New Lower Bounds for Distributed Graph Algorithms

Jukka Suomela
Helsinki Institute for Information Technology HIIT
Department of Computer Science, University of Helsinki

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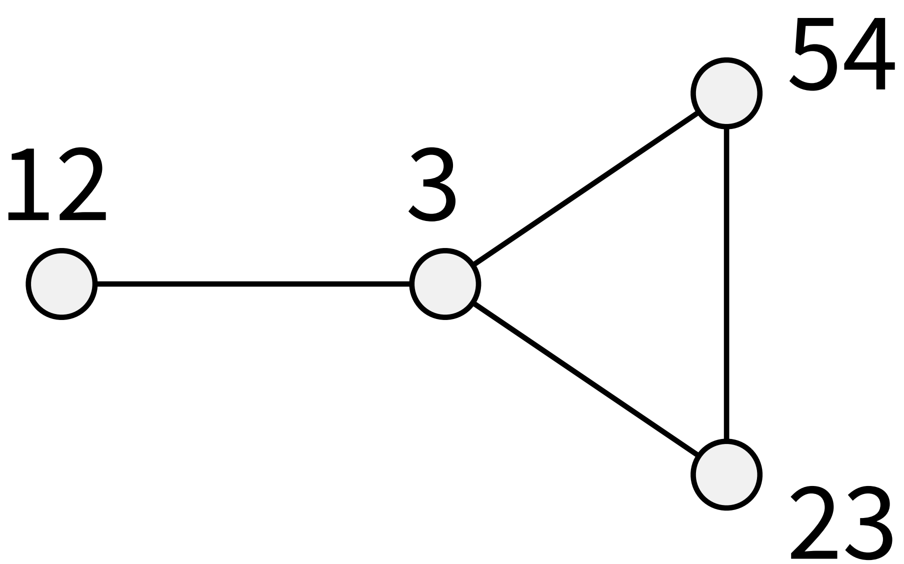
Distributed Graph Algorithm

– mapping from
local neighbourhoods
to local outputs

Distributed Graph Algorithm

- **Input:** simple undirected **graph *G***
  - nodes labelled with unique $O(\log n)$-bit identifiers



Distributed Graph Algorithm

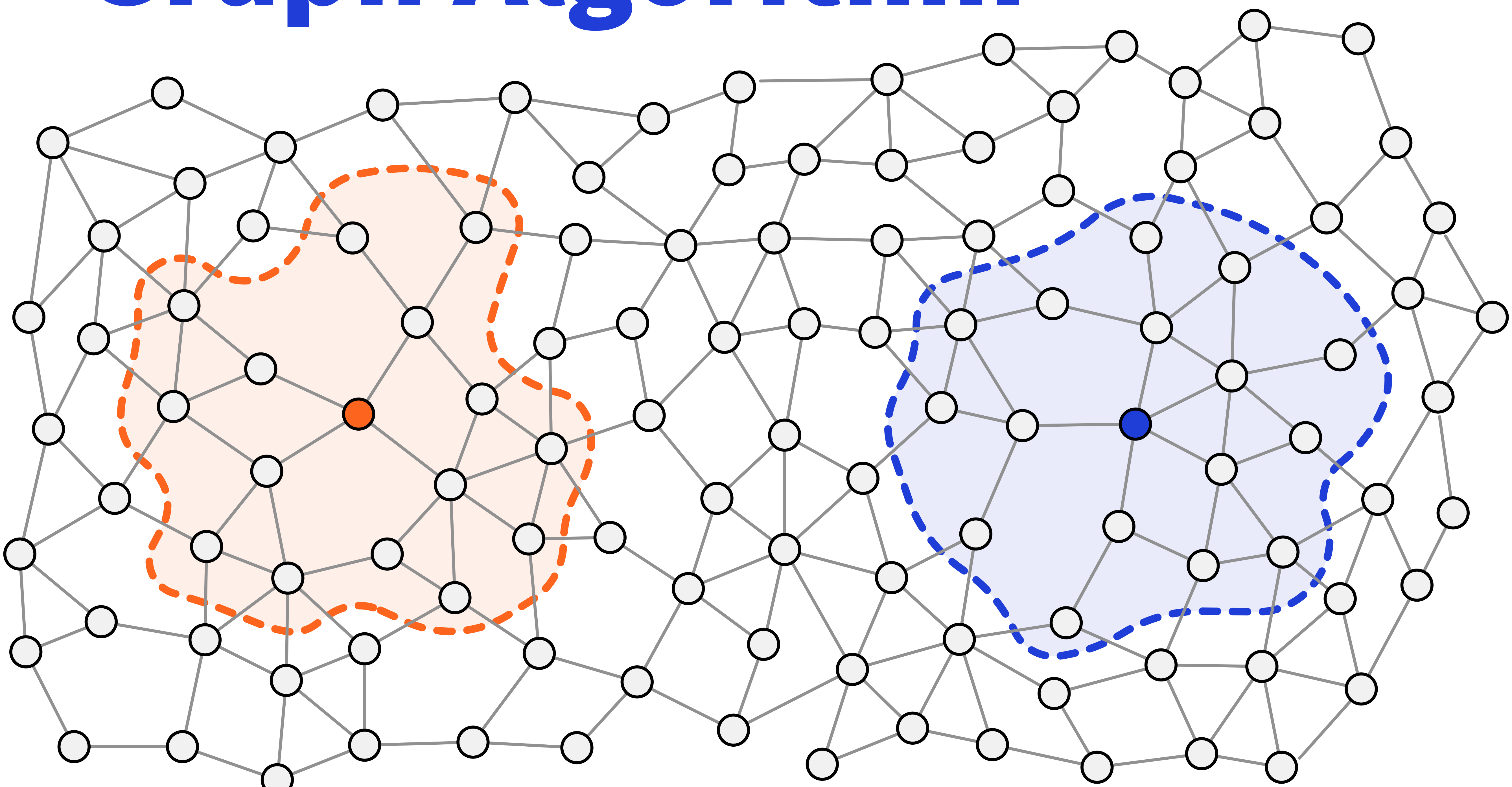
- **Input:** simple undirected graph G
- **Output:** $A(G, v)$ = **local output** of node v
  - graph colouring: $A(G, v)$ = colour of node v
  - vertex cover: $A(G, v) = 1$ if v is in the cover

Distributed Graph Algorithm

- **Input:** simple undirected graph G
- **Output:** $A(G, v)$ = local output of node v
- **Running time is *t*:** local output $A(G, v)$ only depends on **radius-t neighbourhood** of v
  - shortest-path distance at most t

Distributed Graph Algorithm

time $t = 2$



Distributed Graph Algorithm

- **Mapping from local neighbourhoods to local outputs**
  - each node acts based on its radius-t neighbourhood
  - local outputs must form a globally consistent solution

Distributed Graph Algorithm

- **If *G* is a computer network:**
  - time = **number of communication rounds**
  - in *t* communication steps all nodes can learn everything up to distance *t*
- **Fast distributed algorithm = few communication rounds**

Distributed Graph Algorithm

- **Everything trivial in time diam(G)**
  - all nodes see whole G, can compute any function of G
- **What can be solved much faster?**

Our Focus:
Local Algorithms

- **Distributed graph algorithms with running time $O(1)$**
  - extreme limits of distributed algorithms
  - ideal algorithms for large-scale networks
  - fast and fault-tolerant
- **Do these even exist?**

Our Focus: **Local Algorithms**

- **Distributed graph algorithms with running time $O(1)$**
- **We focus on bounded-degree graphs:**
  - n nodes, maximum degree $\Delta = O(1)$
  - running time $t = f(\Delta)$, independently of n

Our Focus: **Local Algorithms**

- **Surprise: many graph problems can be approximated with local algorithms**
  - 2-approximation of minimum vertex cover
  - many linear programming relaxations
  - many problems on bipartite graphs …
- **Today’s main topic: lower bounds**

Lower Bounds

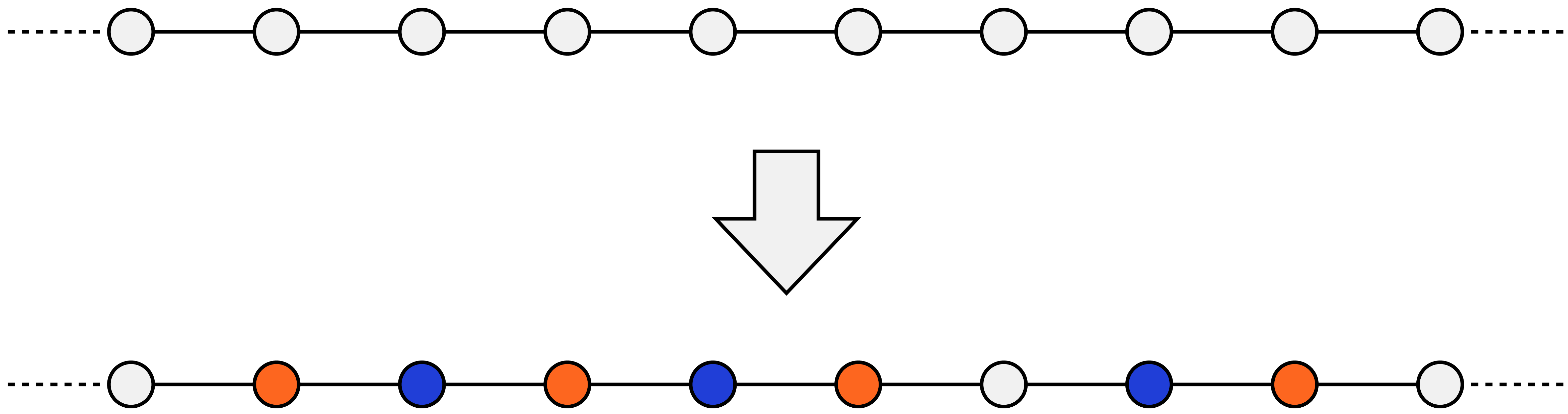
– what cannot
be solved locally?

What Cannot Be Solved Locally?

- **Negative results, even if $\Delta = 2$:**
  - graph colouring
  - maximal matching
  - maximal independent set …
- **Key issue: symmetry breaking**

What Cannot Be Solved Locally?

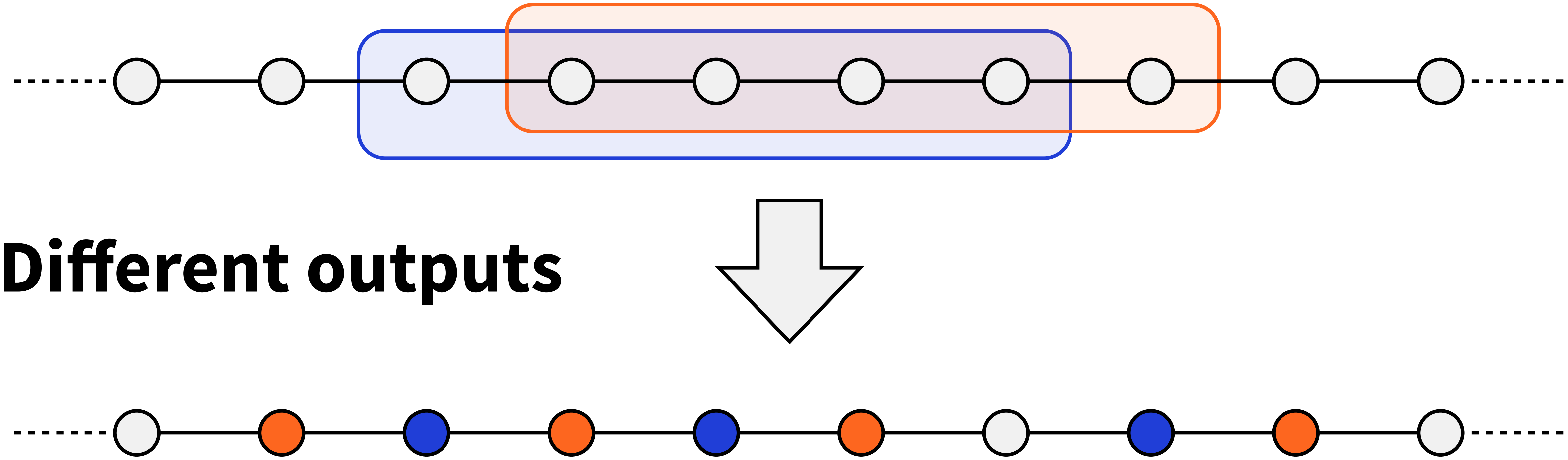
Example: graph colouring (G = long path)



What Cannot Be Solved Locally?

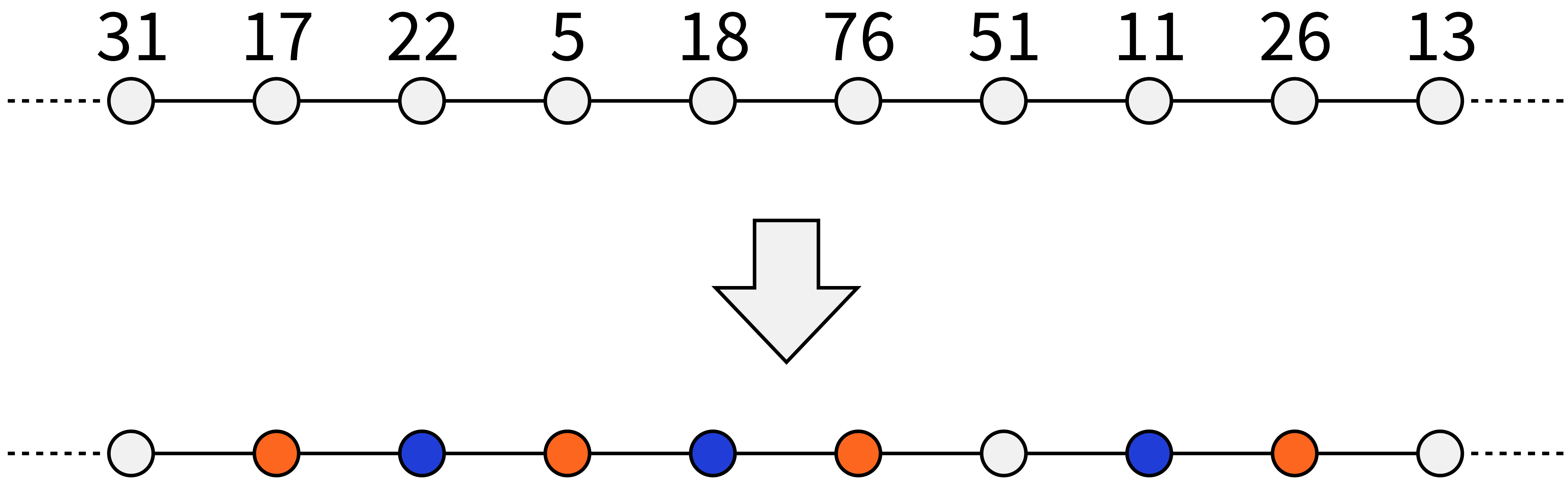
Identical local neighbourhoods

Different outputs



What Cannot Be Solved Locally?

Identical local neighbourhoods — really?

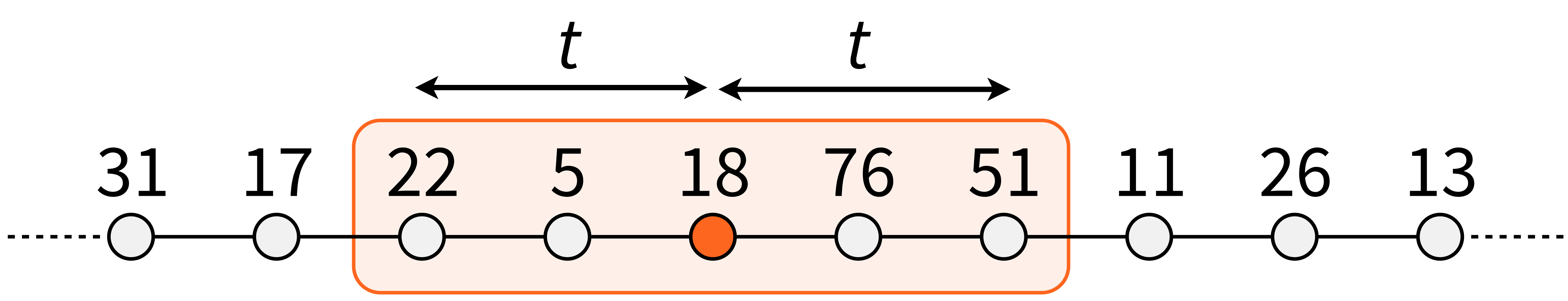


What Cannot Be Solved Locally?

- **Could we somehow use node identifiers to find a graph colouring?**
- **No — would require time $\Omega(\log^* n)$**
  - proof: apply **Ramsey's theorem**…

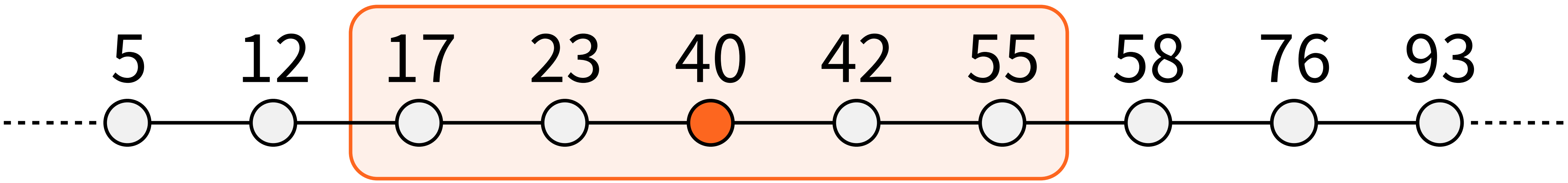
Node Identifiers Do Not Help

- ***G* = path:**
  - local output = f(sequence of k identifiers)
  - $k = 2t + 1$



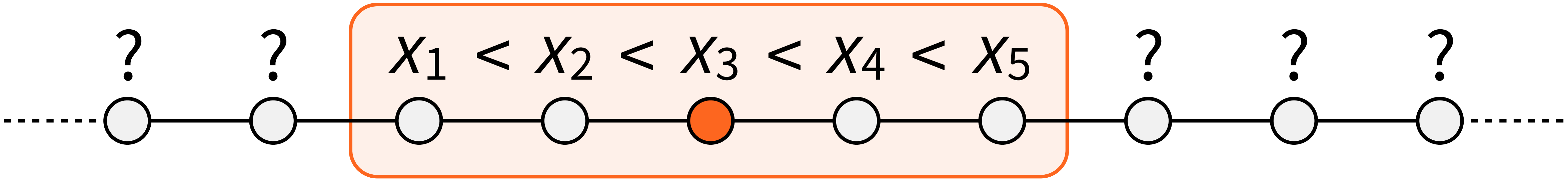
Node Identifiers Do Not Help

- G = **path, identifiers in increasing order:**
  - local output = ***f*(set of *k* identifiers)**
  - $k = 2t + 1$



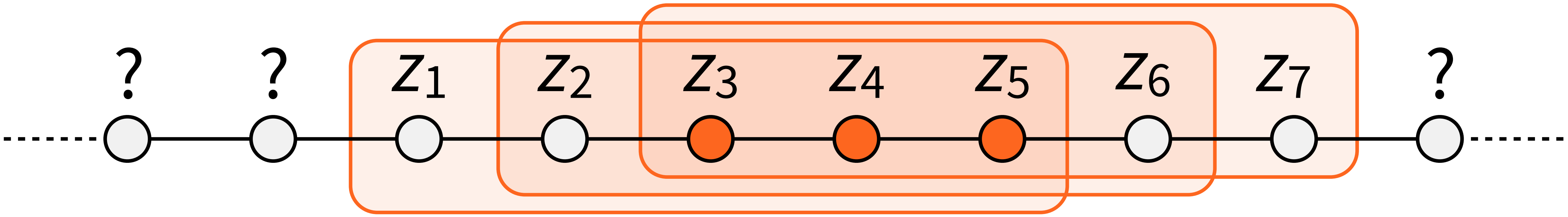
Node Identifiers Do Not Help

- **Universe U = set of all possible identifiers**
- **Colour of k-subset $X = \{x_1, x_2, \ldots, x_k\}$ of U:** output of algorithm A in this neighbourhood



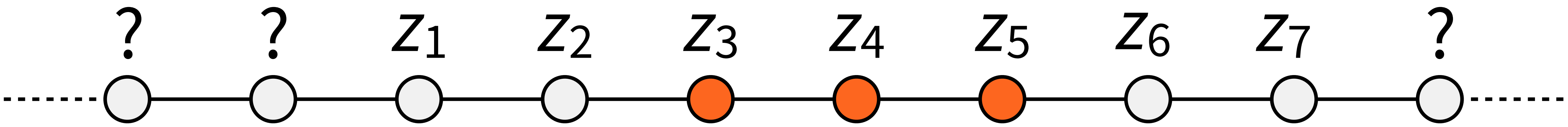
Node Identifiers Do Not Help

- **Ramsey:** there is a **monochromatic** subset $Z = \{z_1, z_2, \ldots, z_m\}$ of universe U
  - all k-subsets of Z have the same colour



Node Identifiers Do Not Help

- **We can construct a path s.t. many adjacent nodes have same local outputs**
  - graph colouring, maximal independent set, maximal matching: **not possible**



Node Identifiers Do Not Help

- **Without unique identifiers:**
  lower bounds often easy to prove
- **With unique identifiers:**
  lower bounds much more difficult
- **Ramsey-like arguments nontrivial for $\Delta > 2$**

Node Identifiers Do Not Help

- **New general result** (Göös et al., 2012):
  - node identifiers do not help with **any local approximation algorithm**
- **Assumption:** “simple” graph problems
  - vertex cover, edge cover, dominating set, matching, independent set, …

Everything Known?

- **Tight upper and lower bounds:**
  - vertex covers & edge covers
  - dominating sets & edge dominating sets
  - independent sets & matchings
  - packing & covering LPs
  - max-min & min-max LPs …

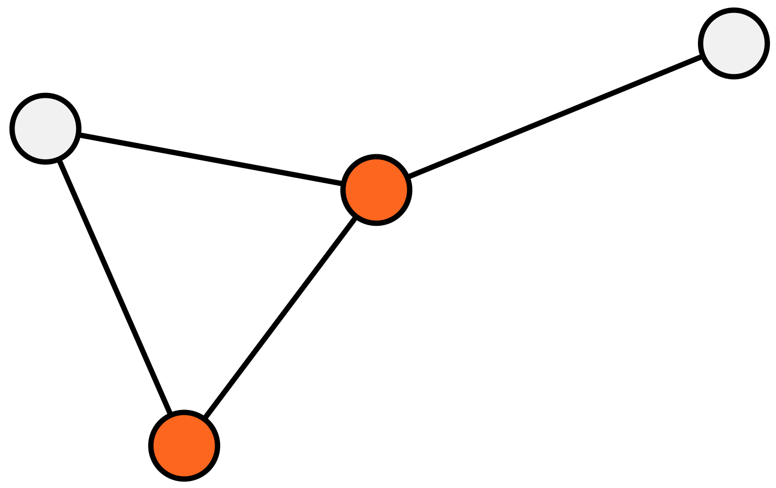
Everything Known?

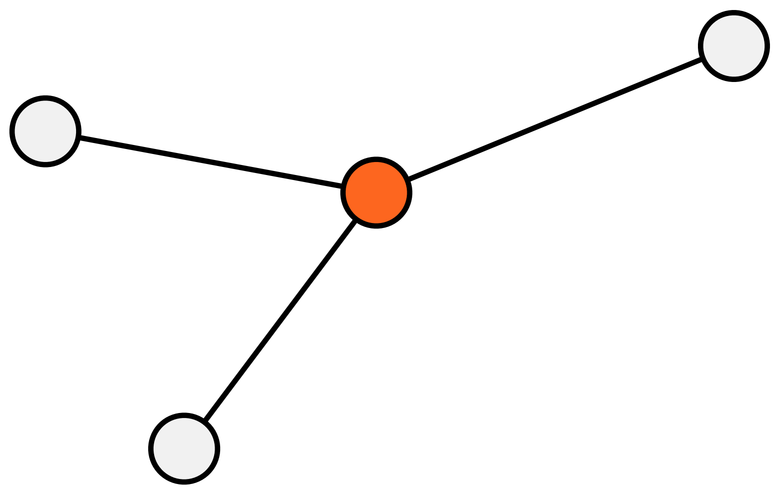
- **Well understood:**
  - what can we do in time that only depends on Δ?
- **Not so well understood:**
  - precisely **how** does t depend on Δ?
  - example: what can we do in time $t = o(\Delta)$?

Time vs. Maximum Degree

– running example: vertex covers and matching

Example:
Vertex Cover

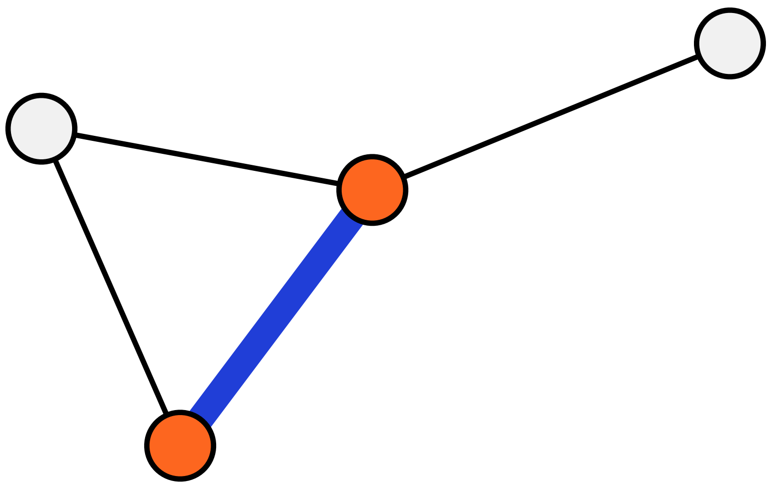




Subset C of nodes that "covers" all edges

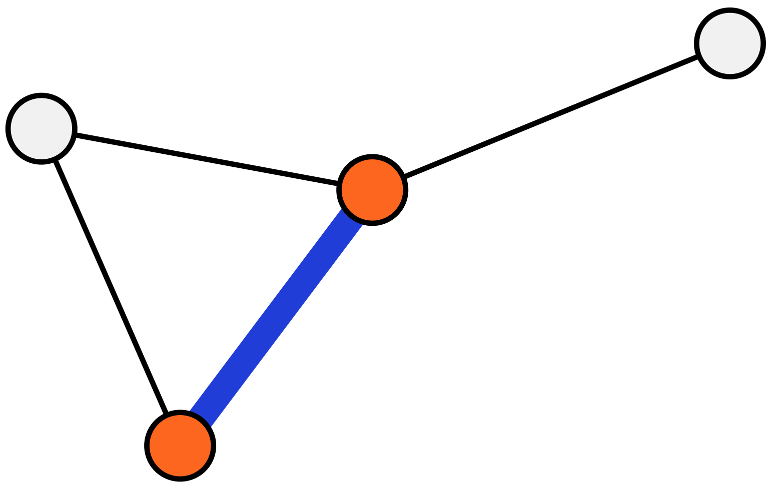
each edge has at least one endpoint in C

Example:
Vertex Cover



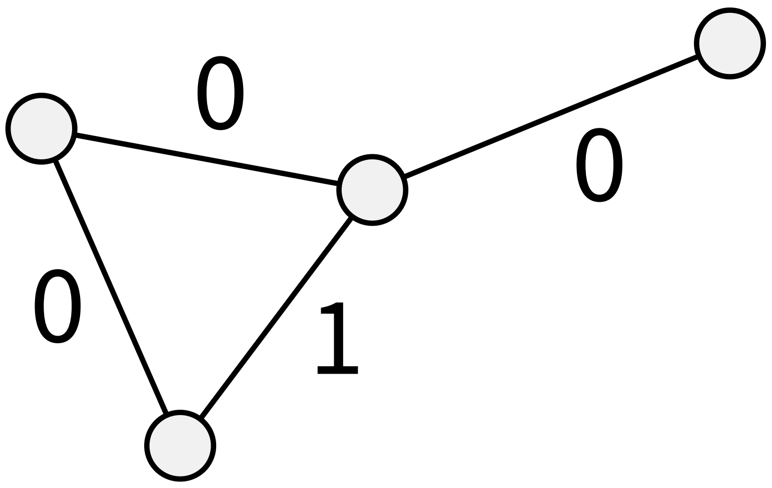
- **2-approximation of minimum vertex cover with centralised algorithms:**
  - find any **maximal matching *M***
  - output **all endpoints** of all edges in *M*
- **Not possible with local algorithms**
  - cannot find a maximal matching

Example:
Vertex Cover



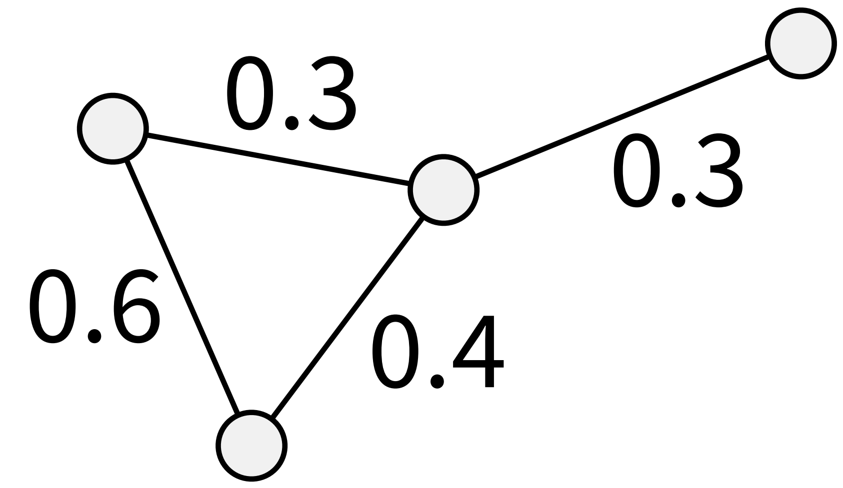
- **Maximal matching:**
  - requires symmetry breaking
- **Maximal fractional matching:**
  - no need to break symmetry
  - still helps with vertex cover approximations!

Matching



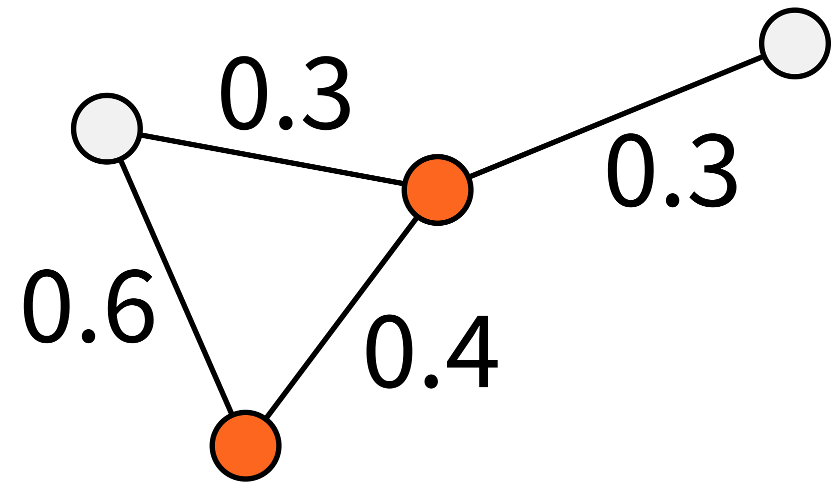
- **Edges labelled with integers {0, 1}**
- **Sum of incident edges at most 1**
- **Maximal matching:**
  cannot increase the value of any label

Fractional Matching



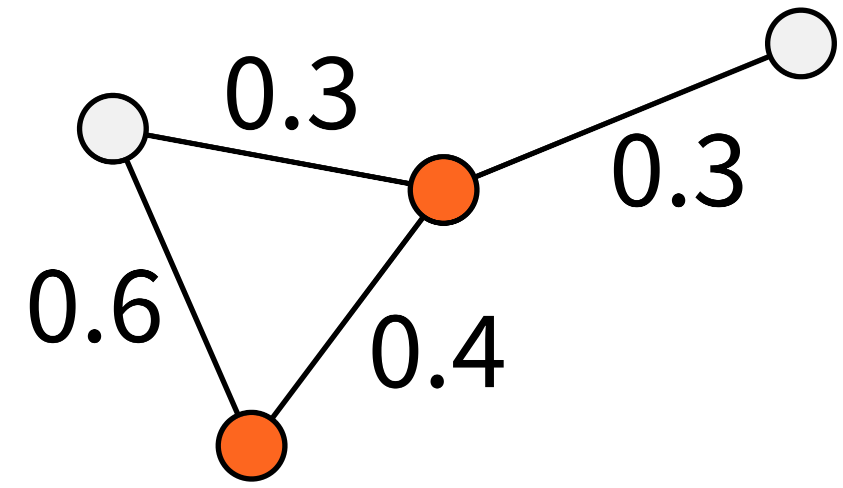
- **Edges labelled with real numbers [0, 1]**
- **Sum of incident edges at most 1**
- **Maximal fractional matching:**
  cannot increase the value of any label

Fractional Matching



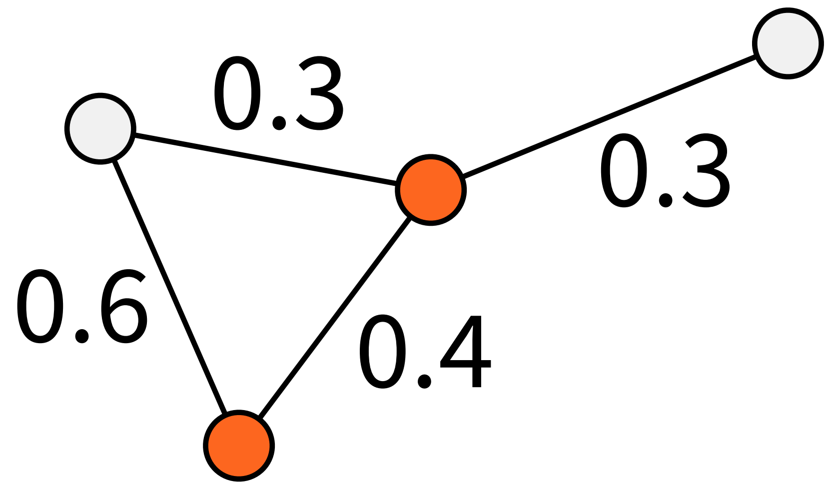
- **Saturated node:**
  - sum of incident edges = 1
- **2-approximation of minimum vertex cover:**
  - find any maximal fractional matching
  - take **all saturated nodes**

Fractional Matching



- **Maximal fractional matching in time $O(\Delta)$** (Åstrand & S., 2010)
  - does **not** require symmetry breaking
  - d-regular graph: label all edges with $1/d$
- **Nontrivial part:** graphs that are not regular…

Fractional Matching



- **Maximal fractional matching in time $O(\Delta)$**
  - 2-approximation of minimum vertex cover in time $O(\Delta)$
- **Can we do it faster?**

Maximal Fractional Matching

– new lower bound

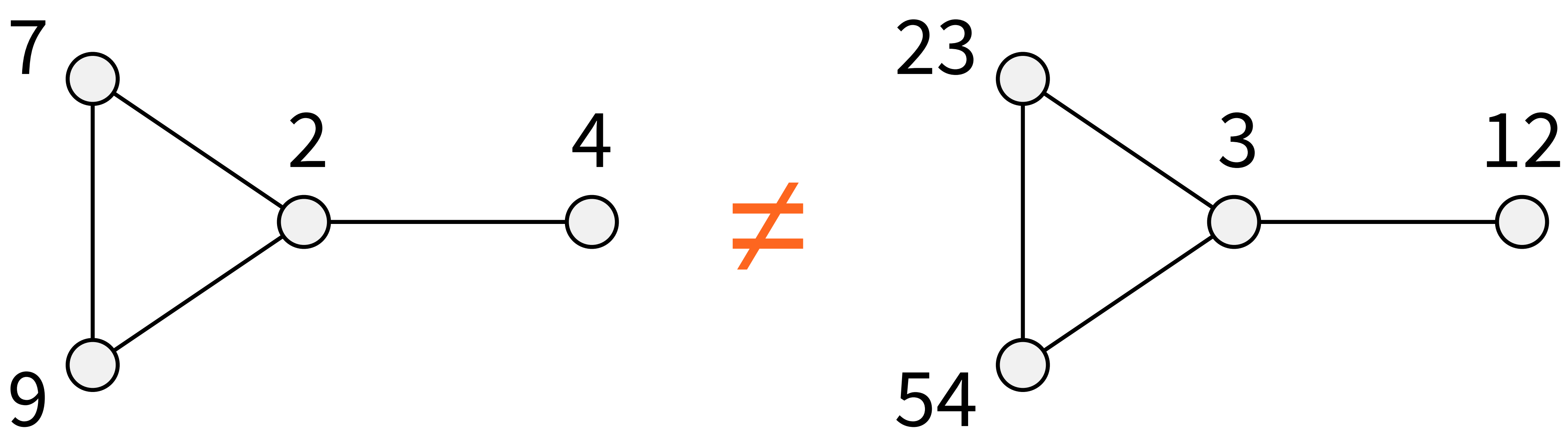
Maximal Fractional Matching

- **Cannot be solved in time $o(\Delta)$**
  (Göös et al., 2013)

- **Key ingredient of the proof:**
  analyse many **different models** of distributed computing

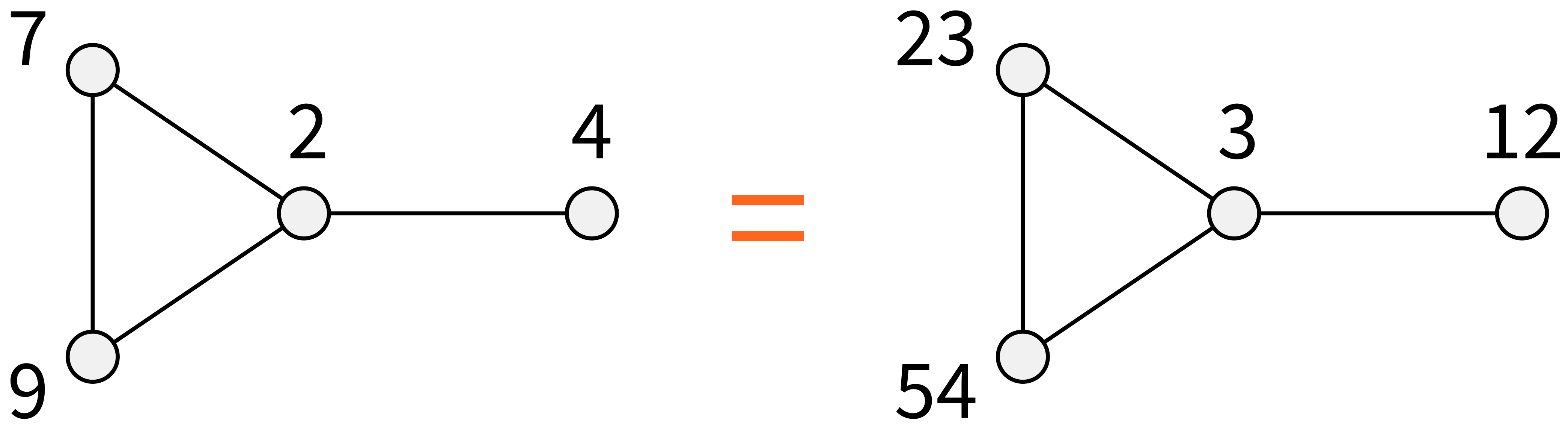
ID:
Unique Identifiers

Nodes have unique identifiers,
output may depend on them



OI:
Order Invariant

Output does not change if we change identifiers but keep their relative order

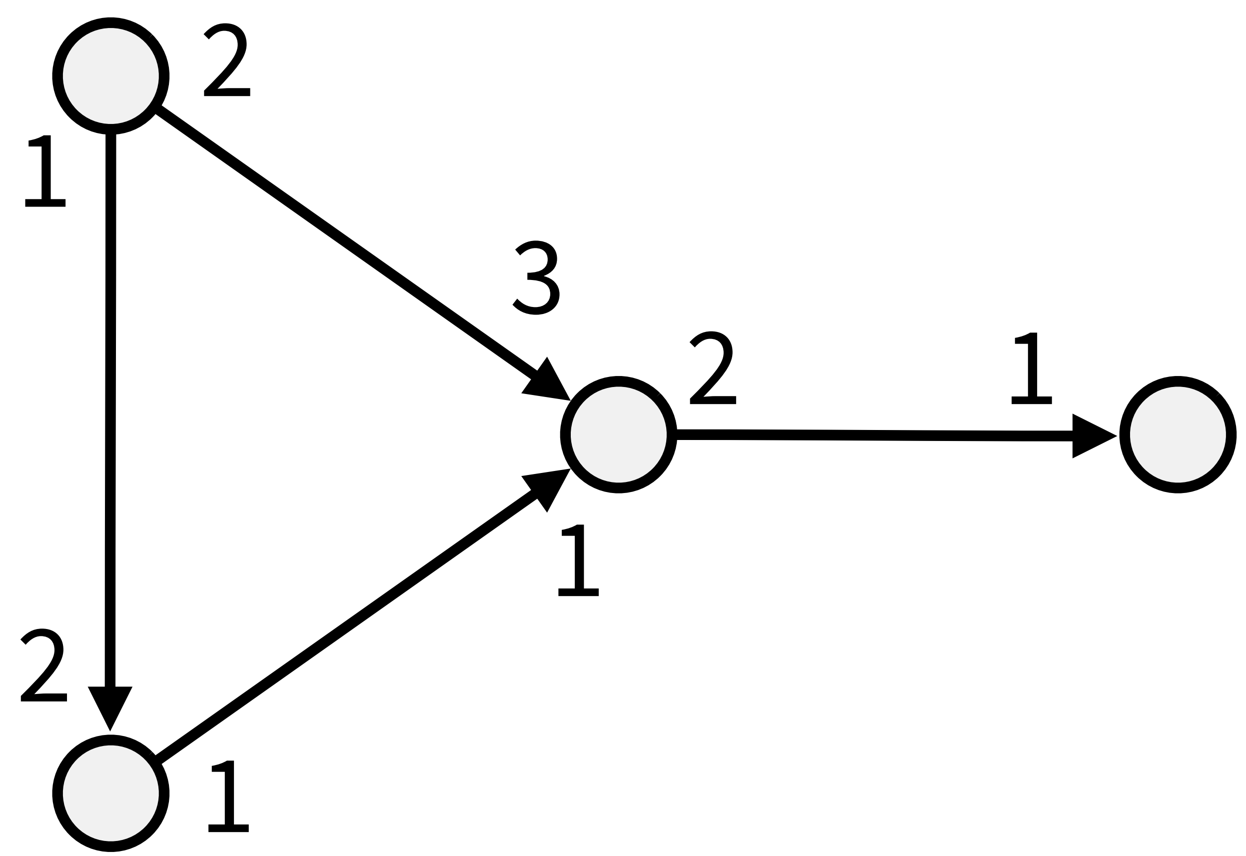


PO:
Ports & Orientation

No identifiers

Node *v* labels incident edges with 1, …, deg(*v*)

Edges oriented

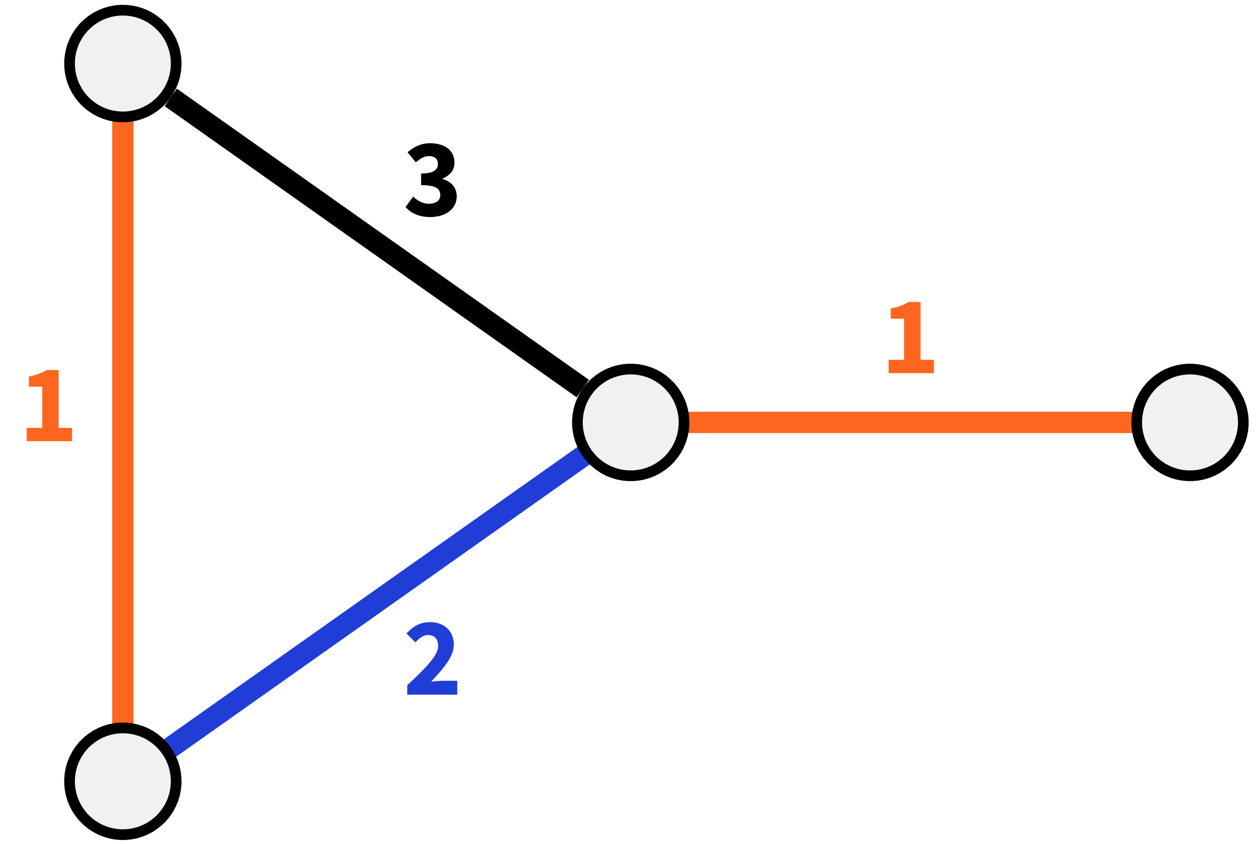


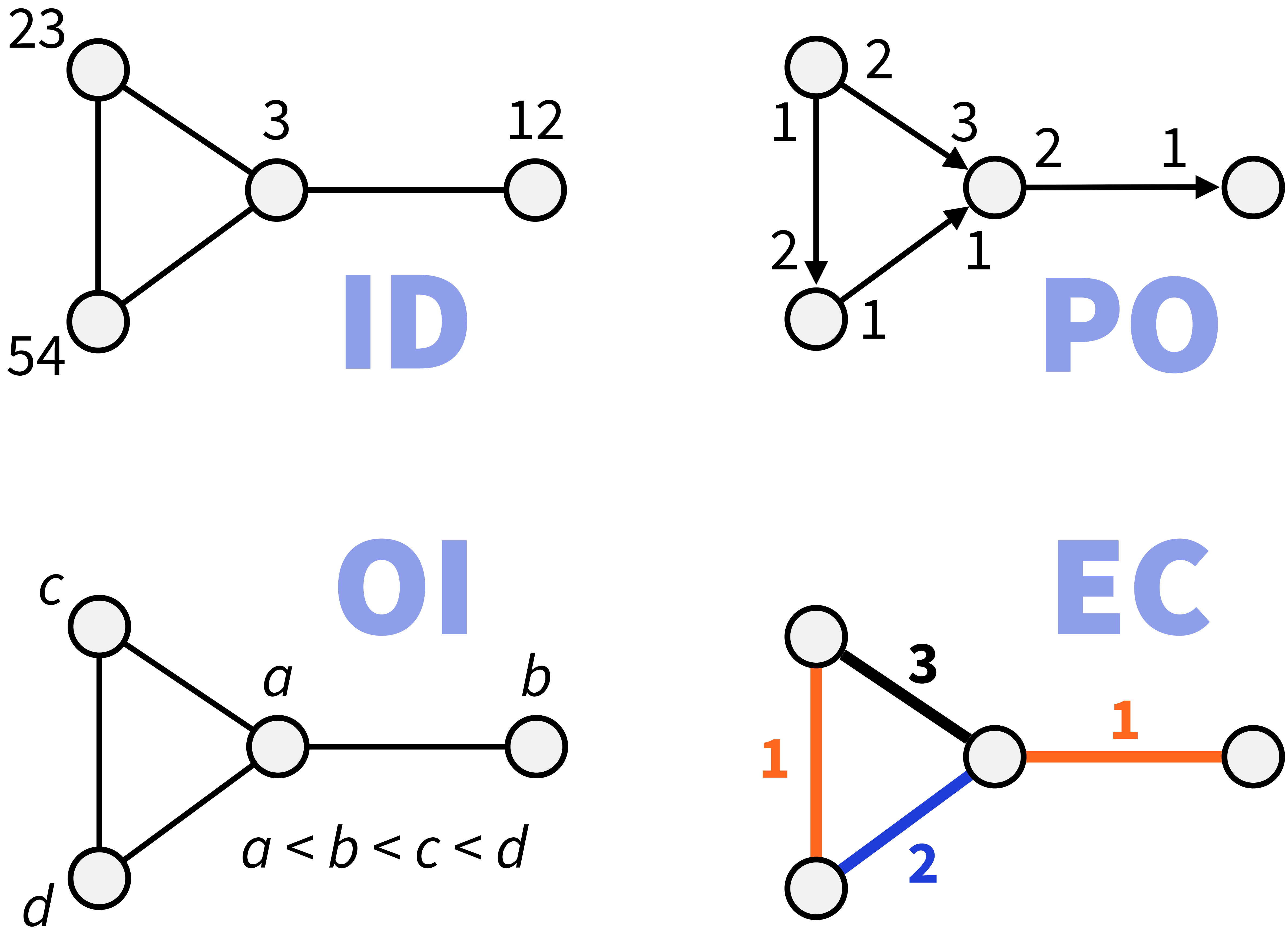
EC:
Edge Colouring

No identifiers

No orientations

Edges coloured
with $O(\Delta)$ colours





23
3
12
54
ID
2
1
3
2
1
2
1
1
PO
OI
c
a
b
d
$a < b < c < d$
EC
3
1
1
2

Oversimplified Proof Overview

- **EC model is very limited**
  - maximal fractional matching requires $\Omega(\Delta)$ time in EC
- **Simulation argument: EC → PO → OI → ID**
  - maximal fractional matching requires $\Omega(\Delta)$ time also in PO, OI, and ID

Bipartite Vertex Cover

– König’s theorem
  is not local

König Duality

- **Bipartite graphs**
- **C^* = minimum vertex cover**
- **M^* = maximum matching**
- **König: $|C^*| = |M^*|$**

Distributed Approximation

Finding (1+ε)-approximation
for constant $\varepsilon > 0$, constant Δ

	Matching	Vertex Cover
Integral	?	?
Fractional	?	?

Distributed Approximation

Kuhn et al. (2004)

Finding (1+ε)-approximation
for constant $\varepsilon > 0$, constant Δ

	Matching	Vertex Cover
Integral	?	?
Fractional	$O(1)$	$O(1)$

Distributed Approximation

Åstrand et al. (2010)

Finding (1+ε)-approximation
for constant ε > 0, constant Δ

	Matching	Vertex Cover
Integral	$O(1)$	?
Fractional	$O(1)$	$O(1)$

Distributed Approximation

Finding (1+ε)-approximation
for constant $\varepsilon > 0$, constant Δ

	Matching	Vertex Cover
Integral	$O(1)$	?
Fractional	$O(1)$	$O(1)$

Distributed Approximation

Göös & S. (2012)

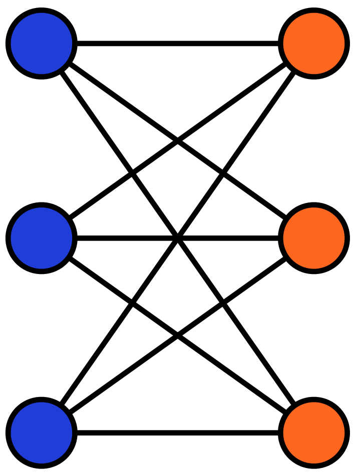
Finding (1+ε)-approximation
for constant $\varepsilon > 0$, constant Δ

	Matching	Vertex Cover
Integral	$O(1)$	$\Omega(\log n)$
Fractional	$O(1)$	$O(1)$

Bipartite Vertex Cover

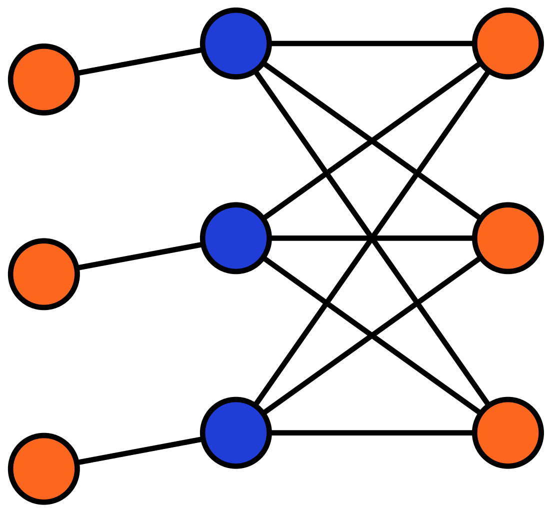
- **No $o(\log n)$-time distributed algorithm for 1.01-approximation of minimum vertex cover**
  - even if we study bipartite graphs of maximum degree 3
  - key ingredient: **expander graphs**

Bipartite Vertex Cover



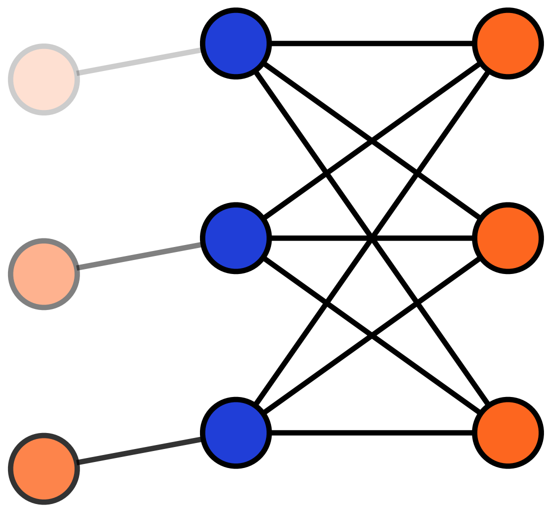
- **Assume we are given an algorithm *A***
- **See what it does in a regular bipartite graph *G***
  - algorithm picks **all orange** or **all blue** nodes
  - w.l.o.g., assume that it picks **all orange** nodes

Bipartite Vertex Cover



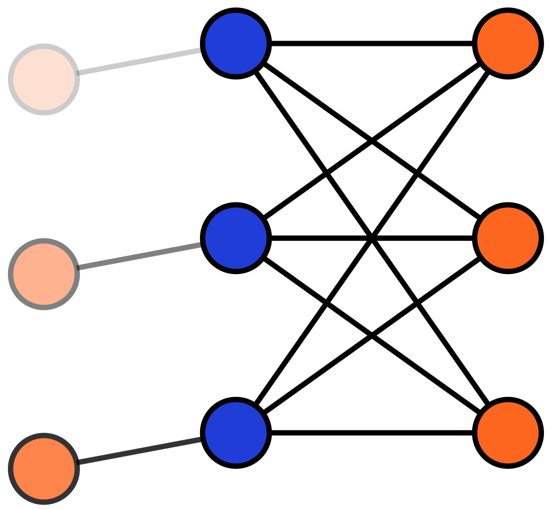
- **Assume we are given an algorithm *A***
- **See what it does in a regular bipartite graph *G***
  - algorithm picks **all orange** nodes
- **Add some gadgets to construct graph *G'***
  - algorithm must pick **all blue** nodes

Bipartite Vertex Cover



- **Construct a sequence of graphs G_0, G_1, … G_n**
- **Start with the regular graph $G_0 = G$**
  - algorithm picks **all orange** nodes in G_0
- **Add gadgets one by one so that $G_n = G'$**
  - algorithm must pick **all blue** nodes in G_n

Bipartite Vertex Cover



- **Construct a sequence of graphs G_0, G_1, … G_n**
  - algorithm picks **all orange** nodes in G_0
  - algorithm must pick **all blue** nodes in G_n
  - only small change from G_i to G_{i+1} if algorithm runs in time $o(\log n)$
  - for some G_k we have **half orange** + **half blue**

Bipartite Vertex Cover

- **For some G_k algorithm outputs**
  half orange + half blue
  - G_k is an expander
  - large **orange**–**blue** boundary
  - many edges redundantly covered
  - poor approximation ratio

Toolbox for
Lower Bound Proofs

- **Highly symmetric graphs:**
  local neighbourhoods "look identical"
- **Ramsey-like arguments:**
  unique identifiers do not help, either
- **Expanders:**
  cannot "hide boundaries"

Toolbox for Lower Bound Proofs

- **Highly symmetric graphs:**
  local neighbourhoods “look identical”
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Thanks!