Local approximation algorithms for vertex cover

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Joint work with Matti Åstrand, Patrik Floréen,
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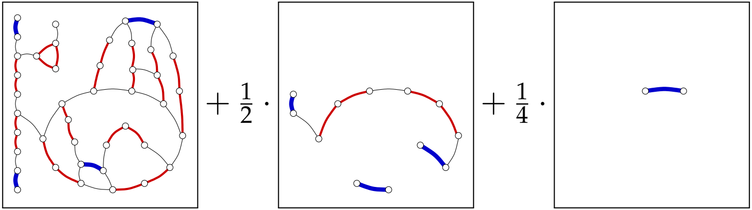
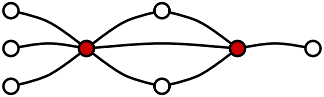


Paderborn, 20 October 2009

□ Part I: Introduction

Vertex cover problem in a distributed setting

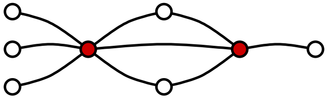


Vertex cover

Given a graph $\mathcal{G} = (V, E)$, find a smallest $C \subseteq V$ that covers every edge of $\mathcal{G}$

- i.e., each edge $e \in E$ incident to at least one node in C

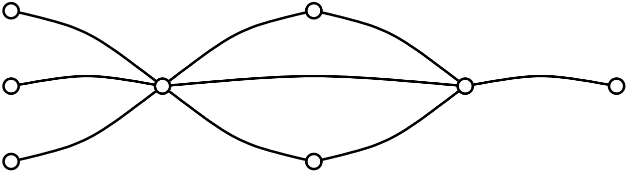
Classical NP-hard optimisation problem



Vertex cover in a distributed setting

Node $=$ computer
Edge $=$ communication link

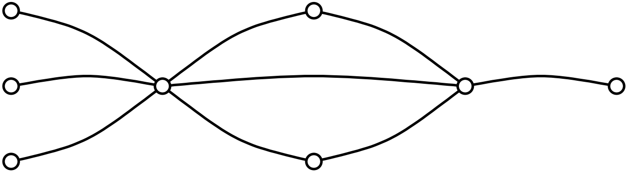
Each node must decide whether it is in the cover C



Vertex cover in a distributed setting

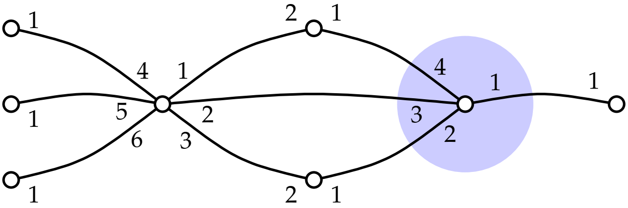
Graph is unknown, all nodes run the same algorithm

Initially: Each node knows its own degree
and the maximum degree Δ



☐ Vertex cover in a distributed setting

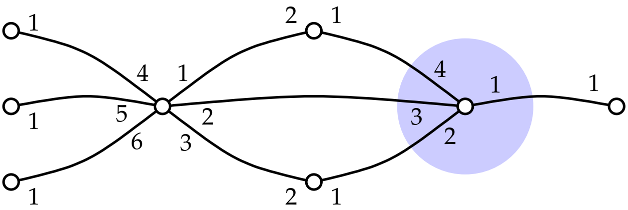
Port numbering: each node has chosen an ordering on its incident edges



Vertex cover in a distributed setting

Communication primitives:

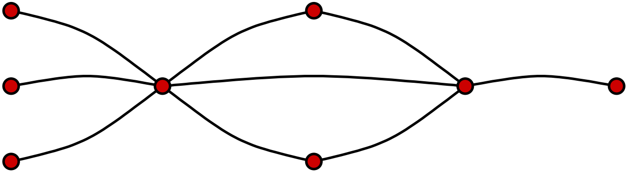
- "send message m to port i"
- "let m be the message received from port i"



Vertex cover in a distributed setting

Synchronous communication round: Each node

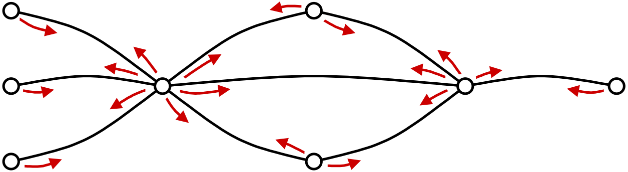
1. performs local computation



Vertex cover in a distributed setting

Synchronous communication round: Each node

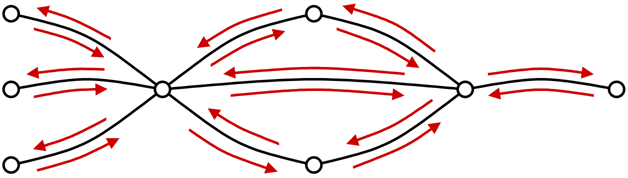
1. performs local computation
2. sends a message to each neighbour



Vertex cover in a distributed setting

Synchronous communication round: Each node

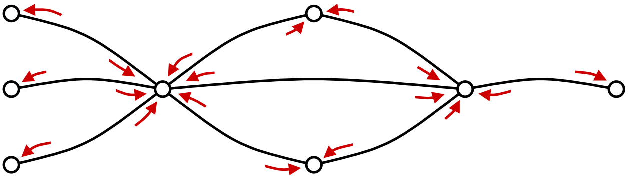
1. performs local computation
2. sends a message to each neighbour
   (message propagation. . .)



Vertex cover in a distributed setting

Synchronous communication round: Each node

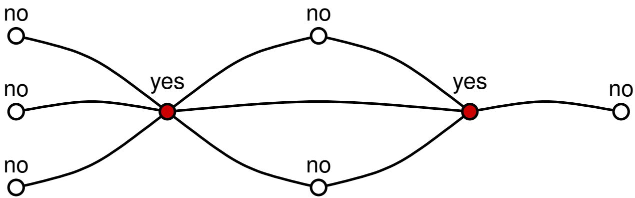
1. performs local computation
2. sends a message to each neighbour
3. receives a message from each neighbour



Vertex cover in a distributed setting

Finally: Each node performs local computation and announces its output: whether it is in the cover C

Running time = number of communication rounds



☐ Vertex cover in a distributed setting

Focus:

- deterministic algorithm
- strictly *local algorithm*,
  running time independent of $n = |V|$
  (but may depend on maximum degree Δ)
- *the best possible approximation ratio*

Prior work

Kuhn et al. (2006):

- $(2 + \epsilon)$-approximation in $O(\log \Delta / \epsilon^4)$ rounds

Czygrinow et al. (2008), Lenzen & Wattenhofer (2008):

- $(2 - \epsilon)$-approximation requires $\Omega(\log^* n)$ rounds, even if $\Delta = 2$

What about 2-approximation?

Is it possible in $f(\Delta)$ rounds, for some f?

☐ Contribution

Deterministic 2-approximation algorithm for vertex cover

- Running time $O(\Delta)$ synchronous rounds

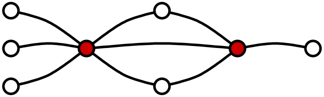
Surprise: node identifiers not needed

- Negative result for $(2 - \epsilon)$-approximation holds even if there are *unique node identifiers*

- Our algorithm can be used in *anonymous networks*

Part II: Background

Maximal matchings and edge packings

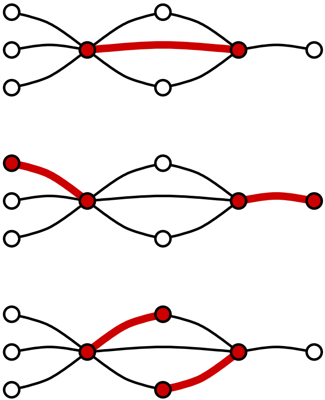


Background: maximal matching

In a centralised setting,
2-approximation is easy:
find a *maximal matching*,
take all matched nodes

But matching requires
$\Omega(\log^* n)$ rounds
and unique identifiers

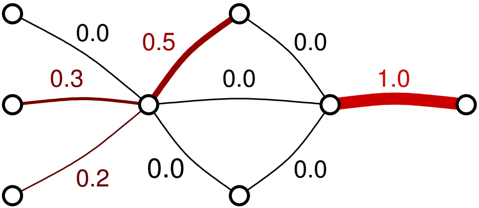
- symmetry breaking!



Background: maximal edge packing

Edge packing = nonnegative edge weights,
for each $v \in V$, total weight on incident edges ≤ 1

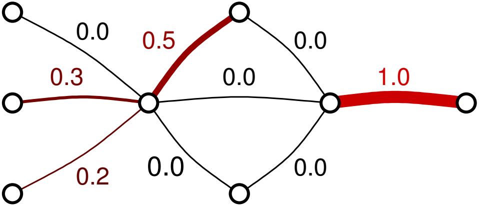
Maximal, if no weight can be increased



□ Background: maximal edge packing

Weighted edge packing = nonnegative edge weights,
for each $v \in V$, total weight on incident edges $\leq w_v$

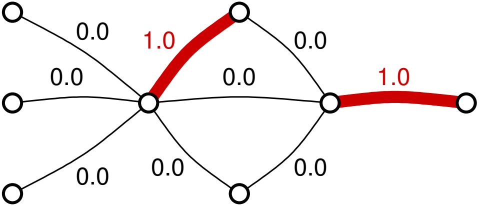
Maximal, if no weight can be increased



Background: maximal edge packing

Maximal matching $\implies$ maximal edge packing

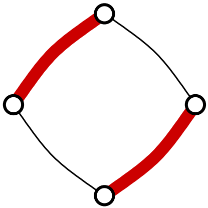
(matched: weight 1, unmatched: weight 0)

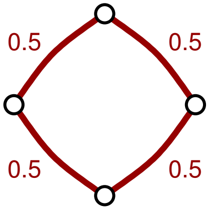


Background: maximal edge packing

Maximal matching requires symmetry breaking

Maximal edge packing does not

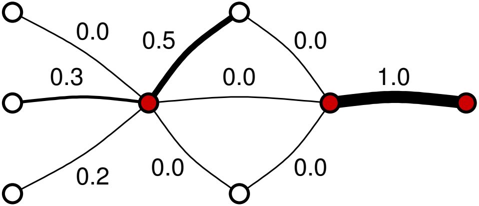




Background: maximal edge packing

Node *saturated* if total weight on incident edges $= 1$

Saturated nodes in a maximal edge packing $=$
2-approximation of vertex cover (proof: LP duality)



Background: maximal edge packing

Node *saturated* if total weight on incident edges $= 1$

Saturated nodes in a maximal edge packing $=$
2-approximation of vertex cover

$* * *$

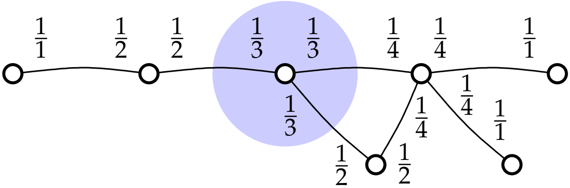
So we only need to design a distributed algorithm that finds a maximal edge packing

Warm-up: how to find a (non-trivial) edge packing?

Finding an edge packing

A simple approach: a node of degree d *offers* $1/d$ of its residual capacity to each incident edge

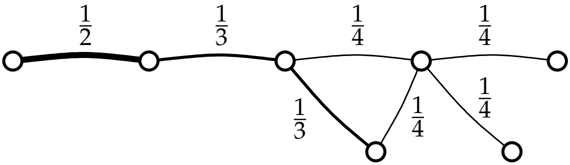
Residual capacity $= 1 -$ total weight of incident edges
$=$ how much we could increase the weights of incident edges



Finding an edge packing

Each edge *accepts* the minimum of the two offers

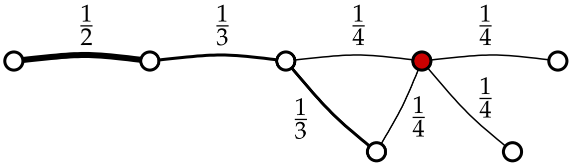
(cf. Khuller et al. 1994, Papadimitriou and Yannakakis 1993)



Finding an edge packing

Looks good, some progress is guaranteed,
and we might even saturate some nodes

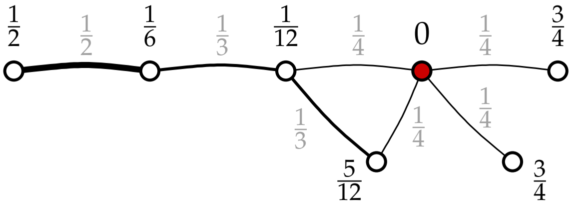
But this is not a maximal edge packing yet



Finding an edge packing

Residual capacities are now unwieldy fractions, even though our starting point was unweighted!

Unweighted instance $\Longrightarrow$ *weighted subproblems*



Finding an edge packing

Pessimist's take:

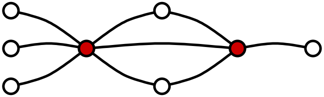
- Solving this will be as hard as finding maximal edge packings in weighted graphs
- Let's try something else

Optimist's take:

- If we solve this, we can also find maximal edge packings in weighted graphs
- Let's do it!

Part III: Pessimist's algorithm

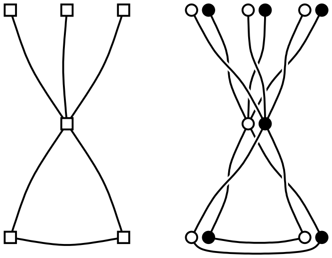
Finding maximal edge packings
in unweighted graphs



Finding an edge packing

Construct a 2-coloured *bipartite double cover*

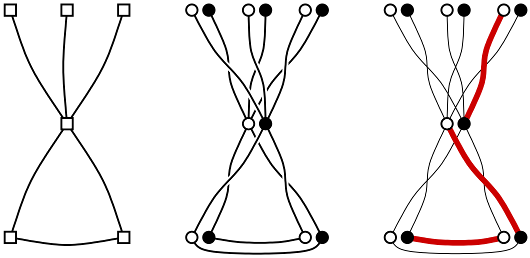
Each original node simulates two nodes of the cover



Finding an edge packing

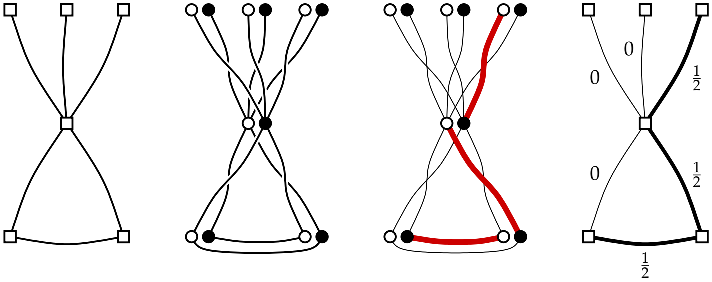
Find a maximal matching in the 2-coloured graph

Easy in $O(\Delta)$ rounds



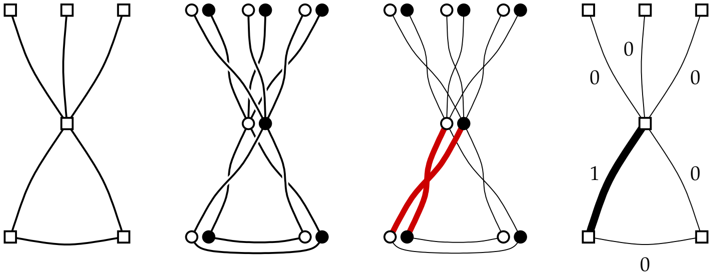
☐ Finding an edge packing

Give $\frac{1}{2}$ units of weight to each edge in matching



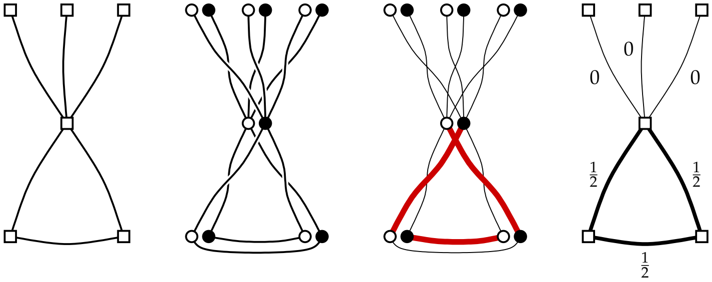
Finding an edge packing

Many possibilities. . .



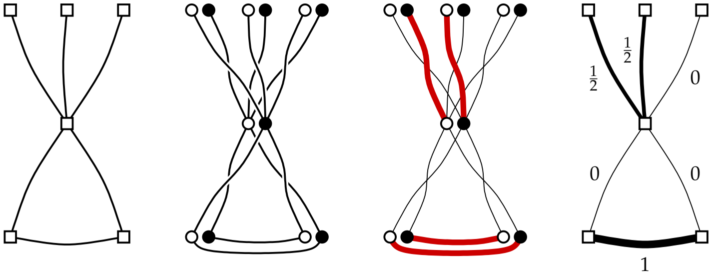
Finding an edge packing

Many possibilities. . .



Finding an edge packing

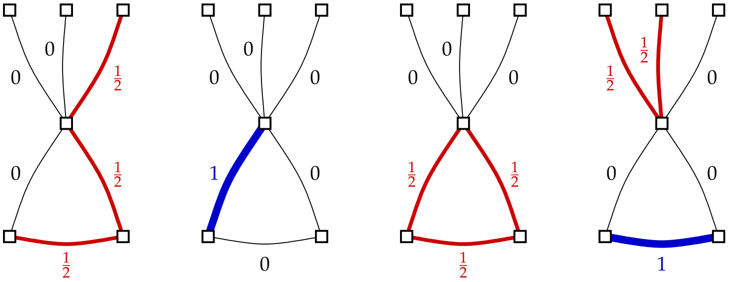
Many possibilities...



Finding an edge packing

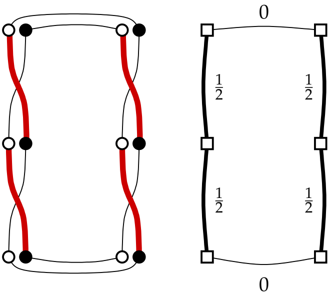
Always: weight $\frac{1}{2}$ paths and cycles and weight 1 edges

Valid edge packing



Finding a maximal edge packing

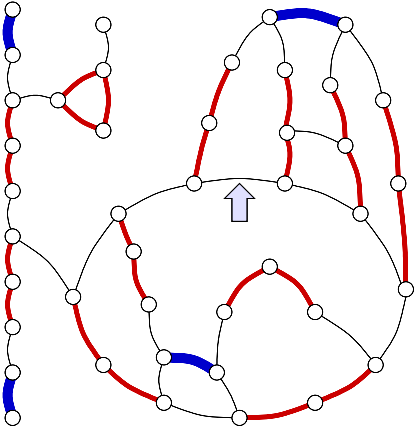
Not necessarily maximal – but all unsaturated edges adjacent to two weight $\frac{1}{2}$ edges



☐ Finding a maximal edge packing

In any graph:

Unsaturated edges adjacent to two weight $\frac{1}{2}$ edges



$\boxed{\Delta = 3}$

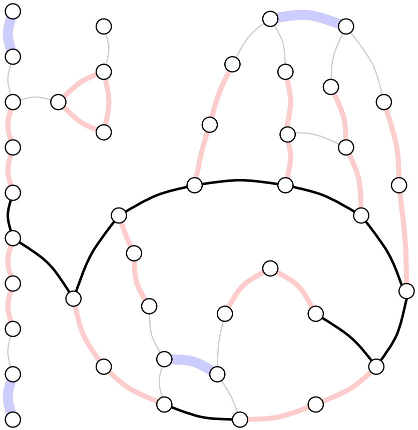
Finding a maximal edge packing

In any graph:

Unsaturated edges adjacent to two weight $\frac{1}{2}$ edges

Delete saturated edges

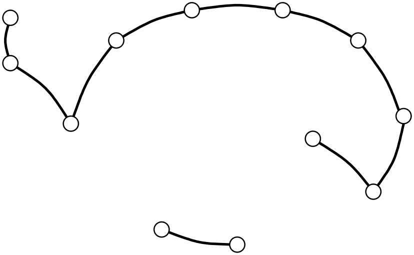
$\boxed{\Delta = 3 \rightarrow \Delta = 2}$



Finding a maximal edge packing

Each node has lost
at least one neighbour

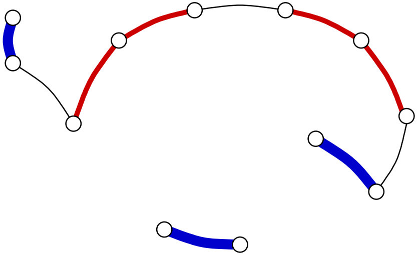
Residual capacity
of each node is
exactly $\frac{1}{2}$



$\boxed{\Delta = 3 \;\rightarrow\; \Delta = 2}$

Finding a maximal edge packing

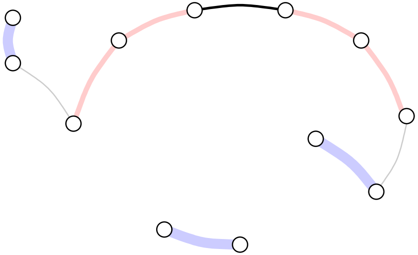
Repeat



$\boxed{\Delta = 2}$

Finding a maximal edge packing

Delete saturated edges



$\boxed{\Delta = 2 \;\rightarrow\; \Delta = 1}$

Finding a maximal edge packing

Each node has lost
at least one neighbour

Residual capacity
of each node is
exactly $\frac{1}{4}$

$\boxed{\Delta = 2 \;\rightarrow\; \Delta = 1}$

Finding a maximal edge packing

Repeat…

$\boxed{\Delta = 1}$

Finding a maximal edge packing

Repeat…

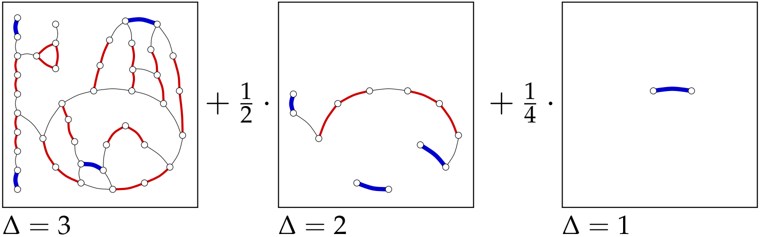
Maximum degree decreases
on each iteration

Everything saturated in
Δ iterations

Finding a maximal edge packing

Maximal edge packing in $(\Delta + 1)^2$ rounds

$\implies$ 2-approximation of vertex cover



□ Finding a maximal edge packing

Maximal edge packing in $(\Delta + 1)^2$ rounds

$\implies$ 2-approximation of vertex cover

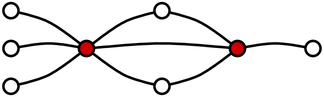
$$* * *$$

But it seems that this cannot be generalised
to approximate minimum-weight vertex cover

A different approach needed

Part IV: Optimist's algorithm

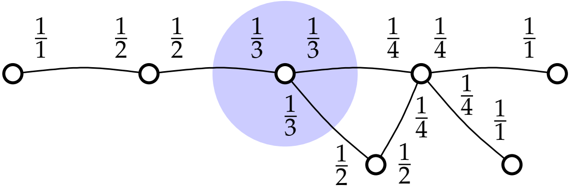
Finding maximal edge packings
in weighted graphs



Finding an edge packing

Recall the simple algorithm: a node of degree d *offers* $1/d$ of its residual capacity to each incident edge

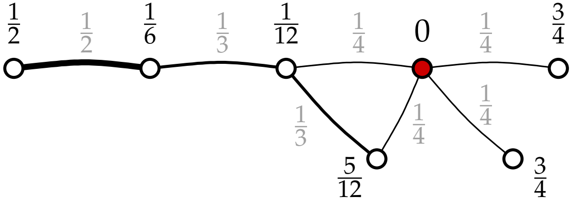
Each edge *accepts* the minimum of the two offers



Finding an edge packing

Starting point has non-uniform capacities,
ok if subproblems have non-uniform capacities!

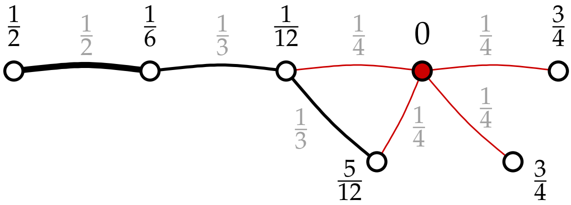
Let's study this approach more carefully…



Finding an edge packing

Key observation: For each node

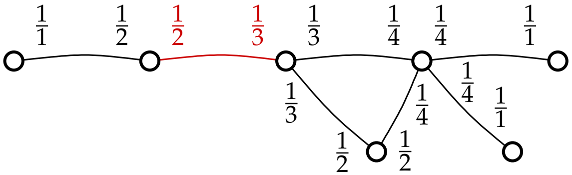
1. at least one incident edge becomes *saturated* (= cannot increase edge weight), or ...



Finding an edge packing

Key observation: for each node

1. at least one incident edge becomes *saturated*, or
2. at least one incident edge got *two different offers*



Finding an edge packing

Key observation: for each node

1. at least one incident edge becomes *saturated*, or
2. at least one incident edge got *two different offers*

We can interpret the offers as “colours”

Progress is guaranteed:
edges become saturated or multi-coloured

Finding an edge packing

After Δ iterations: each edge saturated or multi-coloured

At this point, colours are huge integers

$$1, 2, \ldots, \left(W(\Delta!)^{\Delta}\right)^{\Delta}$$

but Cole–Vishkin (1986) techniques can be used
to reduce the number of colours to $\Delta + 1$ very fast

Then we can use the colours to saturate all edges

($W =$ maximum weight)

Finding an edge packing

In summary, maximal edge packing in $O(\Delta + \log^* W)$ rounds, where $W =$ maximum weight

That is, $O(\Delta)$ rounds in unweighted graphs!

- pessimist's algorithm was $O(\Delta^2)$

Based on a natural "greedy but safe" strategy

- pessimist's algorithm was more ad hoc?

Generalisations: set cover problem, …

Summary

- Two distributed 2-approximation algorithms for the vertex cover problem
- Running times: $O(\Delta^2)$ and $O(\Delta)$ rounds, deterministic, can be self-stabilised
- Strictly local algorithms – running time independent of number of nodes
- Be optimistic: more general problems are sometimes easier to tackle

http://www.cs.helsinki.fi/jukka.suomela/