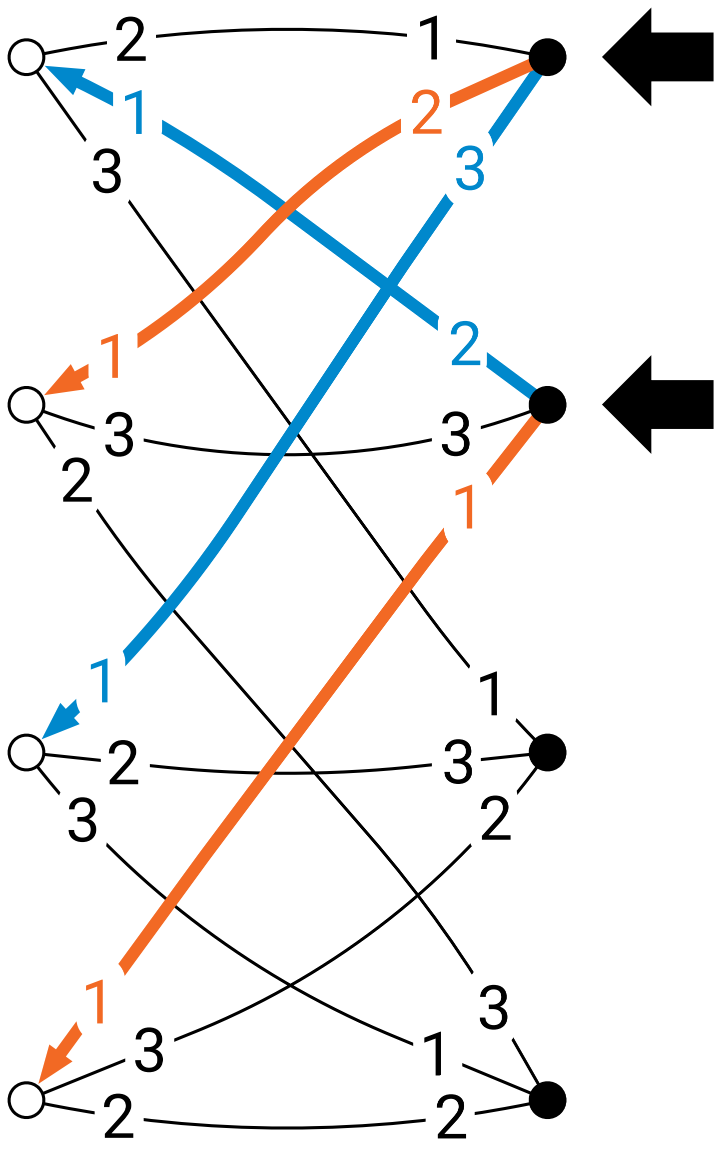
output: ***maximal matching***





Very simple algorithm

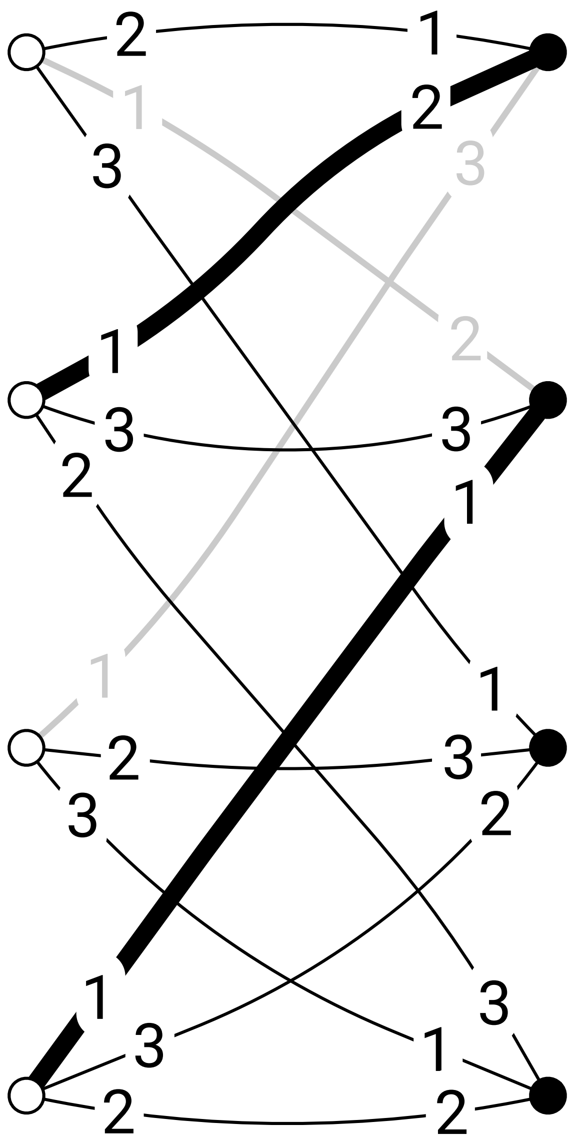
unmatched white nodes:
send ***proposal*** to port 1



Very simple algorithm

unmatched white nodes:
send ***proposal*** to port 1

black nodes:
accept the first proposal you get, ***reject*** everything else (break ties with port numbers)



Very simple algorithm

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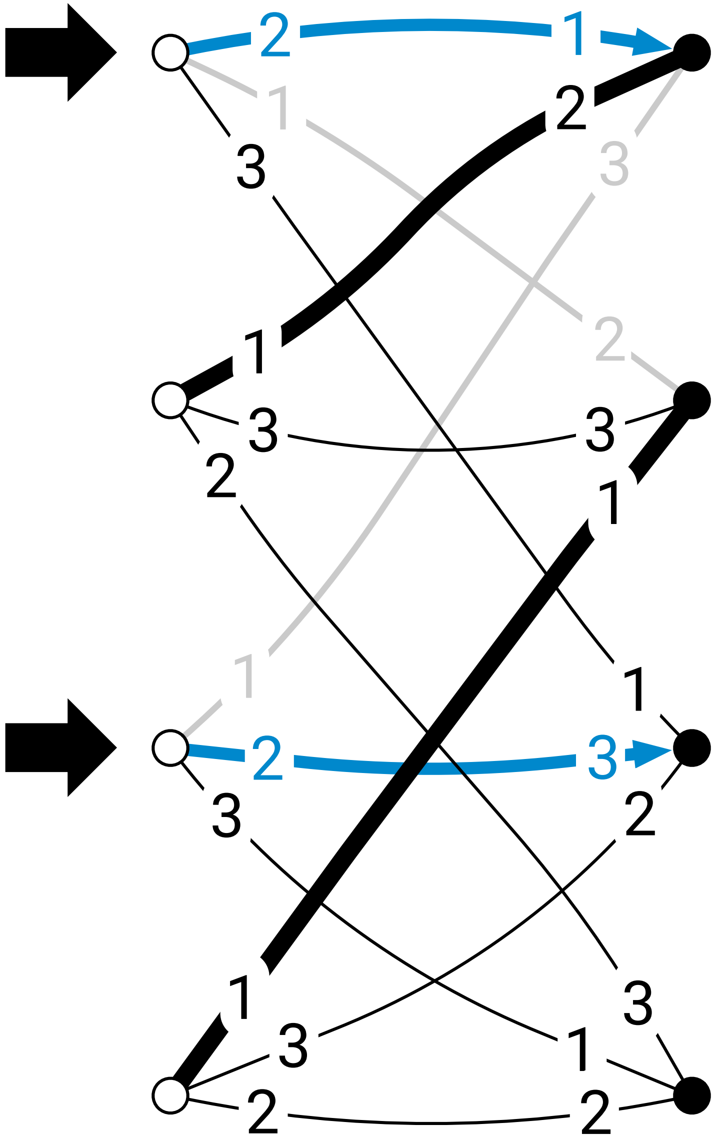
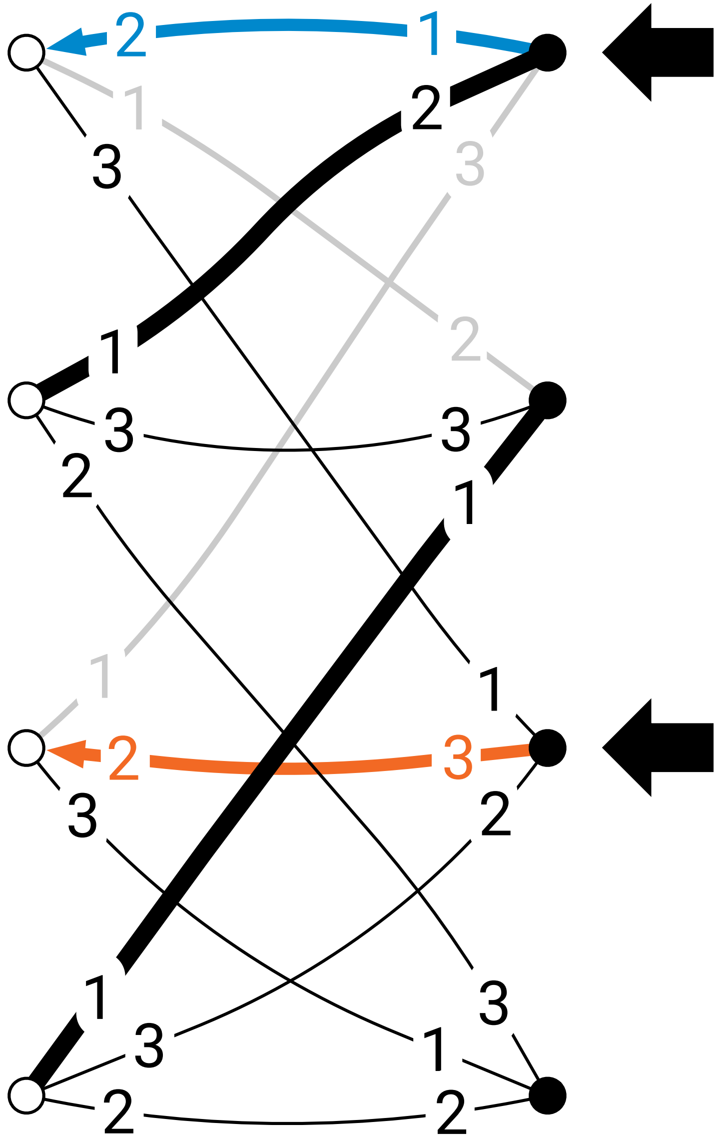


Very simple algorithm

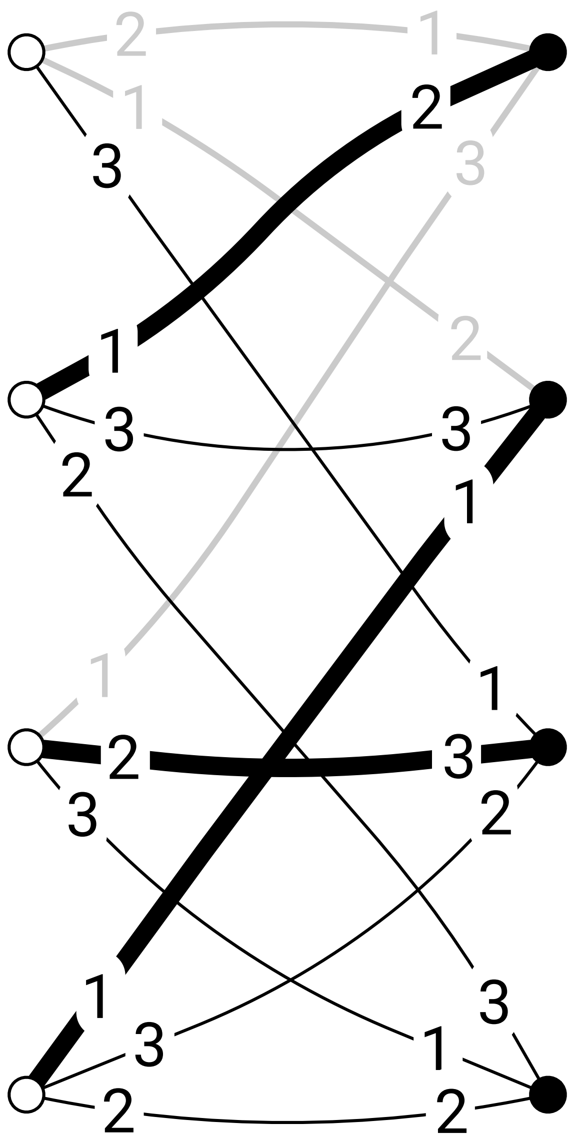
unmatched white nodes:
send ***proposal*** to port 2



Very simple algorithm

unmatched white nodes:
send *proposal* to port 2

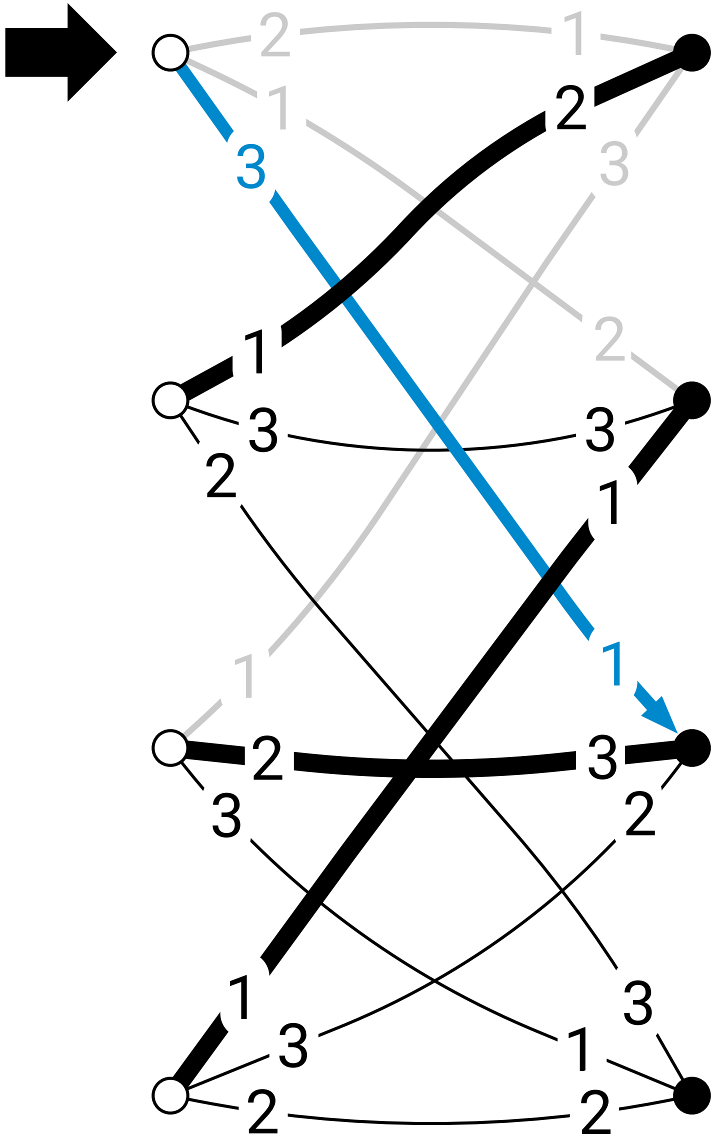
black nodes:
accept the first proposal you get, *reject* everything else
(break ties with port numbers)



Very simple algorithm

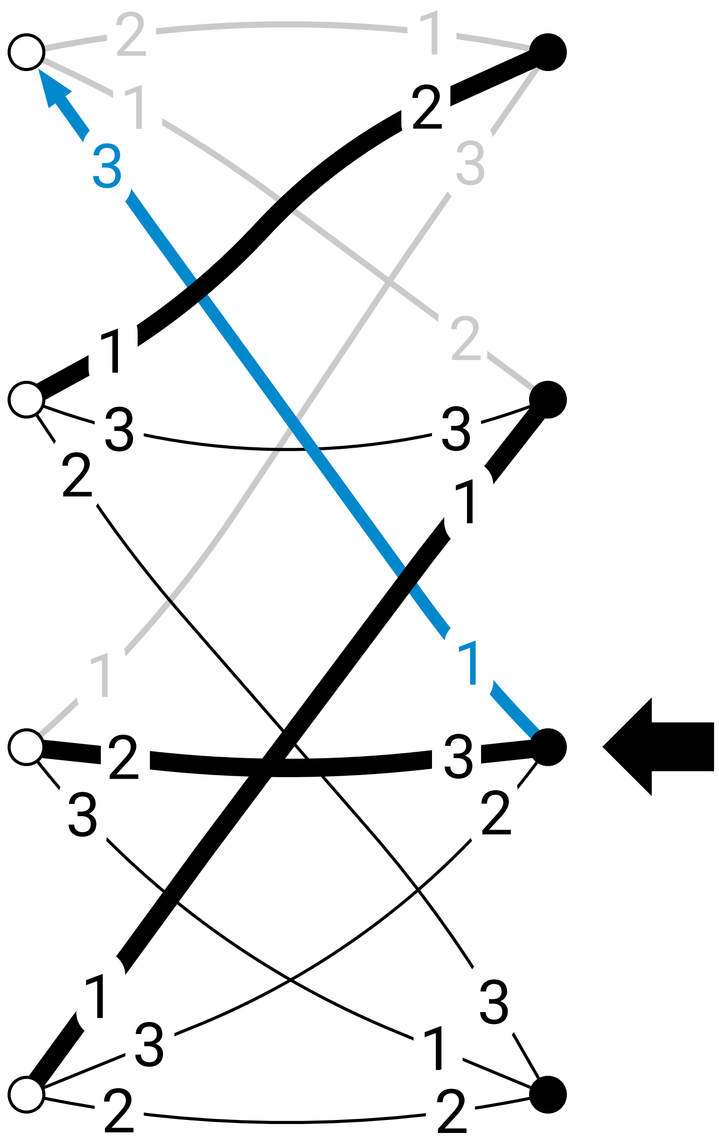
unmatched white nodes:
send ***proposal*** to port 2

black nodes:
accept the first proposal you get, ***reject*** everything else (break ties with port numbers)



Very simple algorithm

unmatched white nodes:
send ***proposal*** to port 3



Very simple algorithm

unmatched white nodes:
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black nodes:
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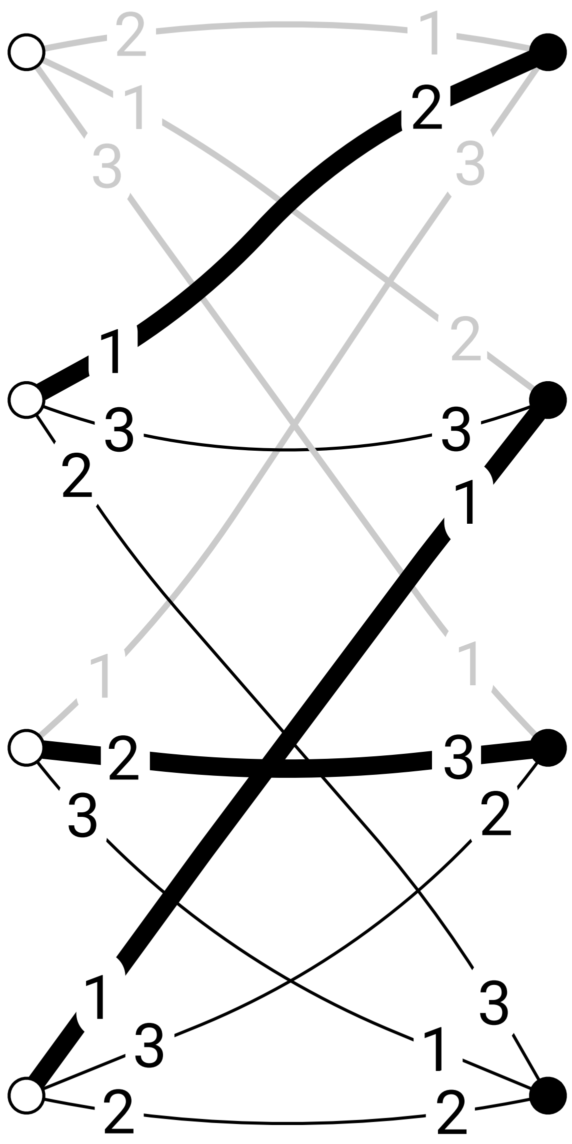

Very simple algorithm

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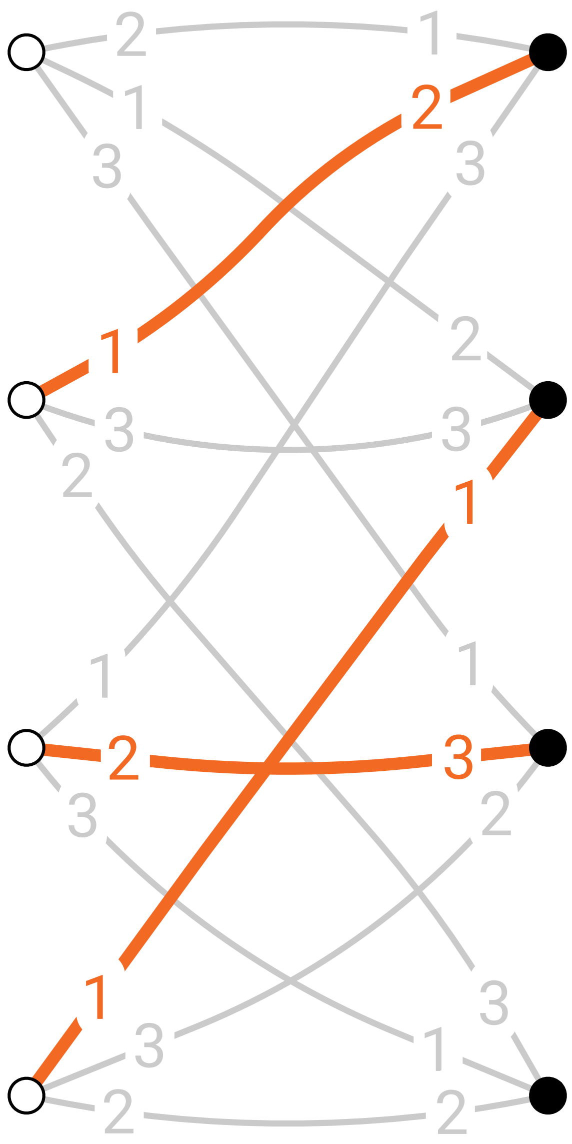


Very simple algorithm

Finds a ***maximal matching*** in $O(\Delta)$ communication rounds

Note: running time does not depend on n

Bipartite maximal matching

- Maximal matching in 2-colored Δ-regular graphs
- Simple algorithm: ***$O(\Delta)$ rounds***, independently of n
  = if each node sees its radius-$O(\Delta)$ neighborhood, it can choose its own part of the solution (whether it is matched and with whom)
- ***Is this optimal?***
  - **$o(\Delta)$** rounds?
  - **$O(\log \Delta)$** rounds??
  - **4** rounds???

Bipartite maximal matching

- Seemingly simple toy problem
  - no need for randomness, unique identifiers
- Promising starting point?
  - hypothesis: "$O(\Delta)$" in the proposal algorithm is there "for the same reason" as in much more complicated $O(\Delta + \log^* n)$-time algorithms

Progress since 2011

- **PODC 2012:** maximal matching not possible in $o(\Delta)$ time in the “***edge-coloring model***”
  - *doesn’t tell anything about **bipartite maximal matching***
- **PODC 2014:** maximal ***fractional*** matching not possible in $o(\Delta)$ time in the usual LOCAL model
  - *doesn’t tell anything about **bipartite maximal matching***

We had a lower-bound technique,
but it couldn’t handle 2-colored graphs

Progress since 2011

- In the meantime, there were new upper bounds!
- **Barenboim, PODC 2015**:
  (Δ+1)-vertex coloring and **(2Δ−1)-edge coloring**
  in $O(\Delta^{3/4} + \log^* n)$ time
- Could it be the case that also maximal matching and maximal independent set are solvable in $o(\Delta)$ time using similar techniques??

Progress since 2011

- We kept working on the ***bipartite maximal matching*** problem
- And it certainly wasn't a secret!
  - e.g. in ADGA 2014 I gave a talk outlining the whole research program: solve the complexity of bipartite maximal matching, it would probably tell us something about all these problems
  - every time we had visitors, I annoyed them with questions about bipartite maximal matchings
- But zero new progress, until late 2018
  - or so we thought…

ADGA 2014

Summary

- **Distributed time complexity, LOCAL model**
- **$O(\log^* n)$: "symmetry breaking", OK**
- **$O(\Delta)$: "local coordination", poorly understood**
- **Maximal *fractional* matching solved, next step: *bipartite* maximal matching**

Linial (1987, 1992): coloring cycles

- **Given:**
  - algorithm A_0 solves 3-coloring in $T = o(\log^* n)$ rounds
- **We construct:**
  - algorithm A_1 solves 2^3-coloring in $T - 1$ rounds
  - algorithm A_2 solves 2^{2^3}-coloring in $T - 2$ rounds
  - algorithm A_3 solves $2^{2^{2^3}}$-coloring in $T - 3$ rounds

  …
  - algorithm A_T solves $o(n)$-coloring in 0 rounds
- But $o(n)$-coloring is nontrivial, so A_0 cannot exist

Brandt et al. (2016): sinkless orientation

- **Given:**
  - algorithm A_0 solves **sinkless orientation** in $T = o(\log n)$ rounds
- **We construct:**
  - algorithm A_1 solves **sinkless coloring** in $T - 1$ rounds
  - algorithm A_2 solves **sinkless orientation** in $T - 2$ rounds
  - algorithm A_3 solves **sinkless coloring** in $T - 3$ rounds

    …
  - algorithm A_T solves **sinkless orientation** in **0** rounds
- But **sinkless orientation** is nontrivial, so A_0 cannot exist

Brandt (2019): this can be automated

- Always possible for **any graph problem P_0** that is "locally verifiable"
- If problem P_0 has complexity T, we can always find in a mechanical manner problem P_1 that has complexity $T - 1$
  - holds for tree-like neighborhoods (e.g. high-girth graphs)
- This technique is nowadays known as "**round elimination**"

Late 2018 research meeting…

- *Sebastian Brandt* told us about his new lower bound technique that he had applied to **weak 2-coloring**
- We invited Sebastian for a 4-day visit so that he could present his proof
- He started by presenting the general round elimination technique, and before he could continue, we had already got sidetracked into discussing **bipartite maximal matching**…
- Could we use round elimination to prove a lower bound?

Late 2018 research meeting…

- Challenge when you try to apply round elimination:
  - you start with P_0 = bipartite maximal matching
  - you apply round elimination for a few steps
  - P_1 = something that still makes some sense

    …
  - P_3 = a complicated mess that fills two whiteboards, and nobody has any idea what the problem is about — how to continue?
- We need to discover a family of ***simpler*** problems Q_0, Q_1, Q_2, …
  - Q_i is a relaxation of P_i (a lower bound for Q_i gives a lower bound for P_i)
  - Q_i isn't too easy to solve (there exists a nontrivial lower bound for Q_i)

Late 2018 research meeting…

- Given a complicated graph problem with lots of possible output labels, one can try to ***simplify it in systematic ways*** by e.g. merging labels
- But this is very slow (and very error-prone) to do by hand
  - round elimination is mechanical, but lots of work
- ***Dennis Olivetti*** implemented **round elimination as a computer program** in one evening
  - we could start to quickly explore what happens if we try out different simplification schemes — **this led to the breakthrough**!

Main results

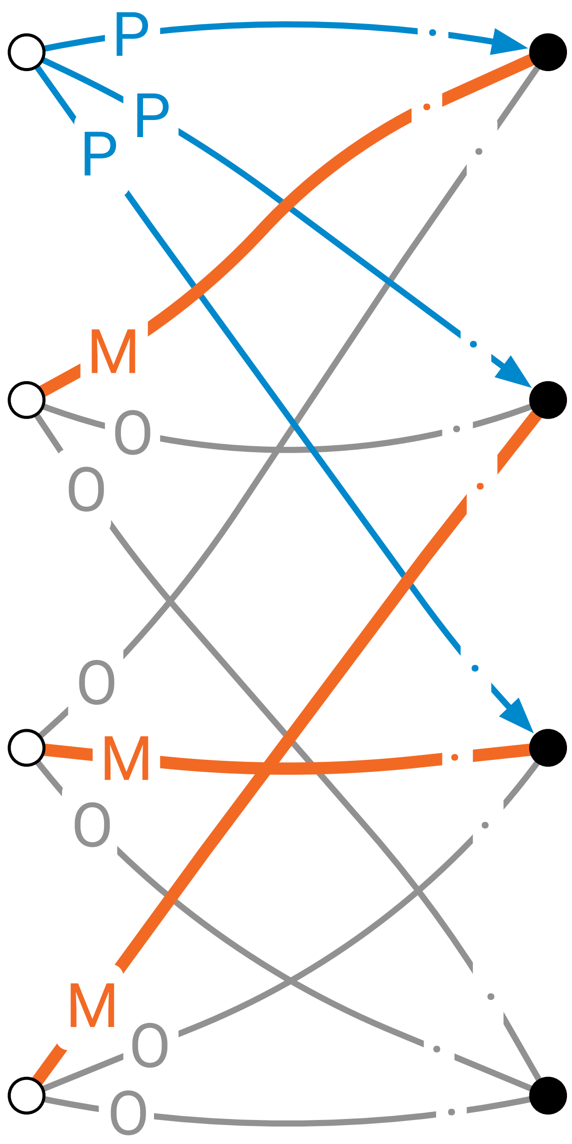
Maximal matching and **maximal independent set** cannot be solved in

- **$o(\Delta + \log \log n / \log \log \log n)$** rounds with randomized algorithms
- **$o(\Delta + \log n / \log \log n)$** rounds with deterministic algorithms

Latest news

- The complexity of both **maximal matching** and **maximal independent set** now well-understood, thanks to the network decomposition algorithm by Rozhoň & Ghaffari (STOC 2020)
- **“Round Eliminator”** program freely available online, with a web user interface
  - github.com/olidennis/round-eliminator
- ***Still wide open: complexity of graph coloring***
  - can you find a $(\Delta+1)$-vertex coloring in $O(\log \Delta + \log^* n)$ rounds??

Representation for maximal matchings



M = “matched”
P = “pointer to matched”
O = “other”

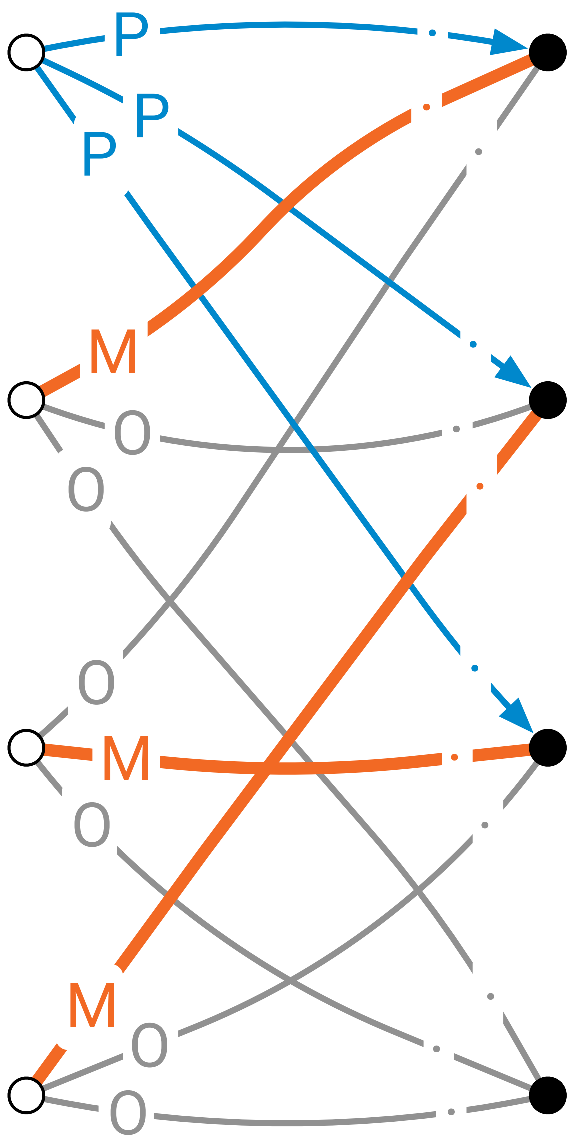
white nodes “active”

output one of these:
· 1 × M and (Δ−1) × O
· Δ × P

black nodes “passive”

accept one of these:
· 1 × M and (Δ−1) × {P, O}
· Δ × O

Representation for maximal matchings



M = “matched”
P = “pointer to matched”
O = “other”

white nodes “active”

output one of these:

- 1 × M and (Δ−1) × O
- Δ × P

$$W = \mathsf{M}\mathsf{O}^{\Delta-1} \mid \mathsf{P}^{\Delta}$$

black nodes “passive”

accept one of these:

- 1 × M and (Δ−1) × {P, O}
- Δ × O

$$B = \mathsf{M}[\mathsf{PO}]^{\Delta-1} \mid \mathsf{O}^{\Delta}$$

Parameterized problem family

maximal matching

$$W = \mathsf{MO}^{\Delta-1} \mid \mathsf{P}^{\Delta},$$
$$B = \mathsf{M}[\mathsf{PO}]^{\Delta-1} \mid \mathsf{O}^{\Delta}$$

"weak" matching

$$W_\Delta(x,y) = \left(\mathsf{MO}^{d-1} \,\middle|\, \mathsf{P}^{d}\right)\mathsf{O}^{y}\mathsf{X}^{x},$$
$$B_\Delta(x,y) = \left([\mathsf{MX}][\mathsf{POX}]^{d-1} \,\middle|\, [\mathsf{OX}]^{d}\right)[\mathsf{POX}]^{y}[\mathsf{MPOX}]^{x},$$
$$d = \Delta - x - y$$

Main lemma

- Given: ***A*** solves ***P*(*x*, *y*)** in ***T*** rounds
- We can construct: ***A'*** solves ***P*(*x* + 1, *y* + *x*)** in ***T* − 1** rounds

$$W_\Delta(x,y) = \left(\mathsf{MO}^{d-1} \;\middle|\; \mathsf{P}^d\right)\mathsf{O}^y\mathsf{X}^x,$$

$$B_\Delta(x,y) = \left([\mathsf{MX}]\,[\mathsf{POX}]^{d-1} \;\middle|\; [\mathsf{OX}]^d\right)[\mathsf{POX}]^y\,[\mathsf{MPOX}]^x,$$

$$d = \Delta - x - y$$

Putting things together

Maximal matching in $o(\Delta)$ rounds

→ "$\Delta^{1/2}$ **matching**" in $o(\Delta^{1/2})$ rounds

→ $P(\Delta^{1/2}, 0)$ in $o(\Delta^{1/2})$ rounds

→ $P(O(\Delta^{1/2}), o(\Delta))$ in **0** rounds

→ contradiction

What we really care about

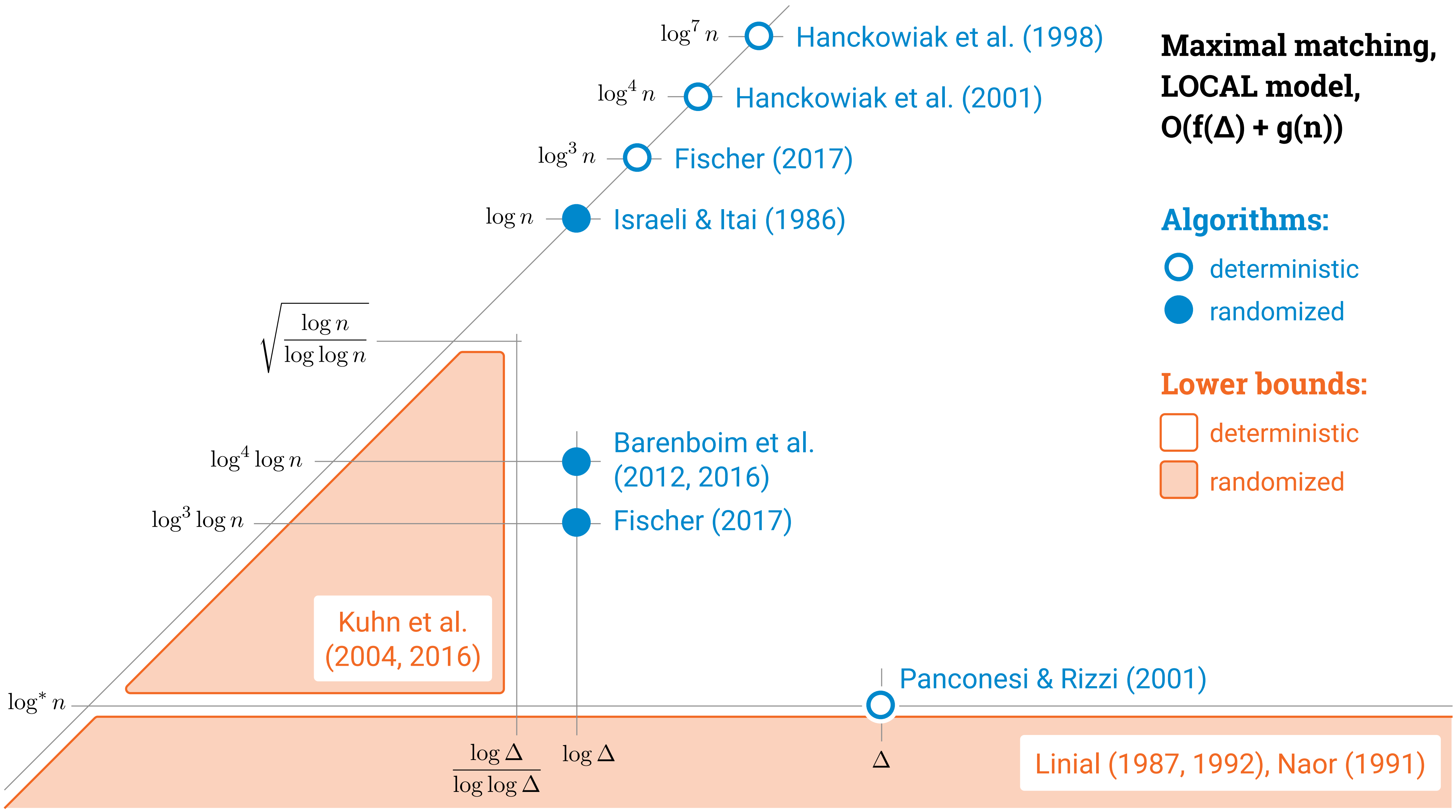
k-matching: select at most k edges per node

Apply speedup simulation $o(\Delta^{1/2})$ times

Putting things together

Proof technique does not work directly with unique IDs

- Basic version:
  - deterministic lower bound, ***port-numbering model***
- Analyze what happens to local failure probability:
  - ***randomized*** lower bound, port-numbering model
- With randomness you can construct unique identifiers w.h.p.:
  - randomized lower bound, ***LOCAL model***
- Fast deterministic → faster deterministic
  - stronger ***deterministic*** lower bound, LOCAL model



log^7 n
Hanckowiak et al. (1998)
log^4 n
Hanckowiak et al. (2001)
log^3 n
Fischer (2017)
log n
Israeli & Itai (1986)
sqrt(log n / log log n)
log^4 log n
Barenboim et al. (2012, 2016)
log^3 log n
Fischer (2017)
Kuhn et al. (2004, 2016)
log* n
Panconesi & Rizzi (2001)
log Δ / log log Δ
log Δ
Δ
Linial (1987, 1992), Naor (1991)
Maximal matching, LOCAL model, O(f(Δ) + g(n))
Algorithms:
deterministic
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Lower bounds:
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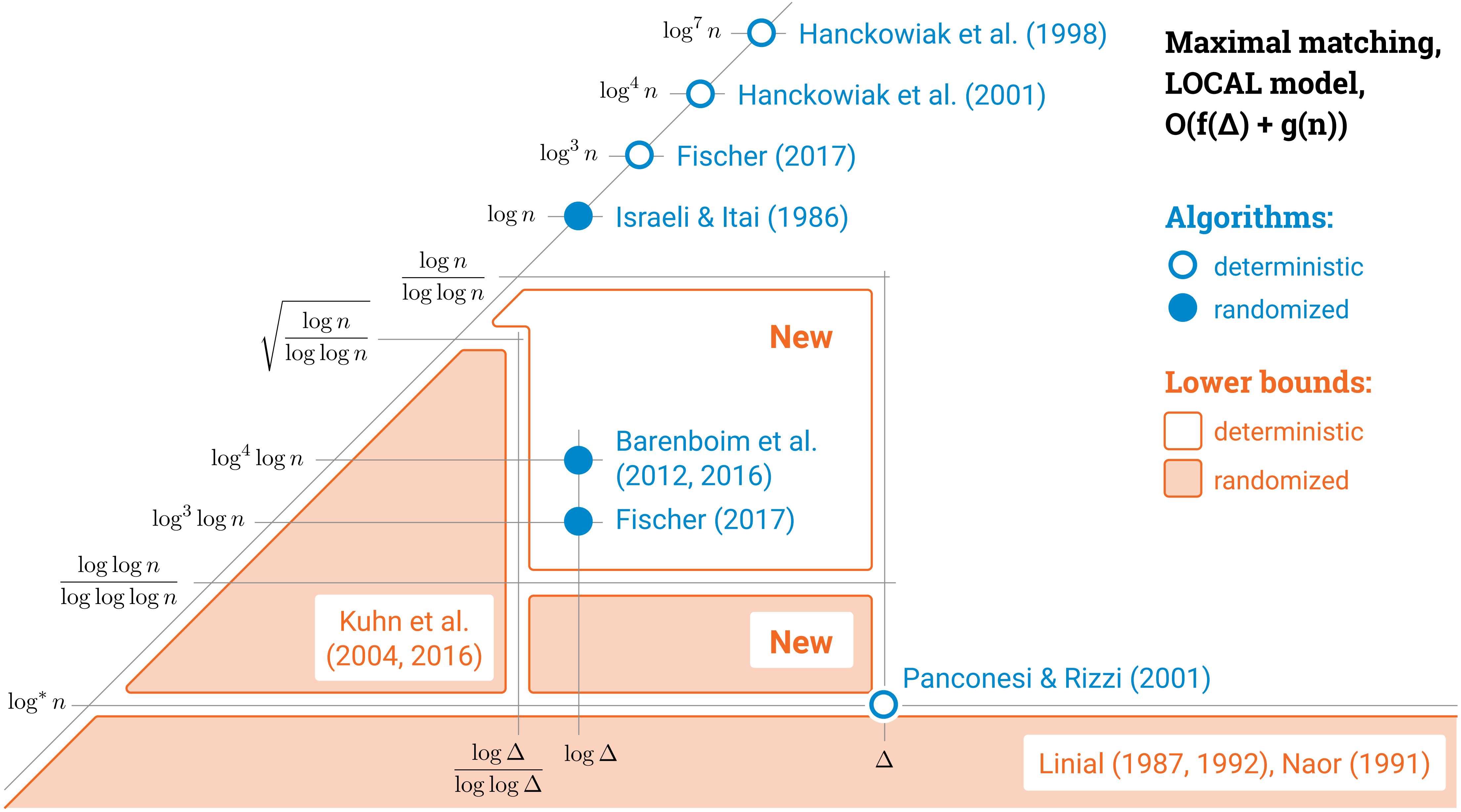
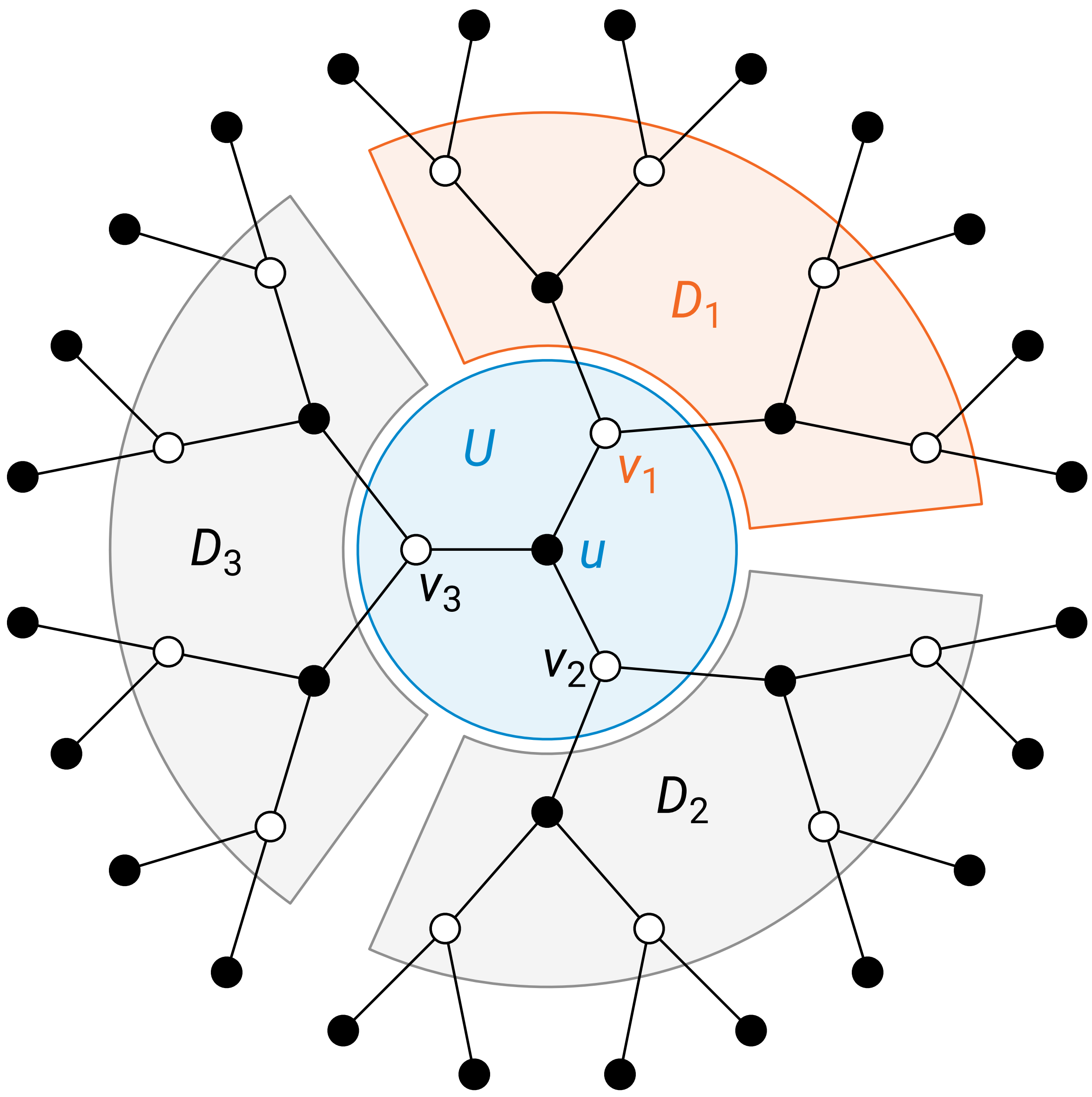
Maximal matching, LOCAL model, O(f(Δ) + g(n))





Round elimination

Given: **white algorithm *A*** that runs in T = 2 rounds

- ***v*₁** in ***A*** sees ***U*** and ***D*₁**

Construct: **black algorithm *A'*** that runs in T − 1 = 1 rounds

- ***u*** in ***A'*** only sees ***U***

A': what is the **set of possible outputs of *A*** for edge {***u***, ***v*₁**} over all possible inputs in ***D*₁**?