Locality lower bounds through round elimination

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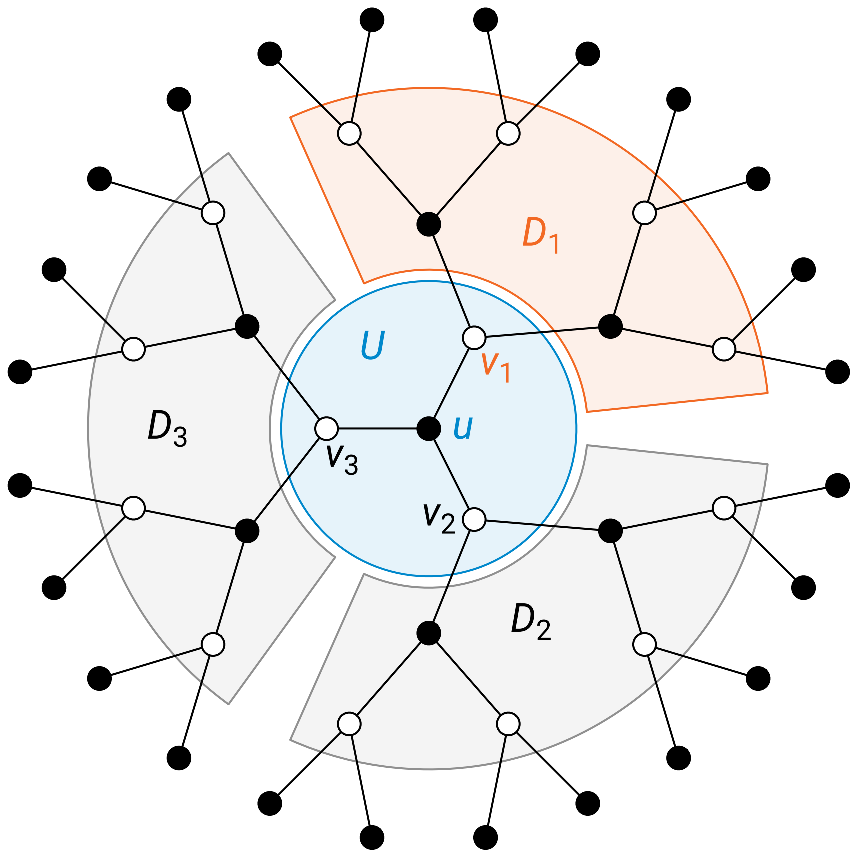
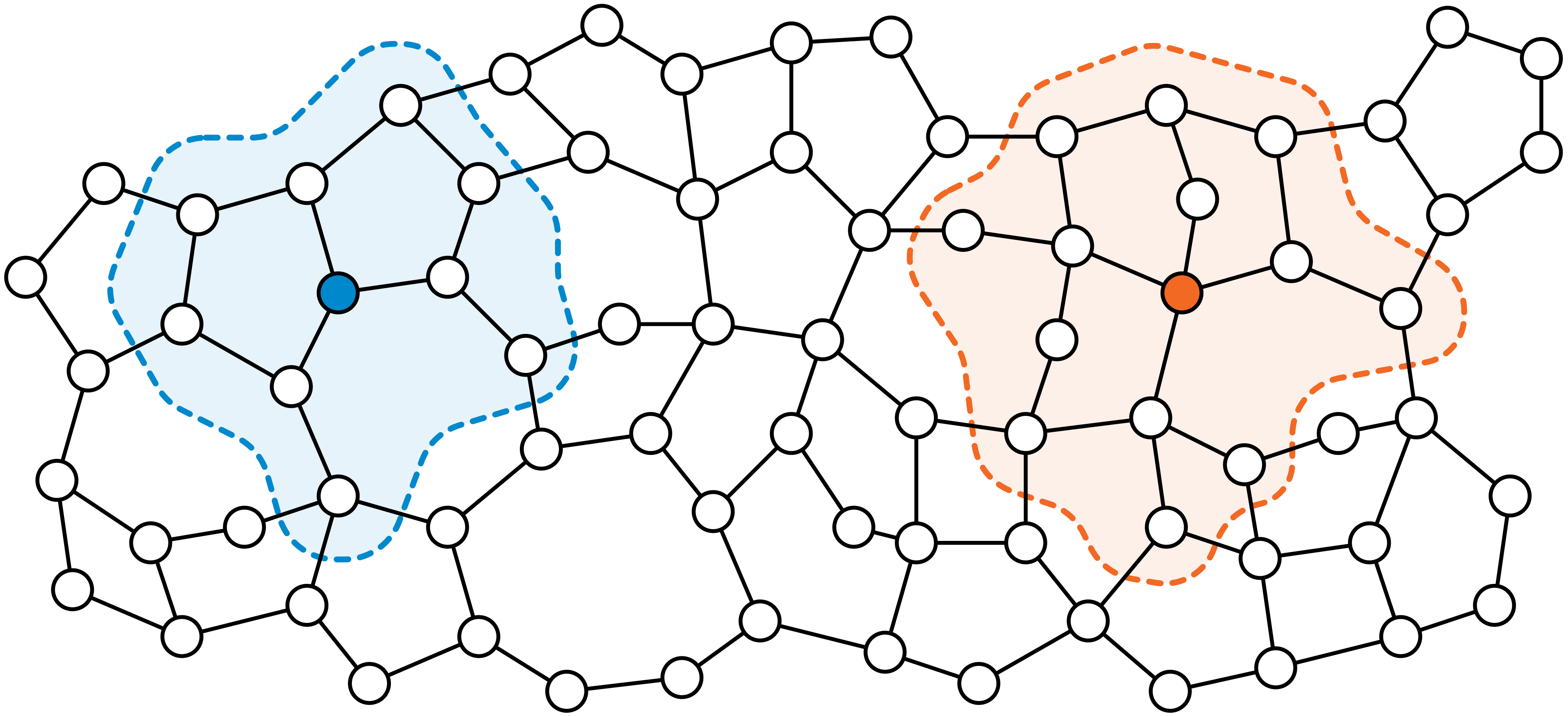


Joint work with

- Alkida Balliu
- Sebastian Brandt
- Orr Fischer
- Juho Hirvonen
- Barbara Keller
- Tuomo Lempiäinen
- Dennis Olivetti
- Mikaël Rabie
- Joel Rybicki
- Jara Uitto

Locality = how far do I need to see to produce my own part of the solution?



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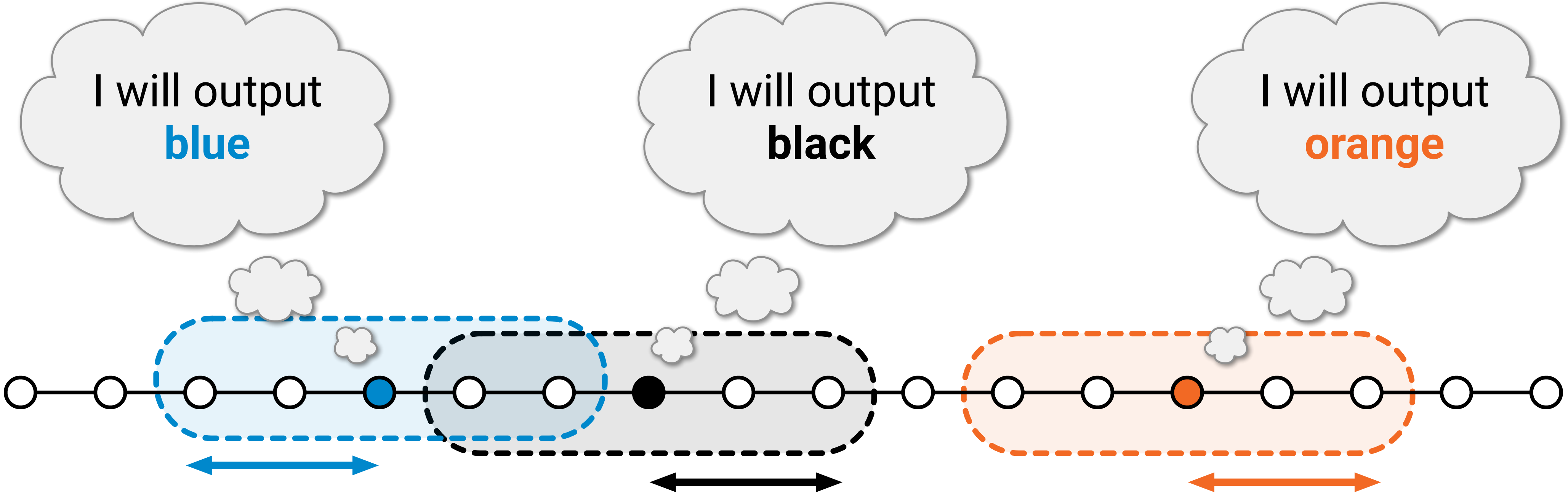
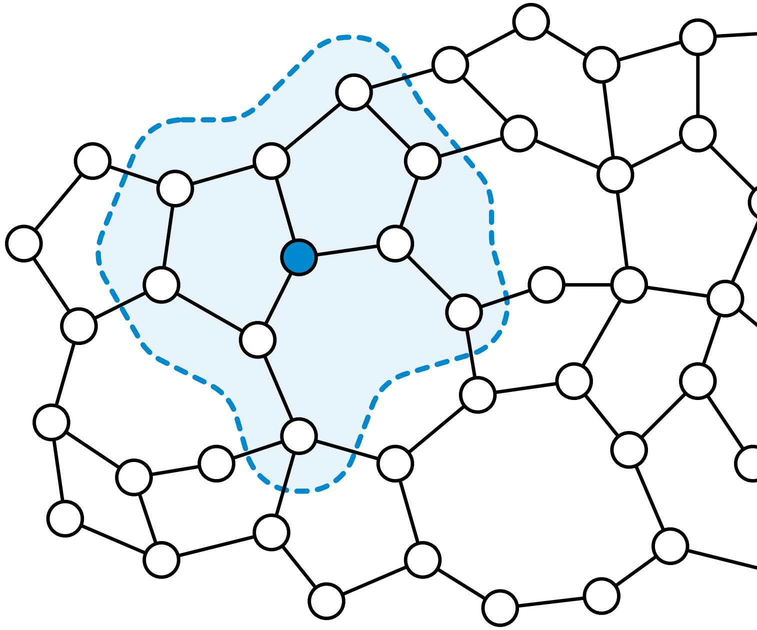


Locality = how far do I need to see to produce my own part of the solution?

Local outputs form a globally consistent solution

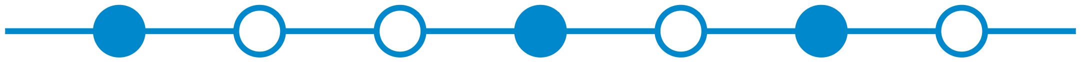
Locality: formalization



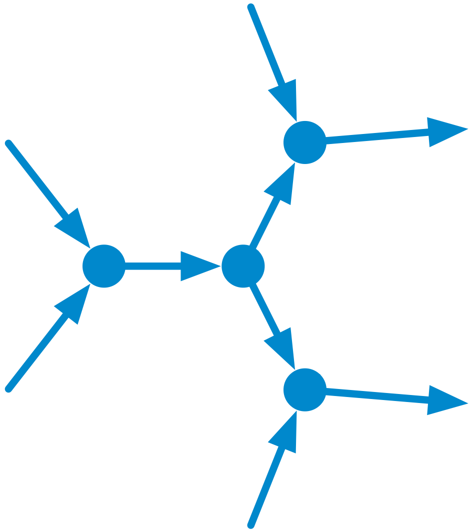
- "**LOCAL**" model of distributed computing:
  - *graph = communication network*
    - **node** = processor
    - **edge** = communication link
    - all nodes have unique identifiers
  - *time = number of communication rounds*
    - **round** = nodes exchange messages with all neighbors
    - 1 communication round: all nodes can learn everything within distance 1
    - *T* communication rounds: all nodes can learn everything within distance *T*
- *Time = distance*

Locality: examples

- Setting: graph with ***n* nodes**, maximum degree **Δ = *O*(1)**
- **Maximal independent set:**
  Θ(log* *n*) randomized,
  Θ(log* *n*) deterministic



- **Sinkless orientation:**
  Θ(log log *n*) randomized,
  Θ(log *n*) deterministic
  - orient edges such that all nodes of degree ≥ 3 have outdegree ≥ 1



How to study locality?

Proving locality upper & lower bounds

Locality: proving upper bounds

- Find a ***function*** that maps local neighborhoods to local outputs
- Design a fast distributed ***message-passing algorithm***
- Design a slow distributed algorithm and apply ***"speedup"*** arguments to turn it into a fast distributed algorithm
  - e.g. $o(n) \to O(\log^* n)$ for "LCL problems" in cycles
- Design a fast ***centralized sequential*** algorithm model and turn it into a fast distributed algorithm
  - e.g. greedy strategy → SLOCAL algorithm → LOCAL algorithm

Locality: proving lower bounds

- ***Indistinguishability***
  - same local view → same output
- ***Adaptive constructions***
  - inductively construct a bad input
    for this specific algorithm
- ***Ramsey-type arguments***
  - "monochromatic set" ≈ bad choice of identifiers
- ***Speedup & derandomization arguments and reductions***
  - locality R → locality R' → not possible

Locality: proving lower bounds

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Today’s focus: “round elimination” technique for proving locality lower bounds

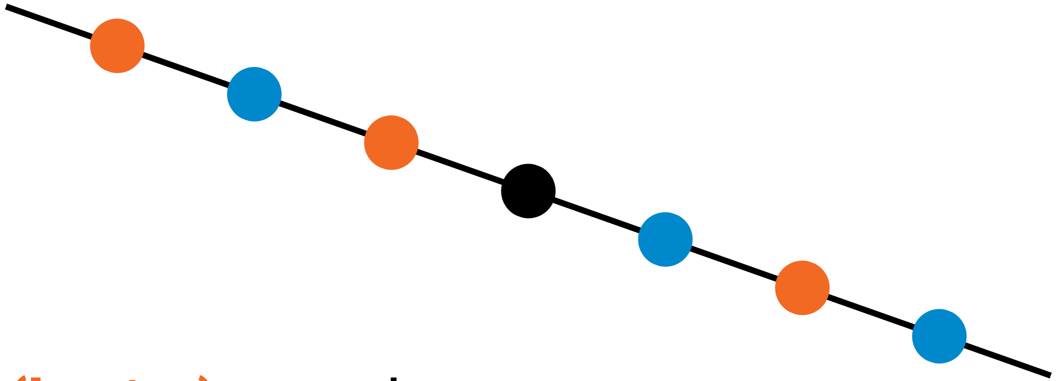
Round elimination

Round elimination technique

- **Given:**
  - algorithm A_0 solves problem P_0 in T rounds
- **We construct:**
  - algorithm A_1 solves problem P_1 in $T - 1$ rounds
  - algorithm A_2 solves problem P_2 in $T - 2$ rounds
  - algorithm A_3 solves problem P_3 in $T - 3$ rounds

  …
  - algorithm A_T solves problem P_T in 0 rounds
- But P_T is nontrivial, so A_0 cannot exist

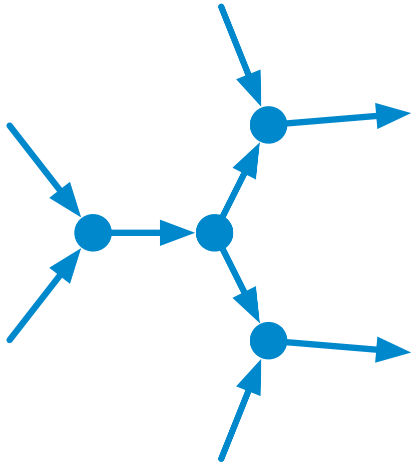
Linial (1987, 1992): coloring cycles



- **Given:**
  - algorithm A_0 solves **3-coloring** in $T = o(\log^* n)$ rounds
- **We construct:**
  - algorithm A_1 solves 2^3**-coloring** in $T - 1$ rounds
  - algorithm A_2 solves 2^{2^3}**-coloring** in $T - 2$ rounds
  - algorithm A_3 solves $2^{2^{2^3}}$**-coloring** in $T - 3$ rounds

    …
  - algorithm A_T solves $o(n)$**-coloring** in **0** rounds
- But $o(n)$**-coloring** is nontrivial, so A_0 cannot exist

Brandt et al. (2016): sinkless orientation



- **Given:**
  - algorithm A_0 solves **sinkless orientation** in $T = o(\log n)$ rounds
- **We construct:**
  - algorithm A_1 solves **sinkless coloring** in $T - 1$ rounds
  - algorithm A_2 solves **sinkless orientation** in $T - 2$ rounds
  - algorithm A_3 solves **sinkless coloring** in $T - 3$ rounds

    …
  - algorithm A_T solves **sinkless orientation** in **0** rounds
- But **sinkless orientation** is nontrivial, so A_0 cannot exist

Round elimination can be automated

Brandt 2019
Olivetti 2019

- **Good news:** always possible for **any graph problem *P_0*** that is "locally checkable"
  - if problem P_0 has complexity T, we can always find in a ***mechanical manner*** problem P_1 that has complexity $T - 1$
  - holds for tree-like neighborhoods (e.g. high-girth graphs)
- **Bad news:** this does not directly give a lower bound
  - P_1 is not necessarily any **natural graph problem**
  - P_1 does not necessarily have a **small description**
  - how do we prove that P_1, P_2, P_3, etc. are **nontrivial problems**?

Round elimination and fixed points

- Sometimes we are very lucky:
  - **P_0 = sinkless orientation**
  - P_1 = something (no need to understand it)
  - **P_2 = sinkless orientation**
- If you are feeling optimistic: just apply round elimination in a mechanical manner for a small number of steps and see if your reach a ***fixed point*** or ***cycle***
  - or you reach a well-known hard problem
- Open question: ***exactly when does this happen?***

Round elimination and "rounding down"

- Sometimes some amount of creativity is needed:
  - **P_0 = k-coloring cycles**
  - P_1 = something complicated with 2^k possible output labels
  - **define: Q_1 = 2^k-coloring cycles**
  - **observation:** solution to P_1 implies a solution to Q_1

P_0 takes ***exactly T*** rounds
$\to$ P_1 takes ***exactly T* – 1** rounds
$\to$ Q_1 takes ***at most T* – 1** rounds
$\to$ …
$\to$ Q_T takes ***at most* 0** rounds

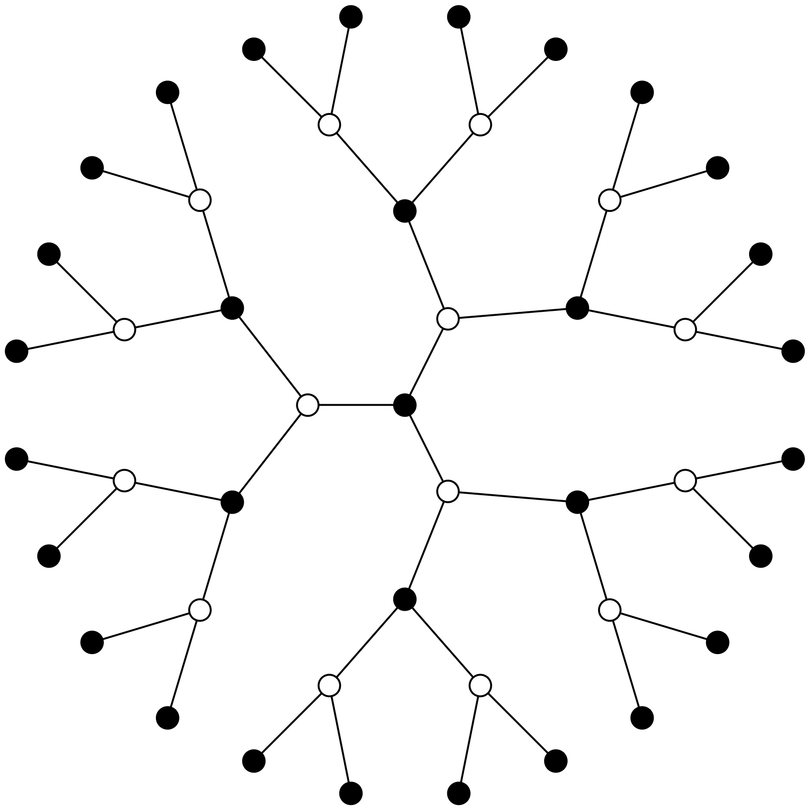
How does it work?

Correct formalism

- We will need the ***right formalism*** for the graph problems that we study
- It will look seemingly arbitrary and very restrictive at first
- No worries, you can ***encode*** a broad range of **locally checkable problems** in this formalism with some effort
  - maximal matching, maximal independent set,
    vertex coloring, edge coloring, sinkless orientation …

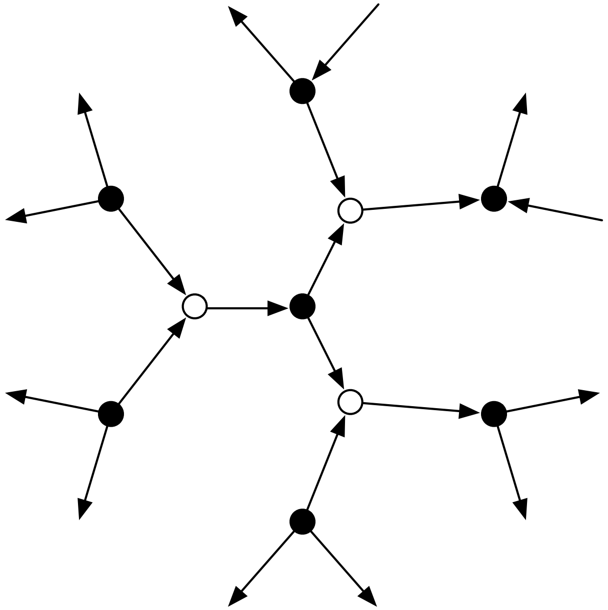
Correct formalism: edge labeling in bipartite graphs

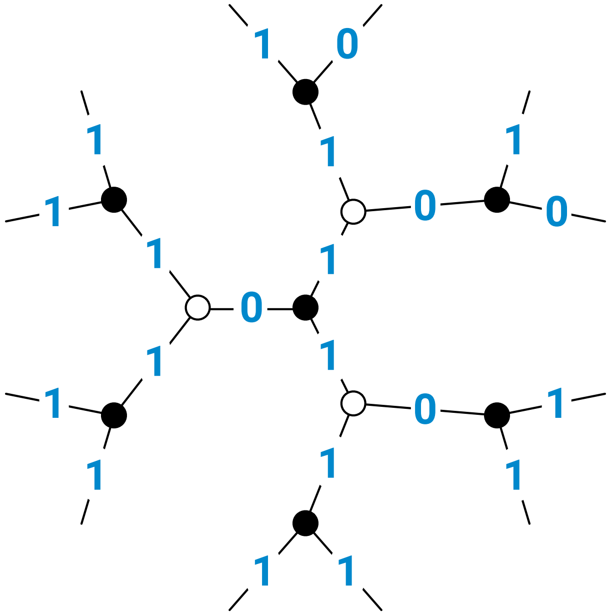
- Assumption: input graph properly 2-colored (“*white*” / “***black***”)
- Finite set of possible **edge labels**
- ***White*** constraint:
  - feasible multiset of labels on edges adjacent to a white node
- ***Black*** constraint:
  - feasible multiset of labels on edges adjacent to a black node



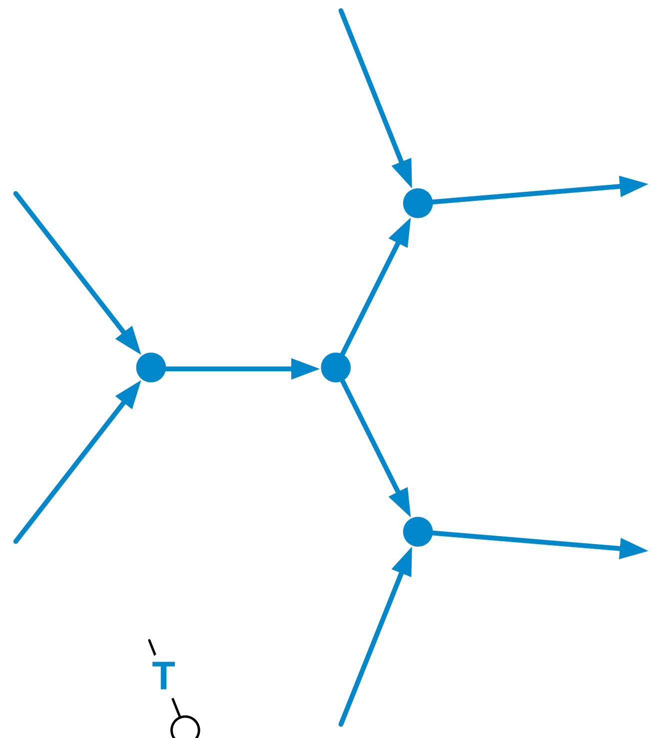
Example 1: sinkless orientation

- Setting: bipartite 3-regular graphs
- Encoding: use original graph
  - "0" = orient from white to black
  - "1" = orient from black to white
- *White* constraint:
  - {0, 0, 0}, {0, 0, 1} or {0, 1, 1}
- ***Black*** constraint:
  - {0, 0, 1}, {0, 1, 1} or {1, 1, 1}

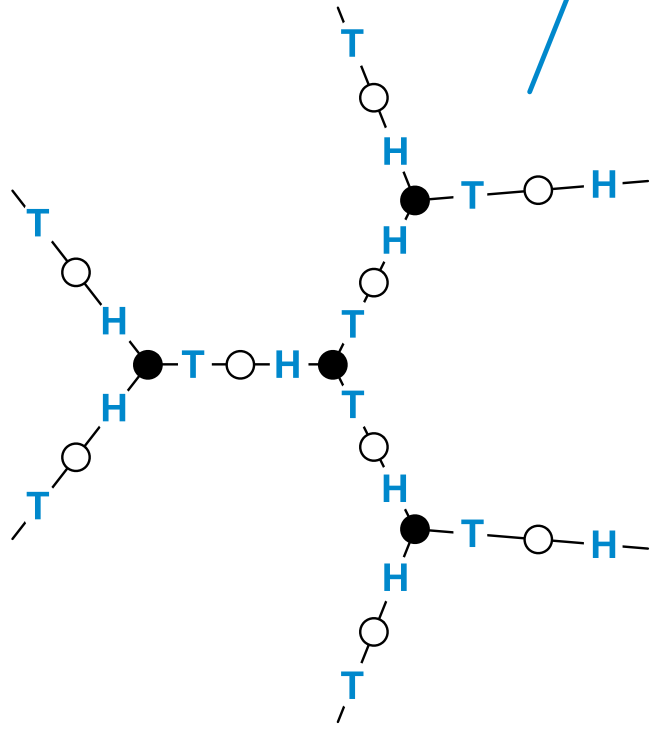




Example 2: sinkless orientation

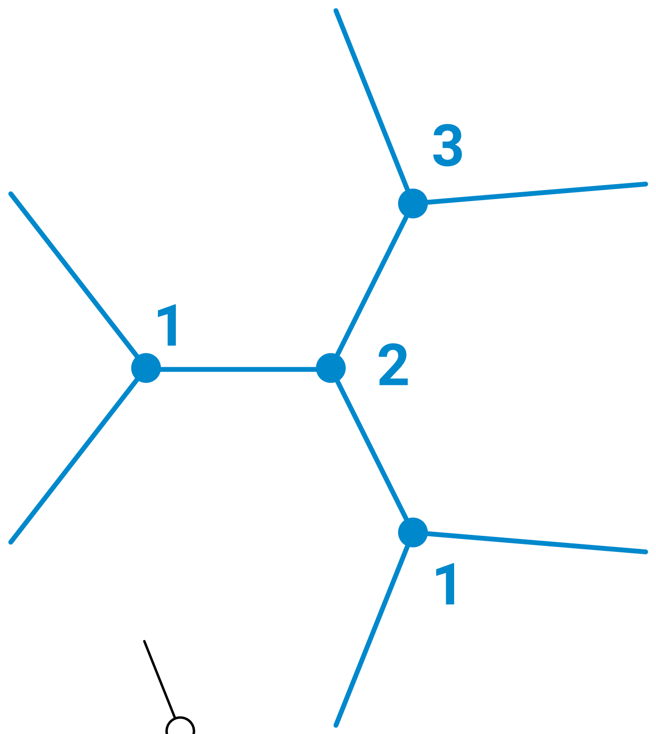


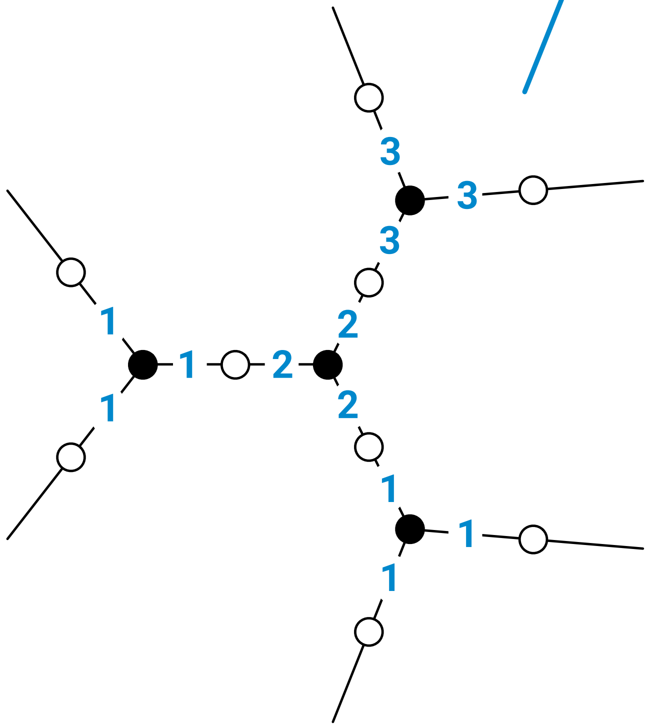
- Setting: 3-regular graphs
- Encoding: subdivide edges
  - ***white*** = edge, ***black*** = node
  - "H" = head, "T" = tail
- ***White*** constraint:
  - {H, T}
- ***Black*** constraint:
  - {H, H, T}, {H, T, T} or {T, T, T}



Example 3: vertex coloring

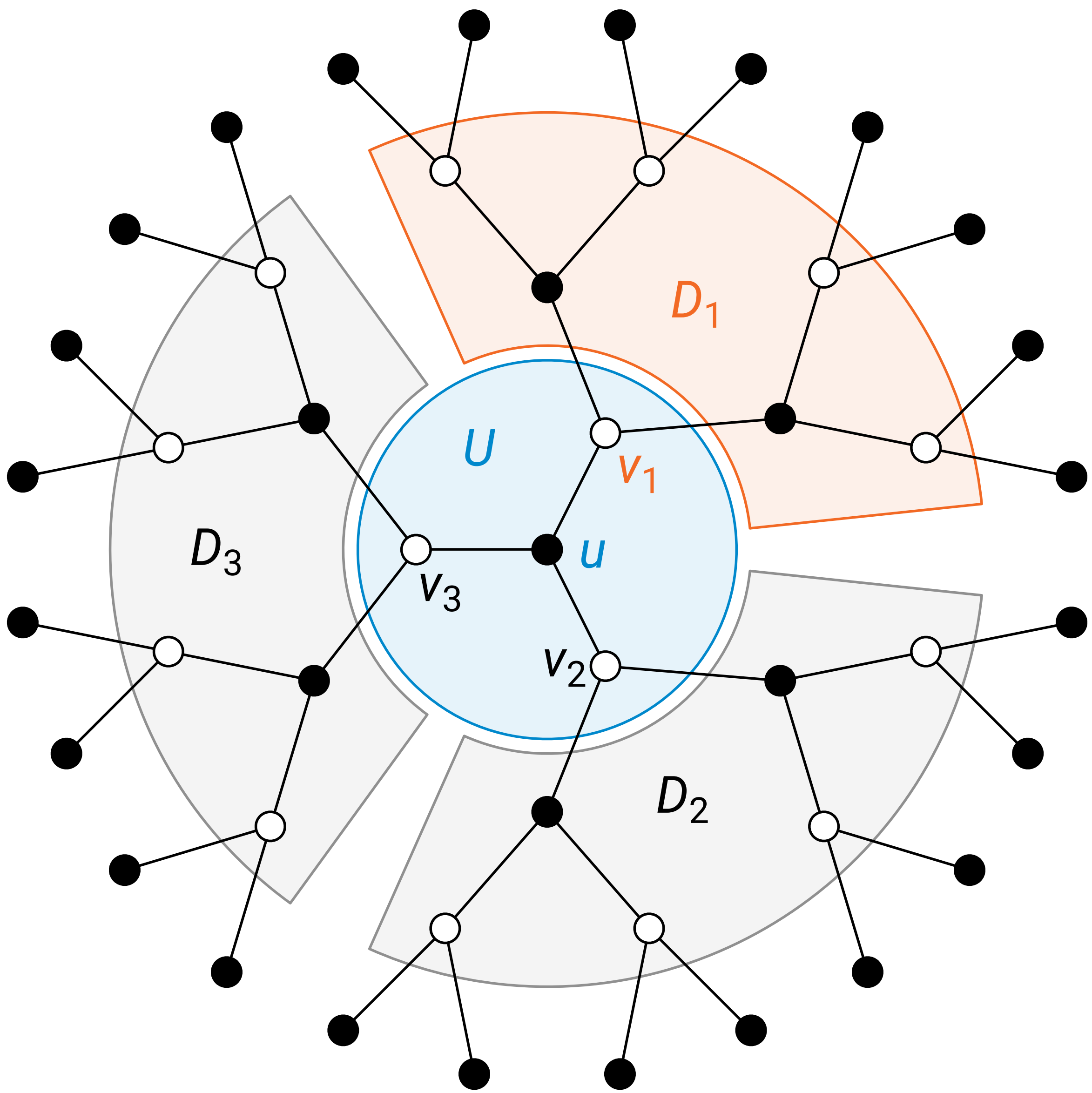
- Setting: 3-regular graphs
- Encoding: subdivide edges
  - *white* = edge, ***black*** = node
  - "1", "2", "3" = color of incident black node
- *White* constraint:
  - {1, 2} or {1, 3} or {2, 3}
- ***Black*** constraint:
  - {1, 1, 1}, {2, 2, 2} or {3, 3, 3}





Correct formalism: white and black algorithms

- ***White*** algorithm:
  - each ***white*** node produces labels on its incident edges
  - ***black*** nodes do nothing
  - satisfies white and black constraints
- ***Black*** algorithm:
  - each ***black*** node produces labels on its incident edges
  - ***white*** nodes do nothing
  - satisfies white and black constraints
- White and black complexity within ±1 round of each other



Round elimination

Given: **white algorithm *A*** that runs in *T* = 2 rounds

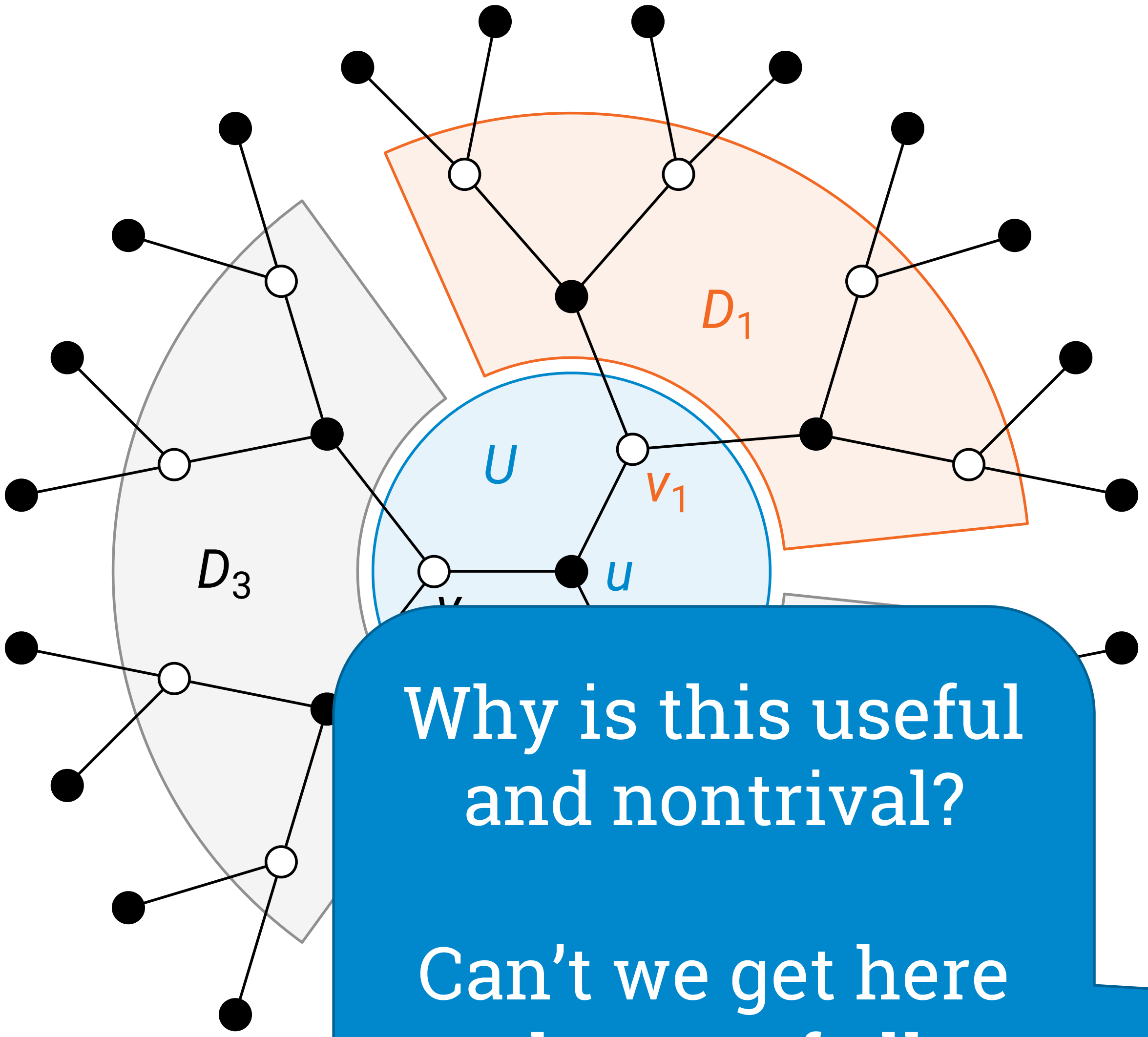
- ***v_1*** in ***A*** sees ***U*** and ***D_1***

Construct: **black algorithm *A*′** that runs in *T* − 1 = 1 rounds

- ***u*** in ***A*′** only sees ***U***

A*′**: what is the **set of possible outputs of *A for edge {***u***, ***v_1***} over all possible inputs in ***D_1***?

Round elimination



Why is this useful and nontrival?

Can’t we get here the set of all possible outputs?

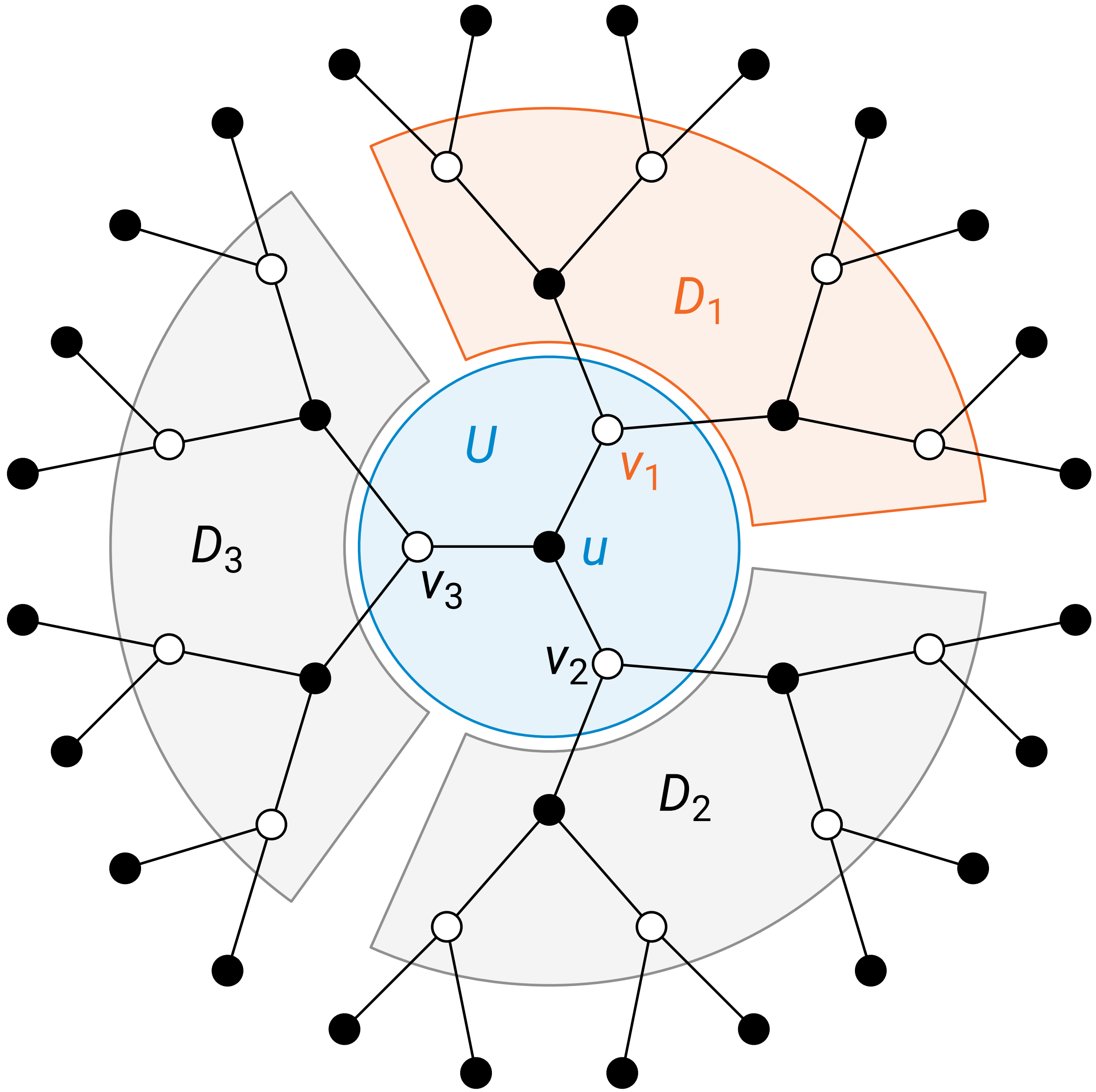
Given: **white algorithm *A*** that runs in $T = 2$ rounds

- v_1 in ***A*** sees ***U*** and D_1

Construct: **black algorithm *A’*** that runs in $T - 1 = 1$ rounds

- ***u*** in ***A’*** only sees ***U***

A’: what is the **set of possible outputs of *A*** for edge $\{u, v_1\}$ over all possible inputs in D_1?



Example: edge coloring

Independence!

- Assume there is some extension $\boldsymbol{D_1}$ such that $\boldsymbol{v_1}$ labels $\{\boldsymbol{u}, \boldsymbol{v_1}\}$ green
- Assume there is some extension $\boldsymbol{D_2}$ such that $\boldsymbol{v_2}$ labels $\{\boldsymbol{u}, \boldsymbol{v_2}\}$ green
- Then we can construct an input in which both $\{\boldsymbol{u}, \boldsymbol{v_1}\}$ and $\{\boldsymbol{u}, \boldsymbol{v_2}\}$ are green

Algorithm A' has to do something nontrivial

Here: sets incident to black nodes have to be non-empty and disjoint

They contain enough information so that we could recover a proper edge coloring in 1 extra round

Example: edge coloring

Independence!

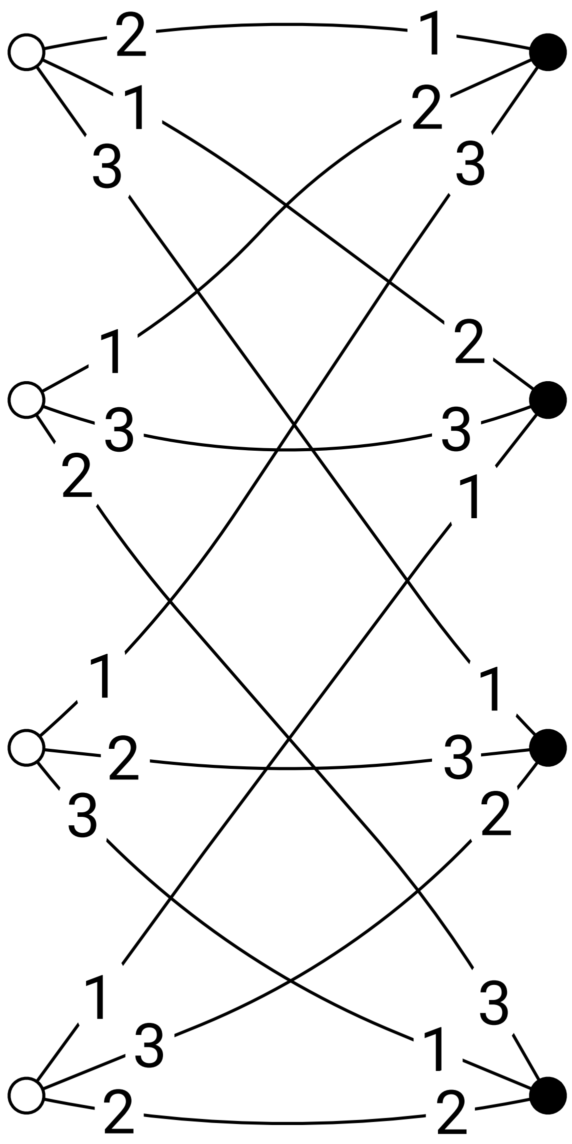
- Assume there is some extension D_1 such that v_1 labels $\{u, v_1\}$ green
- Assume there is some extension D_2 such that v_2 labels $\{u, v_2\}$ green
- Then we can construct an input in which both $\{u, v_1\}$ and $\{u, v_2\}$ are green ⚡

Example:
bipartite maximal matching

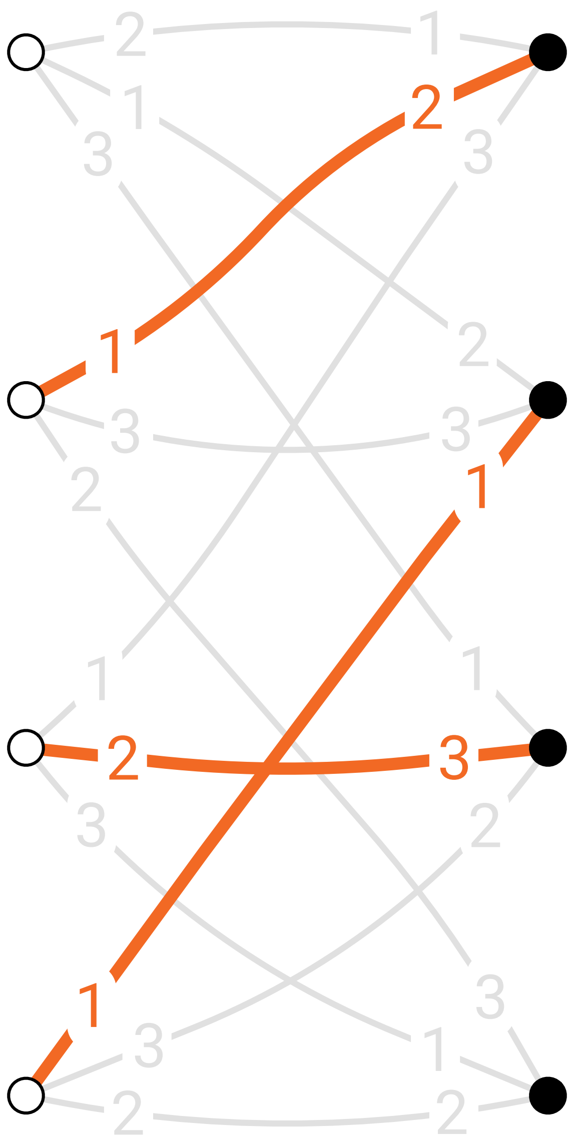
computer
network with
port numbering

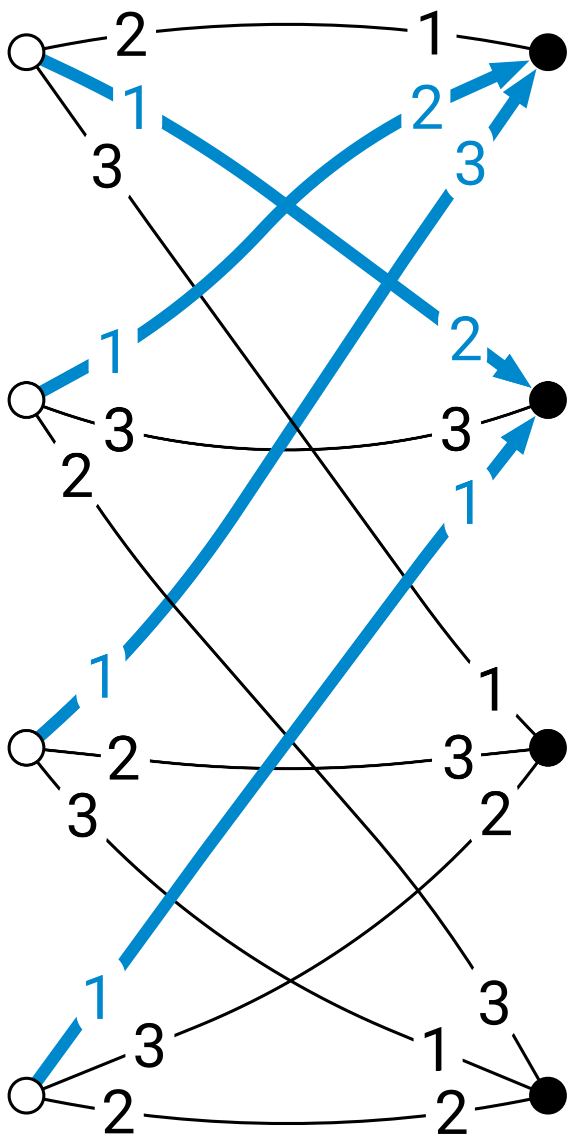
bipartite,
2-colored
graph

Δ-regular
(here Δ = 3)



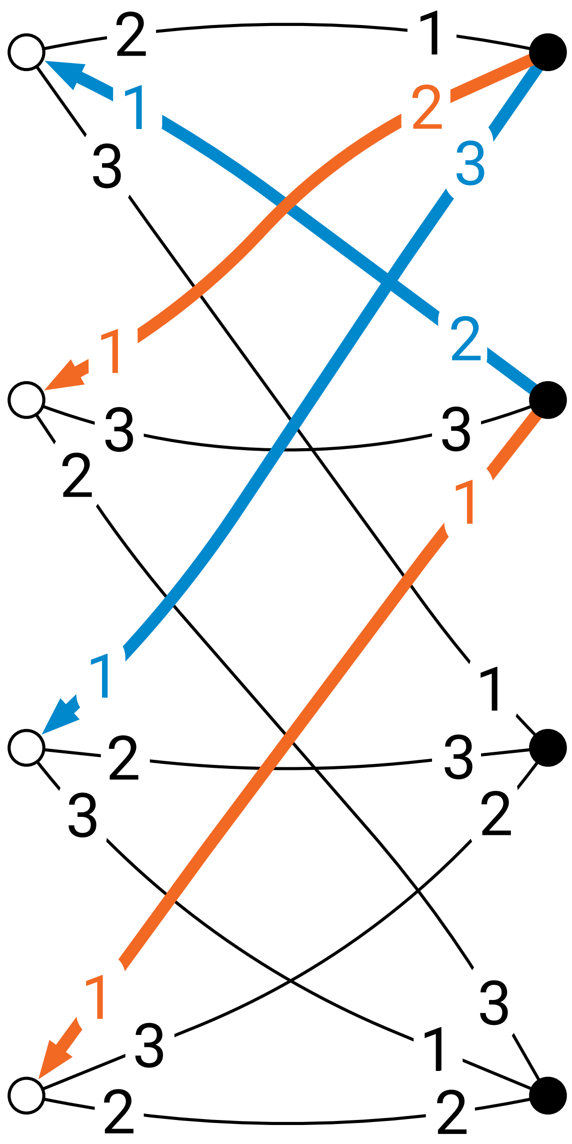
output:
maximal
matching





Very simple algorithm

unmatched white nodes:
send ***proposal*** to port 1



Very simple algorithm

unmatched white nodes:
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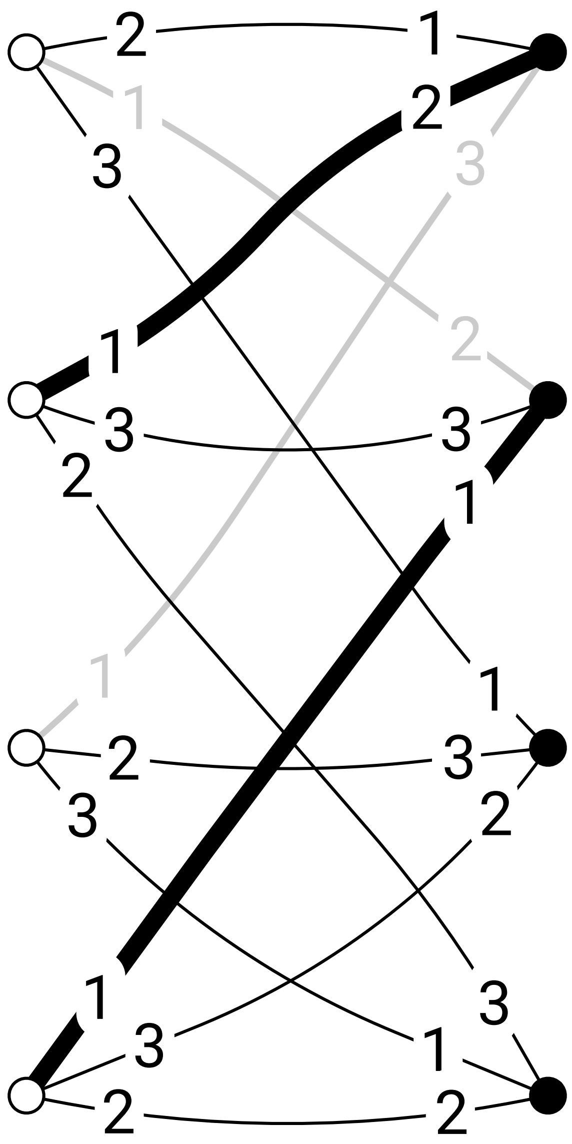
black nodes:
accept the first proposal you get, ***reject*** everything else (break ties with port numbers)



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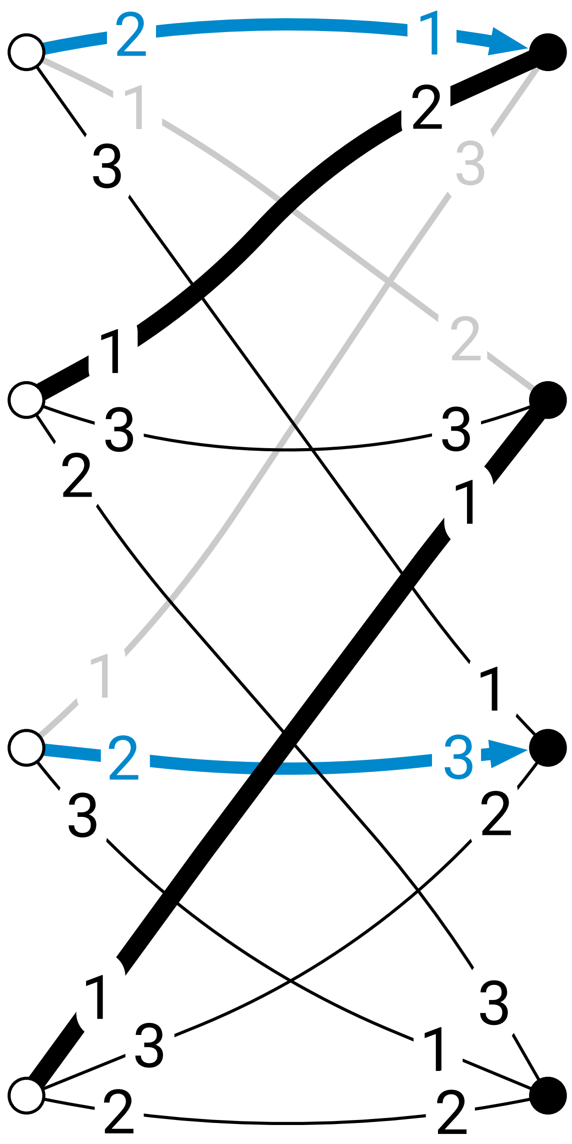
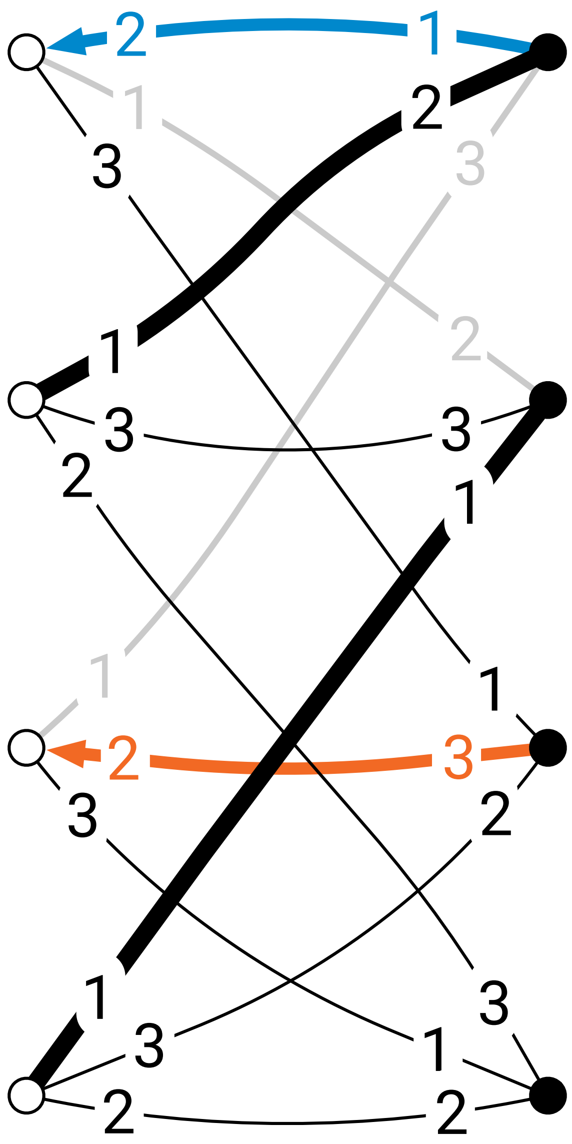


Very simple algorithm

unmatched white nodes:
send ***proposal*** to port 2



Very simple algorithm

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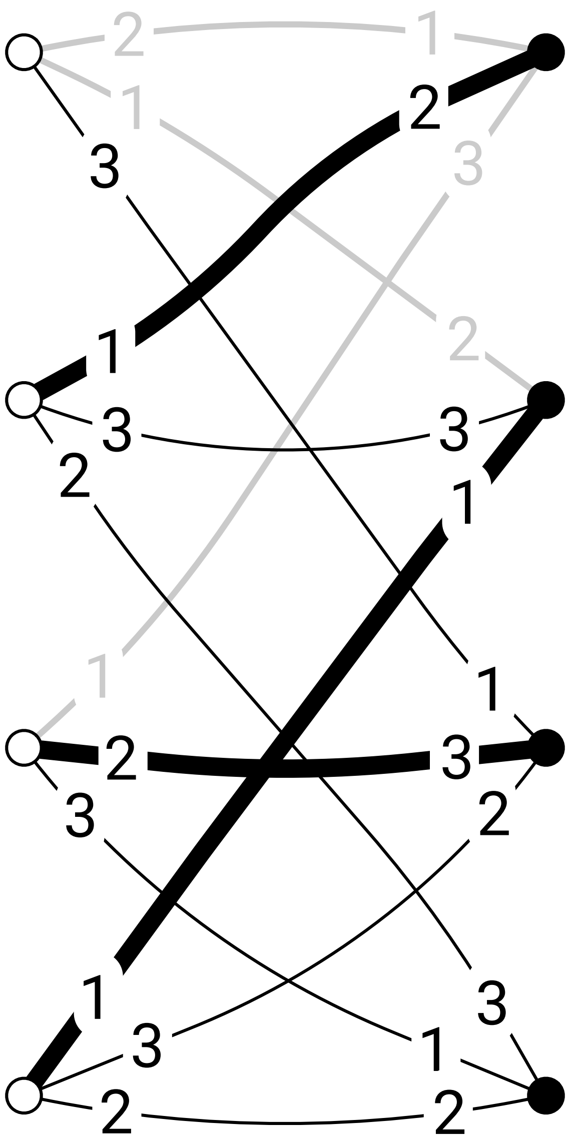

Very simple algorithm

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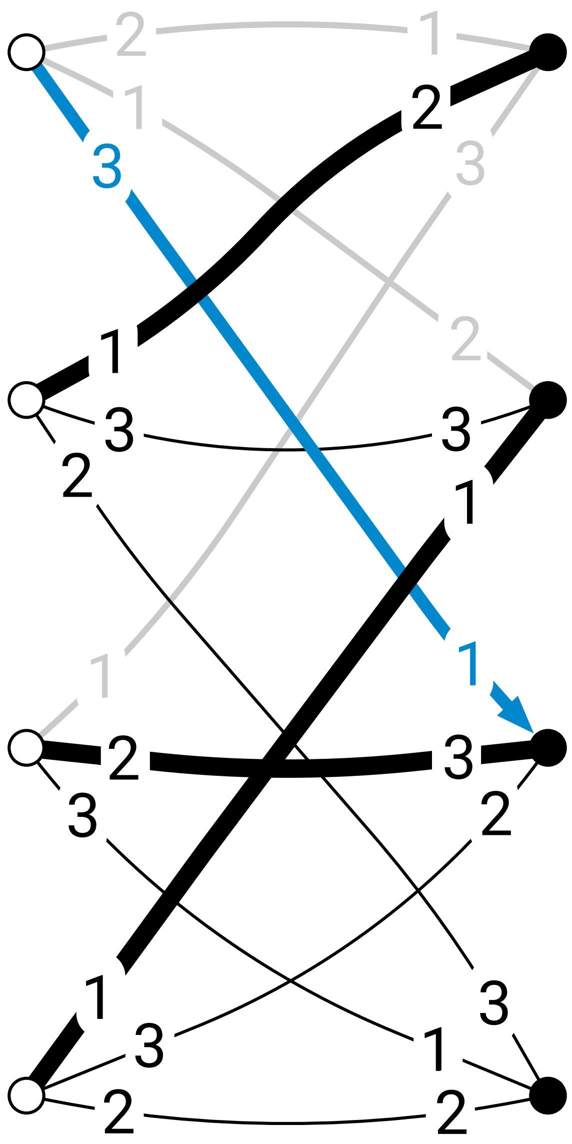
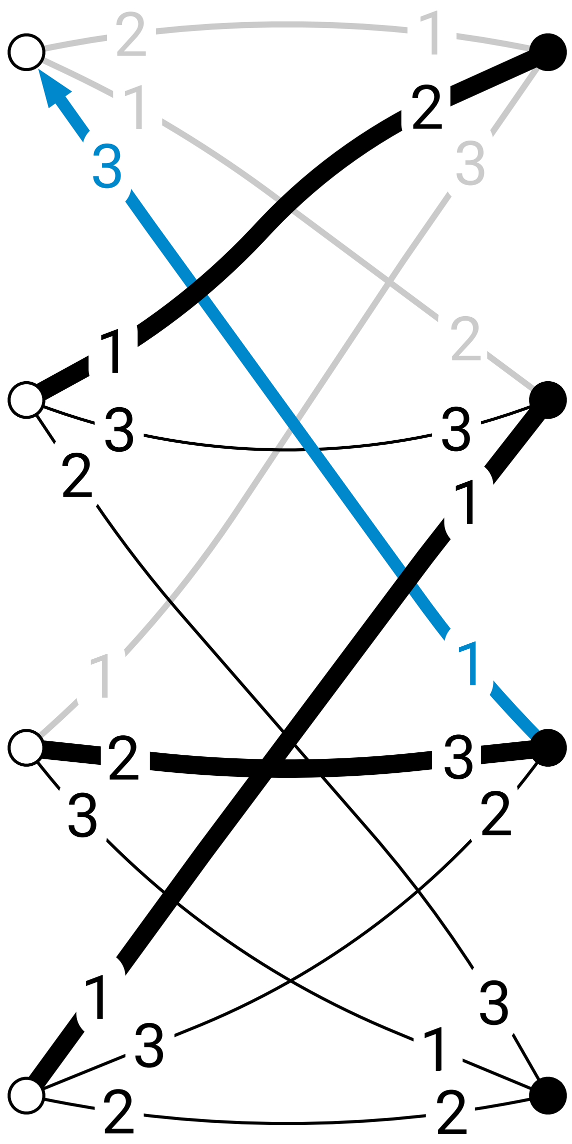


Very simple algorithm

unmatched white nodes:

send ***proposal*** to port 3



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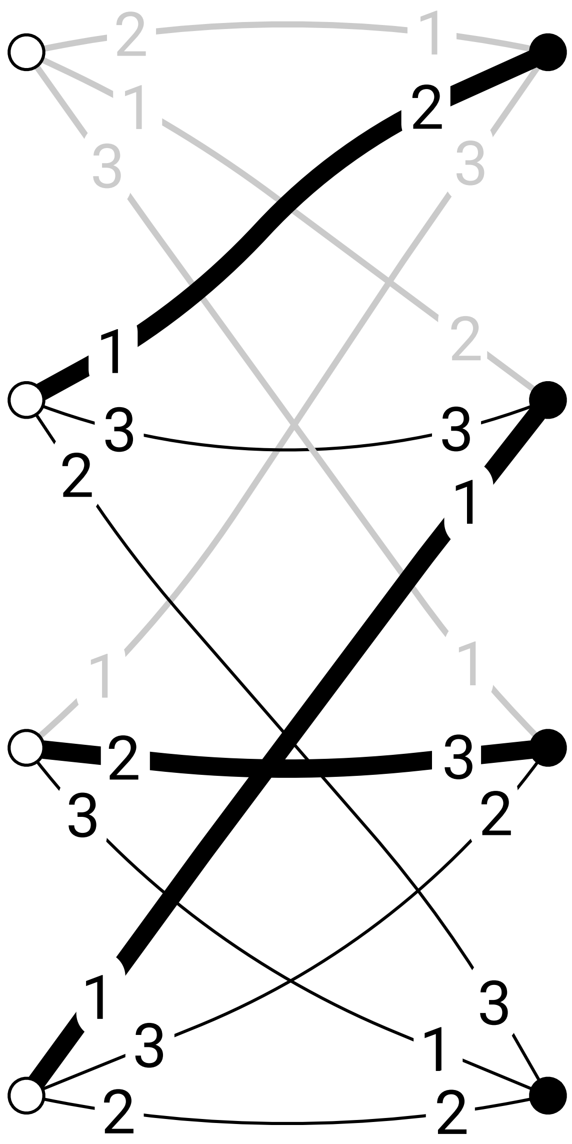

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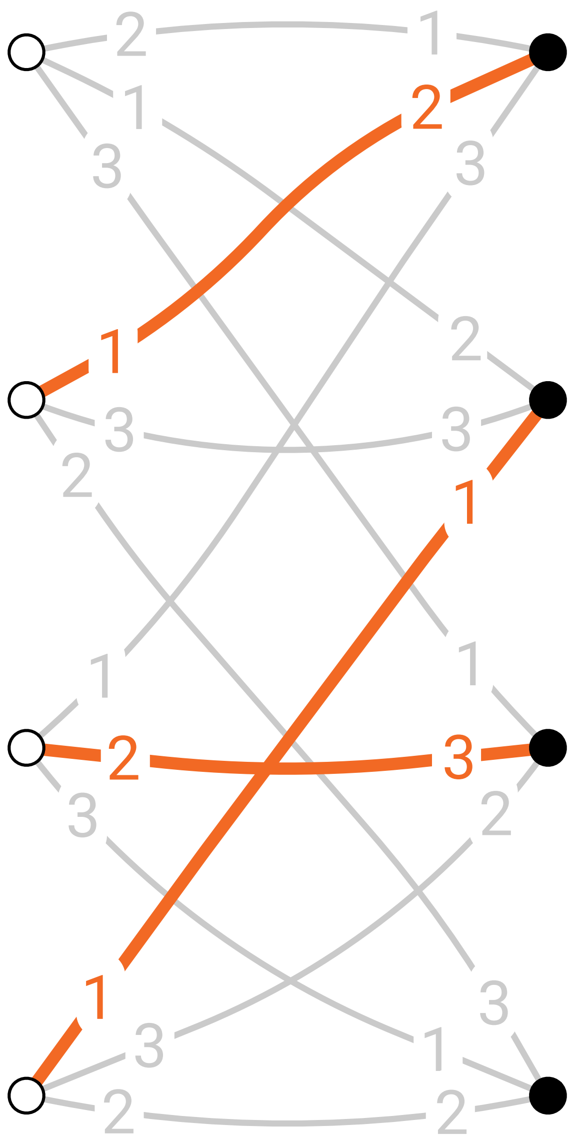

Very simple algorithm

Finds a ***maximal matching*** in $O(\Delta)$ communication rounds

Note: running time does not depend on *n*

Bipartite maximal matching

- Maximal matching in very large 2-colored Δ-regular graphs
- Simple algorithm: ***O*(Δ) *rounds***, independently of *n*
- ***Is this optimal?***
  - $o(\Delta)$ rounds?
  - $O(\log \Delta)$ rounds?
  - 4 rounds??

Lower-bound proof

Round elimination technique for maximal matching

- **Given:**
  - algorithm A_0 solves problem P_0 = **maximal matching** in T rounds
- **We construct:**
  - algorithm A_1 solves problem P_1 in $T - 1$ rounds
  - algorithm A_2 solves problem P_2 in $T - 2$ rounds
  - algorithm A_3 solves problem P_3 in $T - 3$ rounds

    …
  - algorithm A_T solves problem P_T in **0** rounds
- But P_T is nontrivial, so A_0 cannot exist

What are the right problems P_i here?

Round elimination technique for maximal matching

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- But P_T is nontrivial, so A_0 cannot exist

Let's start with P_0 …

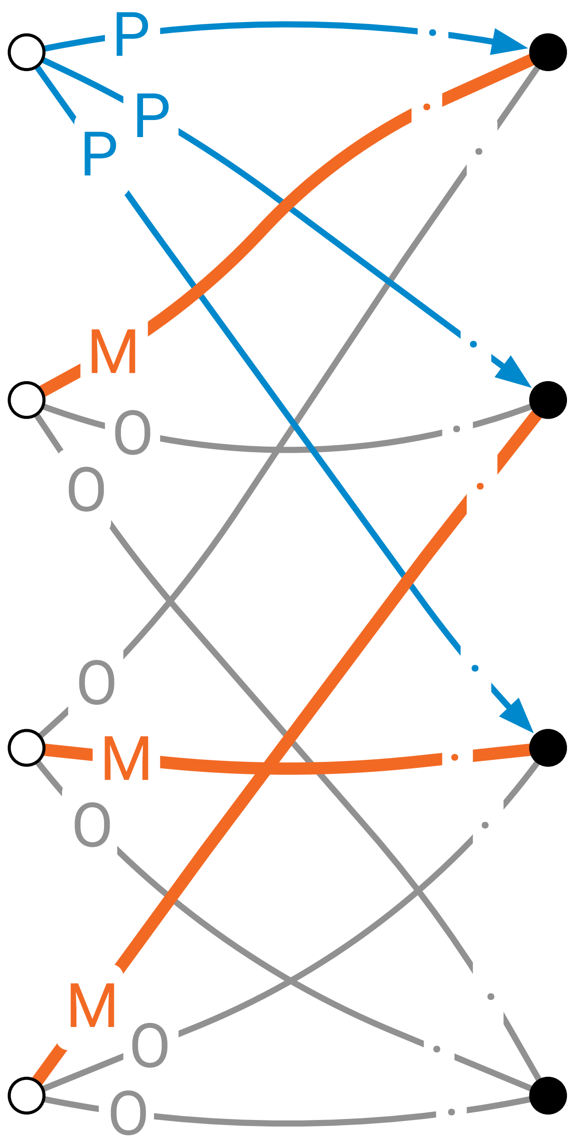
Representation for maximal matchings

white nodes "active"

output one of these:

· 1 × M and (Δ−1) × O

· Δ × P



M = "matched"

P = "pointer to matched"

O = "other"

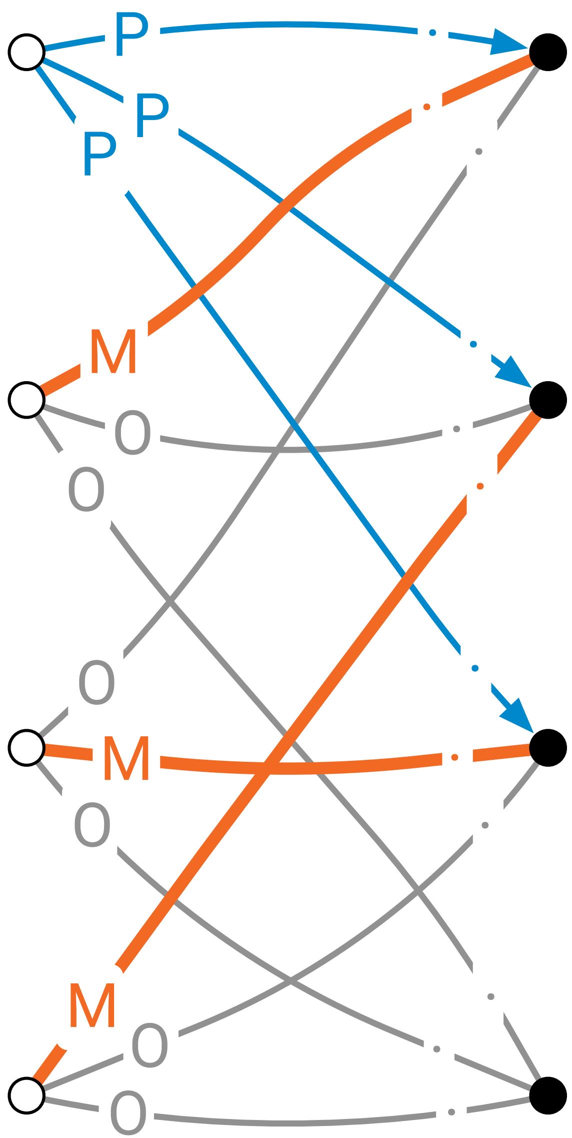
black nodes "passive"

accept one of these:

· 1 × M and (Δ−1) × {P, O}

· Δ × O

Representation for maximal matchings



M = "matched"
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output one of these:
· 1 × M and (Δ−1) × O
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$$W = \mathsf{MO}^{\Delta-1} \mid \mathsf{P}^{\Delta}$$

black nodes "passive"

accept one of these:
· 1 × M and (Δ−1) × {P, O}
· Δ × O

$$B = \mathsf{M}[\mathsf{PO}]^{\Delta-1} \mid \mathsf{O}^{\Delta}$$

Parameterized problem family

$$W = \mathsf{MO}^{\Delta-1} \mid \mathsf{P}^{\Delta},$$
$$B = \mathsf{M}[\mathsf{PO}]^{\Delta-1} \mid \mathsf{O}^{\Delta}$$

maximal matching

$$W_\Delta(x,y) = \left(\mathsf{MO}^{d-1} \,\middle|\, \mathsf{P}^{d}\right) \mathsf{O}^{y} \mathsf{X}^{x},$$
$$B_\Delta(x,y) = \left([\mathsf{MX}]\,[\mathsf{POX}]^{d-1} \,\middle|\, [\mathsf{OX}]^{d}\right) [\mathsf{POX}]^{y}\,[\mathsf{MPOX}]^{x},$$
$$d = \Delta - x - y$$

“weak” matching

Main lemma

- Given: ***A*** solves ***P*(*x*, *y*)** in ***T*** rounds
- We can construct: ***A'*** solves ***P*(*x* + 1, *y* + *x*)** in ***T* − 1** rounds

$$W_\Delta(x,y) = \left(\mathsf{MO}^{d-1} \,\middle|\, \mathsf{P}^{d}\right)\mathsf{O}^{y}\mathsf{X}^{x},$$

$$B_\Delta(x,y) = \left([\mathsf{MX}]\,[\mathsf{POX}]^{d-1} \,\middle|\, [\mathsf{OX}]^{d}\right)[\mathsf{POX}]^{y}\,[\mathsf{MPOX}]^{x},$$

$$d = \Delta - x - y$$

Putting things together

Maximal matching in $o(\Delta)$ rounds

→ **"$\Delta^{1/2}$ matching"** in $o(\Delta^{1/2})$ rounds

→ $P(\Delta^{1/2}, 0)$ in $o(\Delta^{1/2})$ rounds

→ $P(O(\Delta^{1/2}), o(\Delta))$ in **0** rounds

→ contradiction

What we really care about

k-matching: select at most k edges per node

Apply round elimination $o(\Delta^{1/2})$ times

Putting things together

Proof technique does not work directly with unique IDs

- Basic version:
  - deterministic lower bound, ***port-numbering model***
- Analyze what happens to local failure probability:
  - ***randomized*** lower bound, port-numbering model
- With randomness you can construct unique identifiers w.h.p.:
  - randomized lower bound, ***LOCAL model***
- Fast deterministic → faster deterministic → faster randomized
  - stronger ***deterministic*** lower bound, LOCAL model

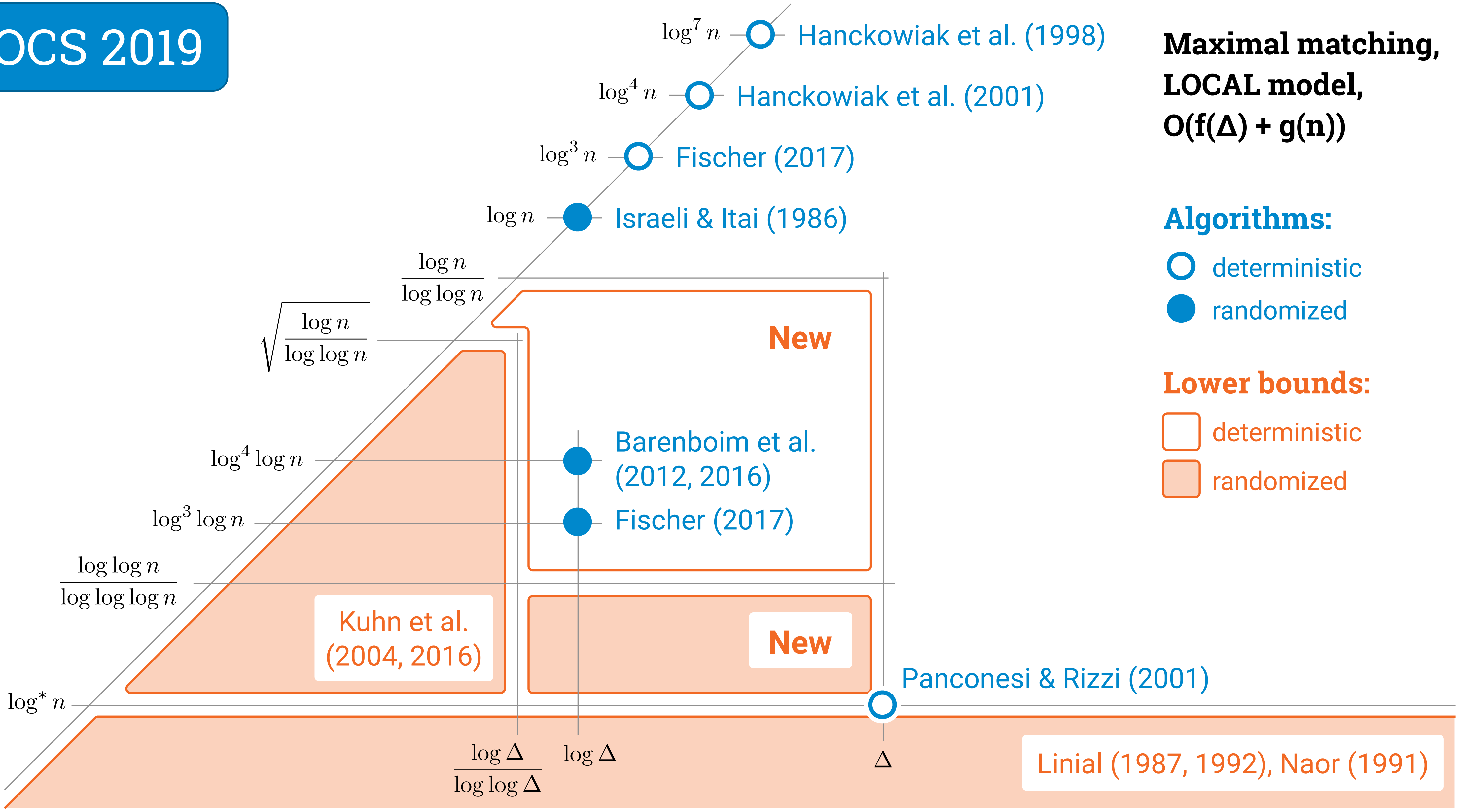
Main results

Maximal matching and **maximal independent set** cannot be solved in

- $o(\Delta + \log \log n / \log \log \log n)$ rounds with randomized algorithms
- $o(\Delta + \log n / \log \log n)$ rounds with deterministic algorithms

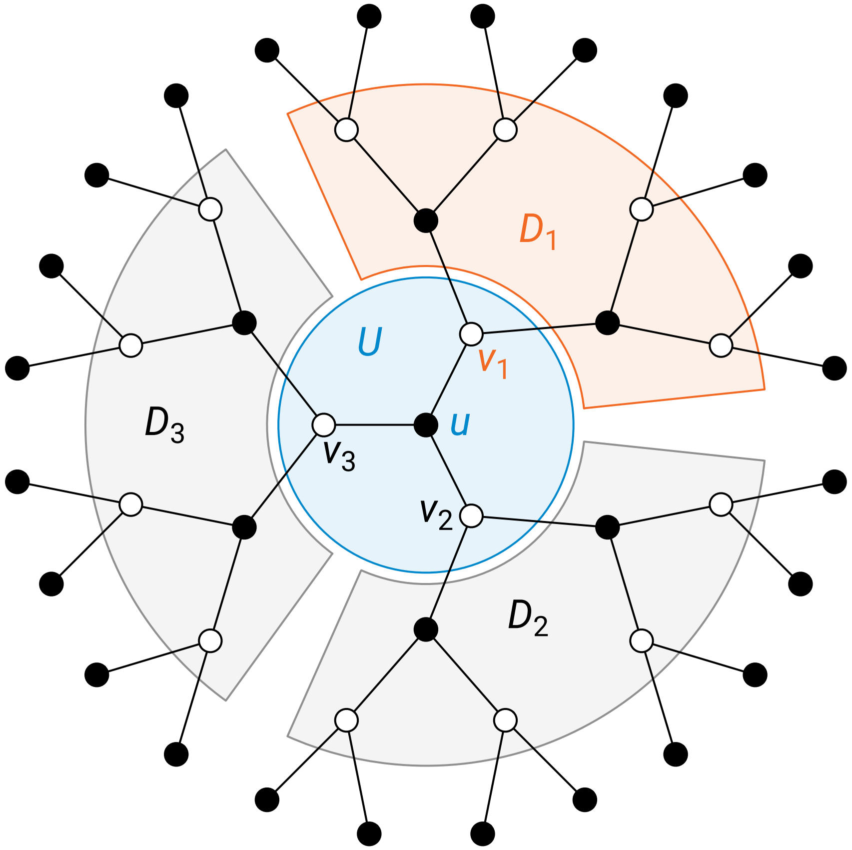
Lower bound for MM implies a lower bound for MIS

FOCS 2019


Maximal matching,
LOCAL model,
O(f(Δ) + g(n))
Algorithms:
deterministic
randomized
Lower bounds:
deterministic
randomized
$\log^7 n$
Hanckowiak et al. (1998)
$\log^4 n$
Hanckowiak et al. (2001)
$\log^3 n$
Fischer (2017)
$\log n$
Israeli & Itai (1986)
$\frac{\log n}{\log \log n}$
$\sqrt{\frac{\log n}{\log \log n}}$
New
$\log^4 \log n$
Barenboim et al. (2012, 2016)
$\log^3 \log n$
Fischer (2017)
$\frac{\log \log n}{\log \log \log n}$
Kuhn et al. (2004, 2016)
New
$\log^* n$
Panconesi & Rizzi (2001)
$\frac{\log \Delta}{\log \log \Delta}$
$\log \Delta$
Δ
Linial (1987, 1992), Naor (1991)

Summary

- Round elimination technique
- Locality lower bounds for a wide range of problems:
  - symmetry breaking in cycles
  - symmetry breaking in regular trees
  - algorithmic Lovász local lemma
  - **maximal matching**, maximal independent set …
- And for a wide range of localities:
  - Ω(log* n), Ω(log log n), Ω(log n), Ω(log* Δ), Ω(Δ) …



Open questions

- Lower bounds for ***volume complexity***?
  - volume lower bounds for **sinkless orientation**?
- Lower bounds for problems related to ***graph coloring***?
  - when is **partial/defective coloring** “easy” and when is it “hard”?
  - nontrivial lower bounds for **(Δ+1)-coloring**?
- Exactly when do we get ***fixed points*** and why?