Fast distributed approximation algorithms for vertex cover and set cover in anonymous networks

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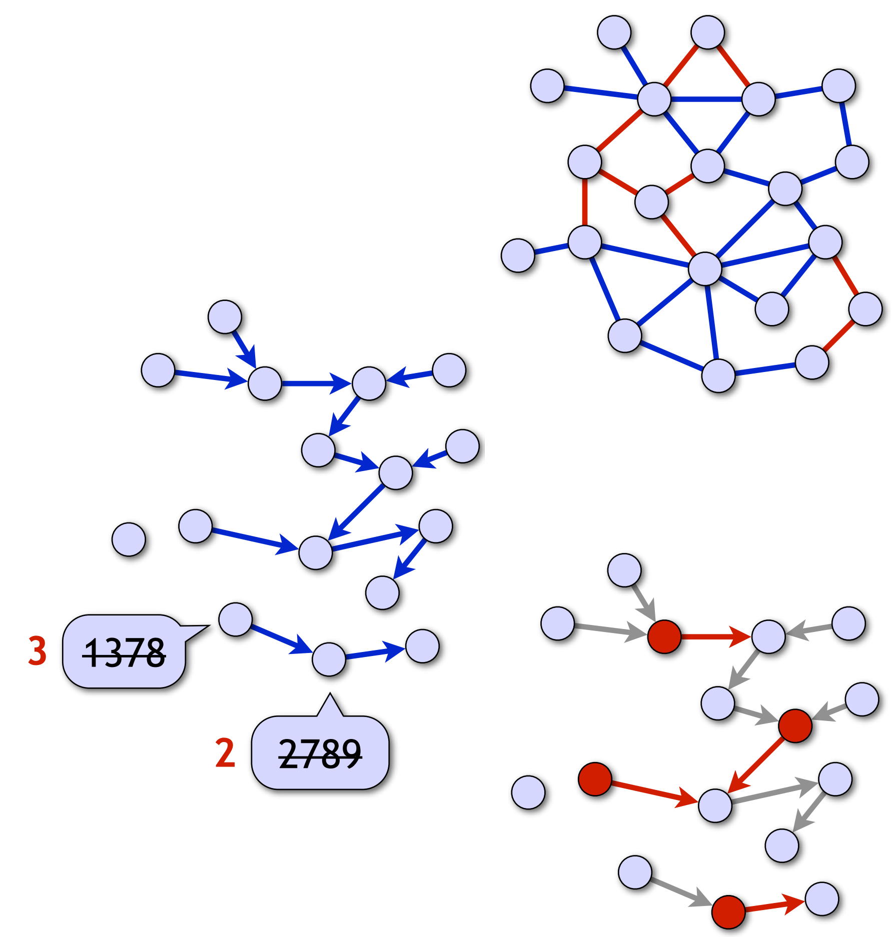
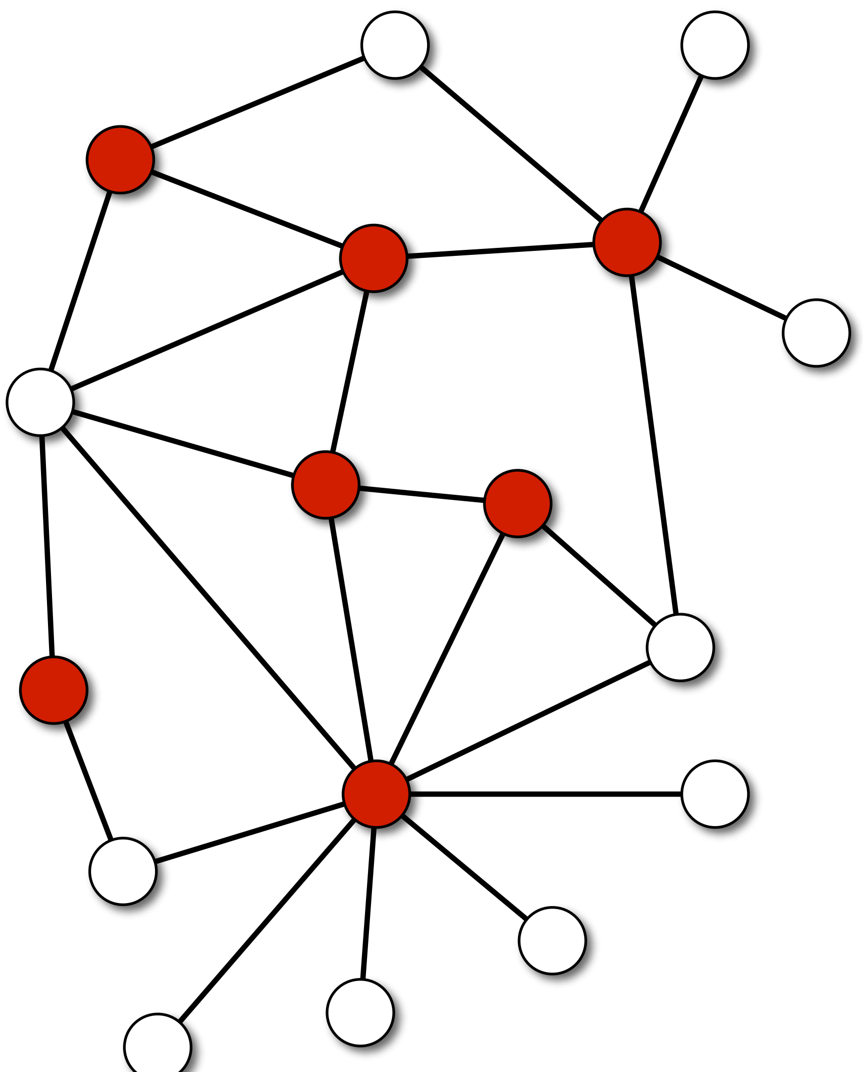


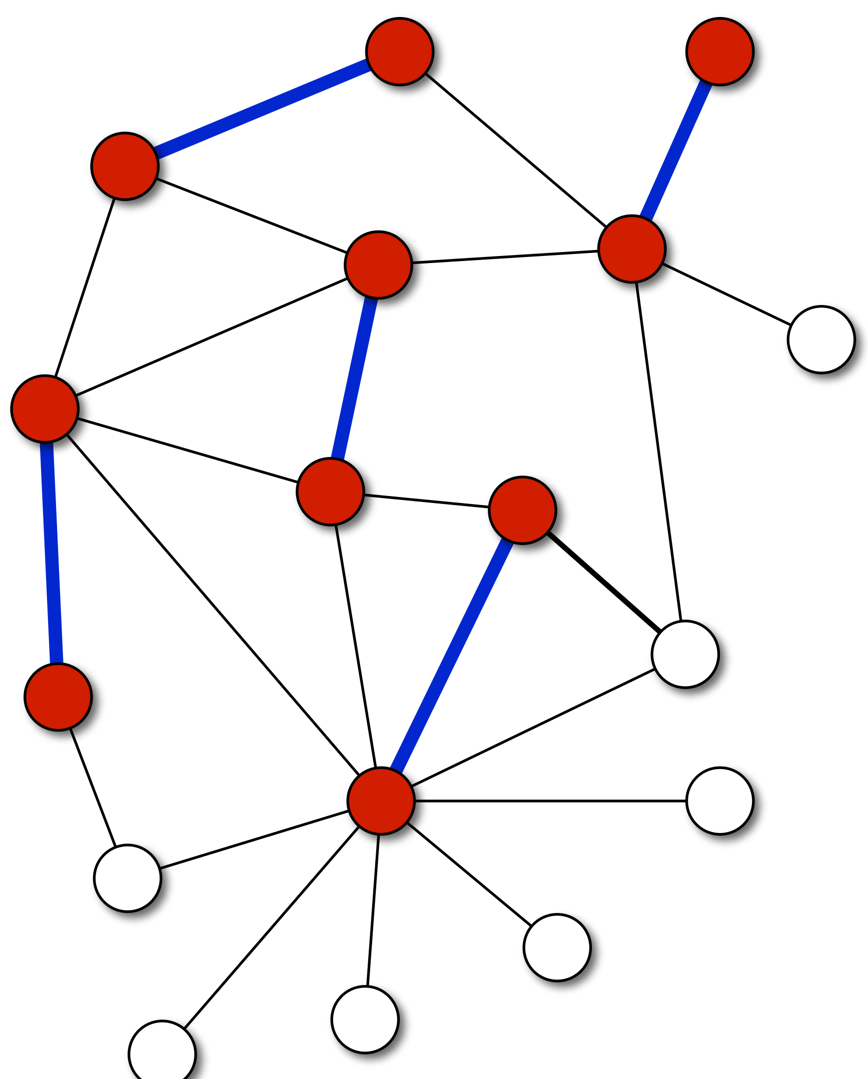
Joint work with Matti Åstrand

Vertex cover problem



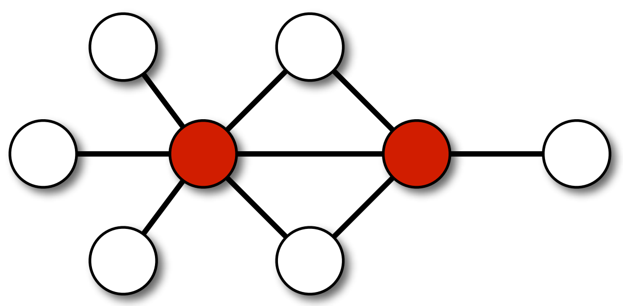
- **Vertex cover *C*** for a graph *G*:
  - Subset *C* of nodes that “covers” all edges of the graph
  - Each edge has at least one endpoint in *C*
- Can we find a ***small*** vertex cover?

Vertex cover problem



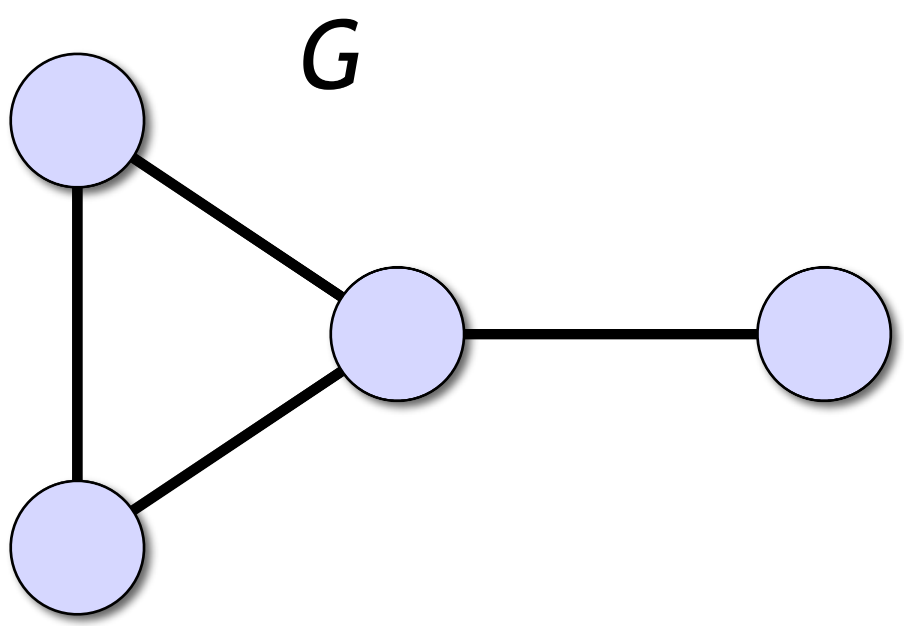
- Classical NP-hard optimisation problem
  - Simple 2-approximation algorithm: endpoints of a ***maximal matching***
  - No polynomial-time algorithm with approximation factor 1.999 known

Research question



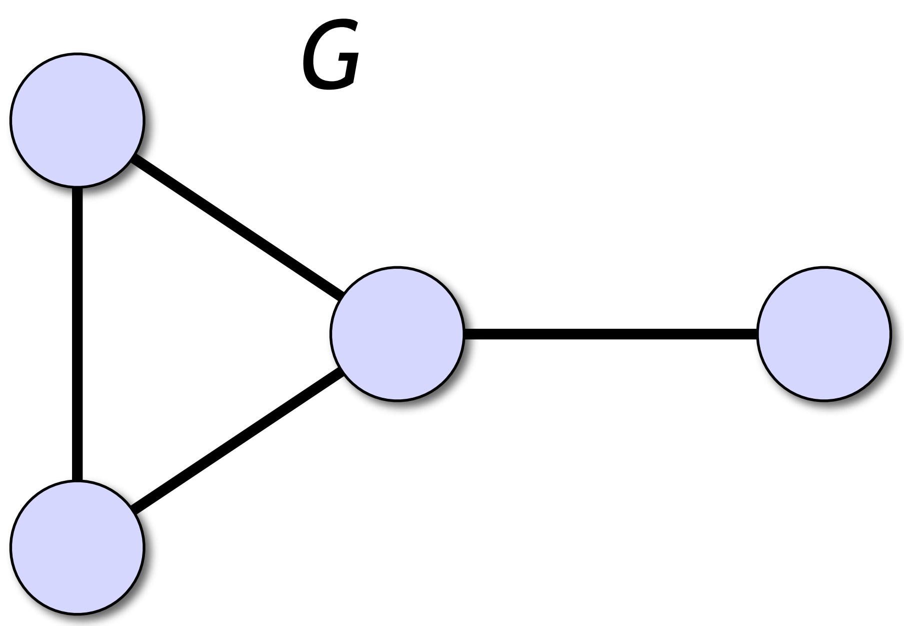
- **Distributed** approximation algorithms for vertex cover
  - Find a small vertex cover in any communication network
  - Best possible approximation ratio
  - As fast as possible: running time independent of *n*
  - Weakest possible models: no randomness, no unique node identifiers
- Let's first define the models...

Distributed algorithms



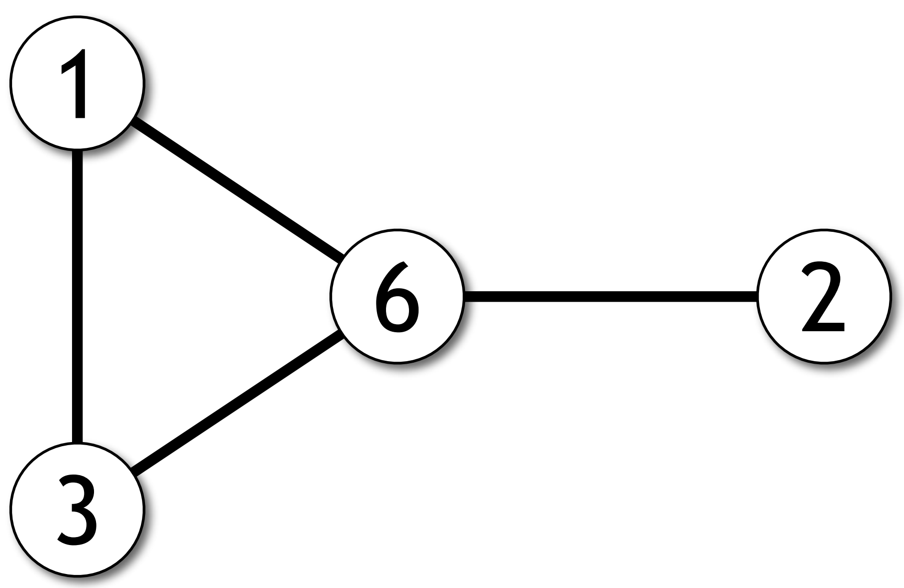
- Communication graph *G*
- Node = computer
- Edge = communication link
- Computers exchange messages and finally decide whether they are in vertex cover *C*
  - "Local output", 0 or 1

Distributed algorithms



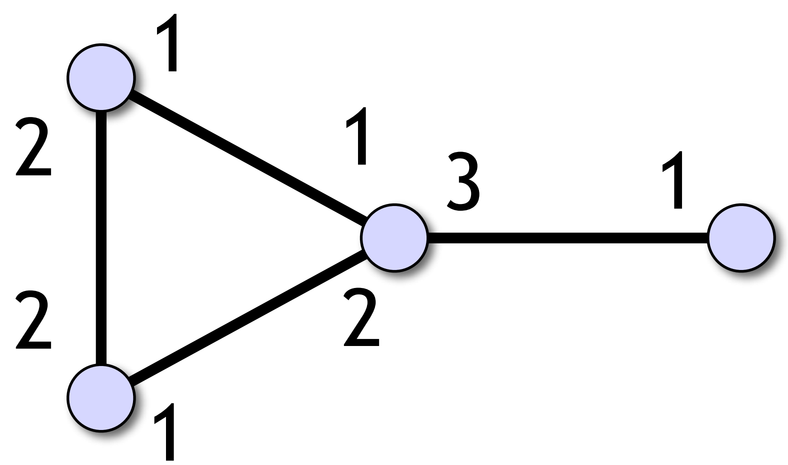
- All nodes are identical, run the **same algorithm**
- ***We*** can choose the algorithm
- An ***adversary*** chooses the structure of *G*
- Our algorithm must produce a valid vertex cover in any graph *G*

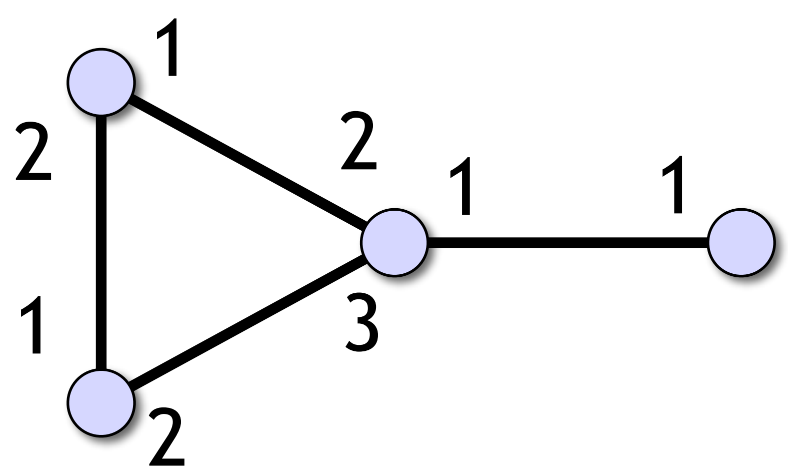
Model 1: Unique identifiers



- The "standard model"
- Node identifiers are a subset of 1, 2, ..., poly(n)
- Subset chosen by adversary

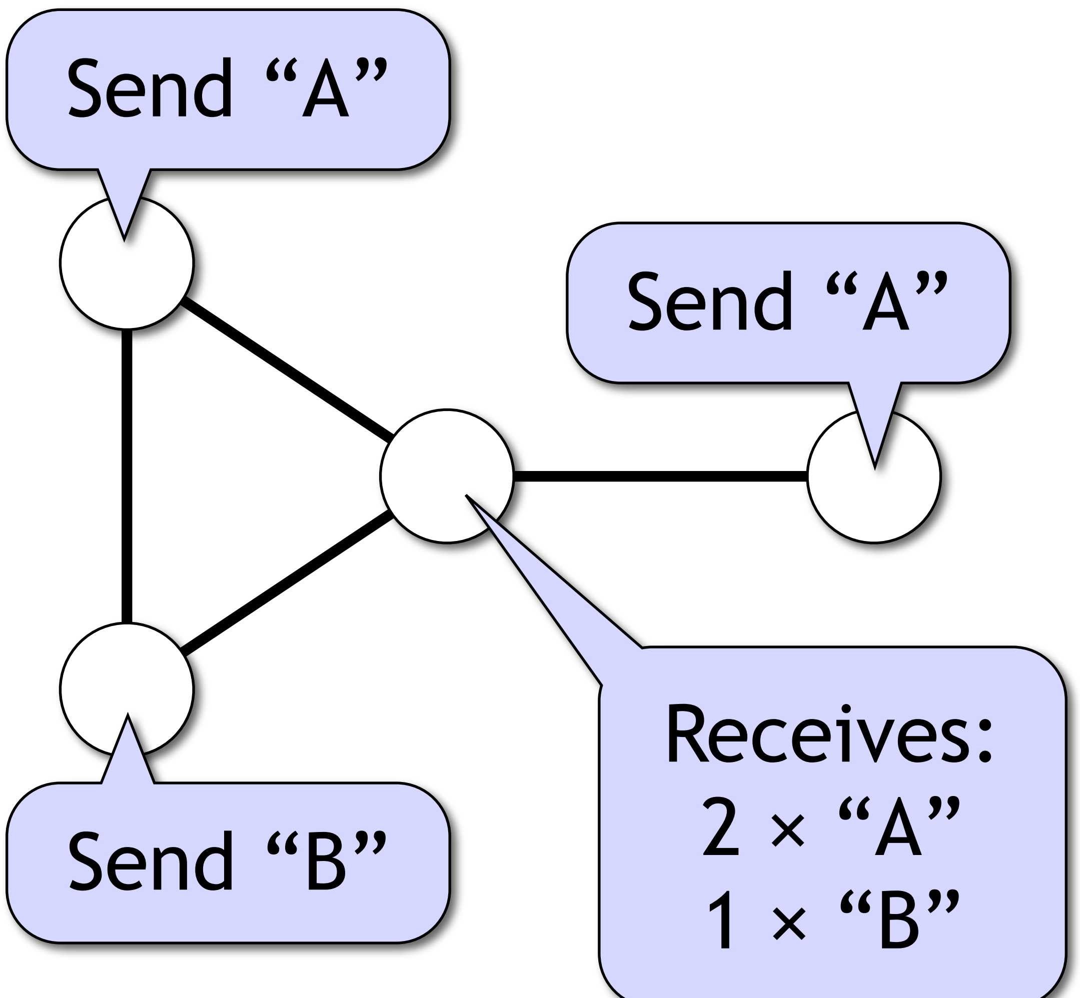
Model 2: Port-numbering model





- No unique identifiers
- A node of degree *d* can refer to its neighbours by integers 1, 2, ..., *d*
- Port-numbering chosen by adversary

Model 3: Broadcast model



- No identifiers, no port numbers
- A node has to send the **same message** to each neighbour
- A node does not know which message was received from which neighbour (*multiset*)

Deterministic distributed algorithms for vertex cover

- Guaranteed approximation ratios?
  - E.g., 2-approximation of minimum vertex cover = at most 2 times as large as the smallest vertex cover
- Fast?
  - Time = number of communication rounds
  - n = number of nodes
  - Δ = maximum degree
- In weak models of distributed computing?

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$						1
$f(\Delta)$ + polylog(n)						
$f(\Delta) + O(\log^* n)$						
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

Trivial algorithm

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$						1
$f(\Delta)$ + polylog(n)						**2**
$f(\Delta) + O(\log^* n)$						**2**
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

Maximal matching

(Panconesi & Rizzi 2001)

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$				**2**		1
$f(\Delta)$ + polylog(n)				**2**		2
$f(\Delta) + O(\log^* n)$						2
$f(\Delta)$						
	Broadcast model		Port numbering		Unique identifiers	

Near-maximal edge packing

(Khuller et al. 1994)

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$				2		1
$f(\Delta)$ + polylog(n)				2		2
$f(\Delta) + O(\log^* n)$				**2 + ε**		2
$f(\Delta)$				**2 + ε**		**2 + ε**
	Broadcast model		Port numbering		Unique identifiers	

Deterministic LP rounding

(Kuhn et al. 2006)

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$				2		1
$f(\Delta)$ + polylog(n)				2		2
$f(\Delta) + O(\log^* n)$				2 + ε		2
$f(\Delta)$	**2**		**2**	2 + ε	**2**	2 + ε
	Broadcast model		Port numbering		Unique identifiers	

Czygrinow et al. 2008

Lenzen & Wattenhofer 2008

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$	**2**		**2**	2		1
$f(\Delta)$ + polylog(n)	**2**		**2**	2		2
$f(\Delta) + O(\log^* n)$	**2**		**2**	2 + ε		2
$f(\Delta)$	**2**		**2**	2 + ε	2	2 + ε
	Broadcast model		Port numbering		Unique identifiers	

Trivial (cycles)

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$	2		2	2		1
$f(\Delta)$ + polylog(n)	2		2	2		2
$f(\Delta) + O(\log^* n)$	2		2	2 + ε		2
$f(\Delta)$	2		2	2 + ε	2	2 + ε
	Broadcast model		Port numbering		Unique identifiers	

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$	2	**?**				1
$f(\Delta)$ + polylog(n)	2	**?**				
$f(\Delta) + O(\log^* n)$	2	**?**	2	**2 + ε**		
$f(\Delta)$	2	**?**	2	**2 + ε**	2	**2 + ε**
	Broadcast model		Port numbering		Unique identifiers	

Anything here?

Could we have 2?

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$	2	**?**	2	2		1
$f(\Delta)$ + polylog(n)	2	**?**	2	2		
$f(\Delta) + O(\log^* n)$	2	**?**	2	**2**		
$f(\Delta)$	2	**?**	2	**2**	2	**2**
	Broadcast model		Port numbering		Unique identifiers	

DISC 2009

Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$	2	**2**				1
$f(\Delta)$ + polylog(n)	2	**2**				
$f(\Delta) + O(\log^* n)$	2	**2**	2	**2**		
$f(\Delta)$	2	**2**	2	**2**	2	**2**
	Broadcast model		Port numbering		Unique identifiers	

Latest results

+ faster and more general solution here

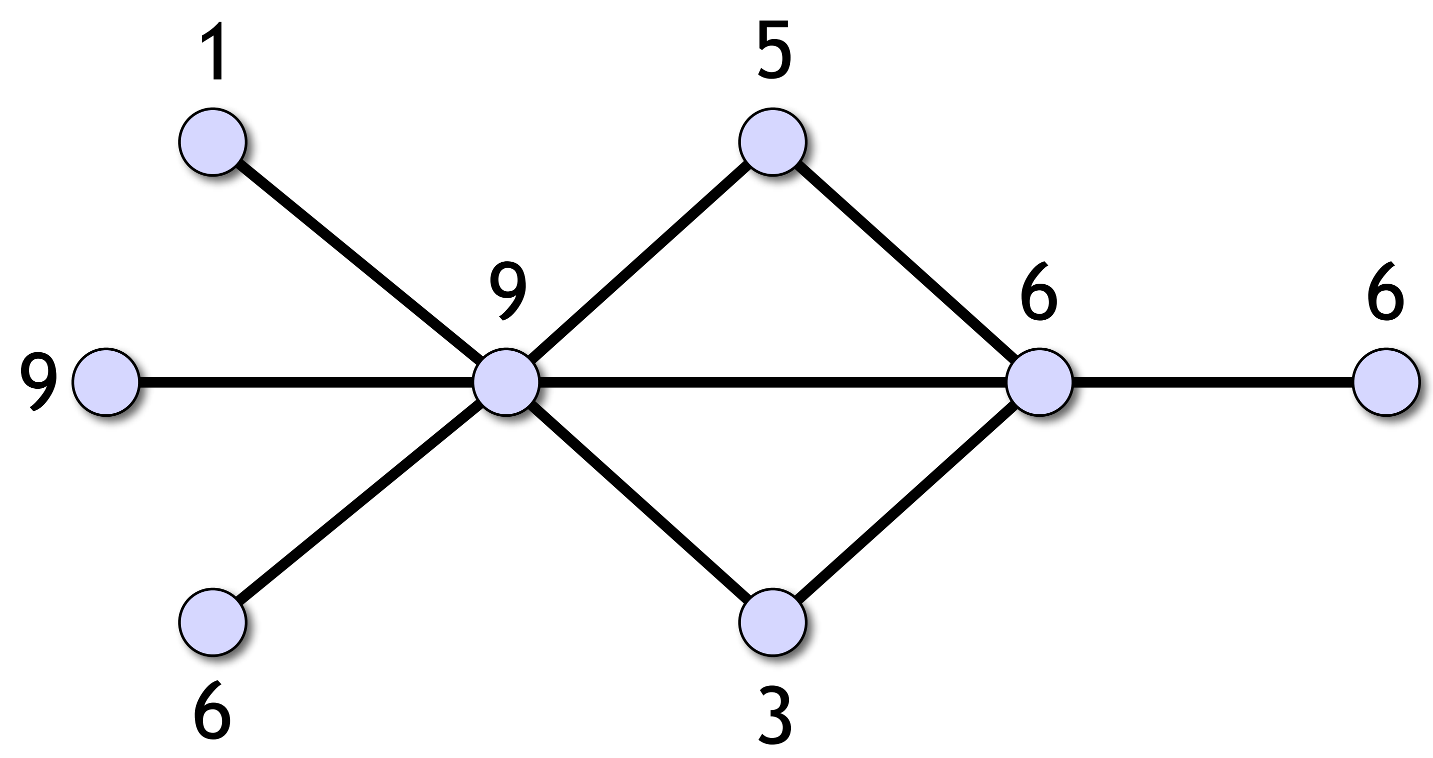
Deterministic distributed algorithms for vertex cover: approximation ratios

Time	lower	upper	lower	upper	lower	upper
$O(n)$	2	**2**	2	2		1
$f(\Delta)$ + polylog(n)	2	**2**	2	2		
$f(\Delta) + O(\log^* n)$	2	**2**	2	**2**		
$f(\Delta)$	2	**2**	2	**2**	2	**2**
	Broadcast model		Port numbering		Unique identifiers	

Let's study this case first...

Vertex cover in the port-numbering model

- Convenient to study a more general problem: minimum-weight vertex cover
  - More general problems are sometimes easier to solve?



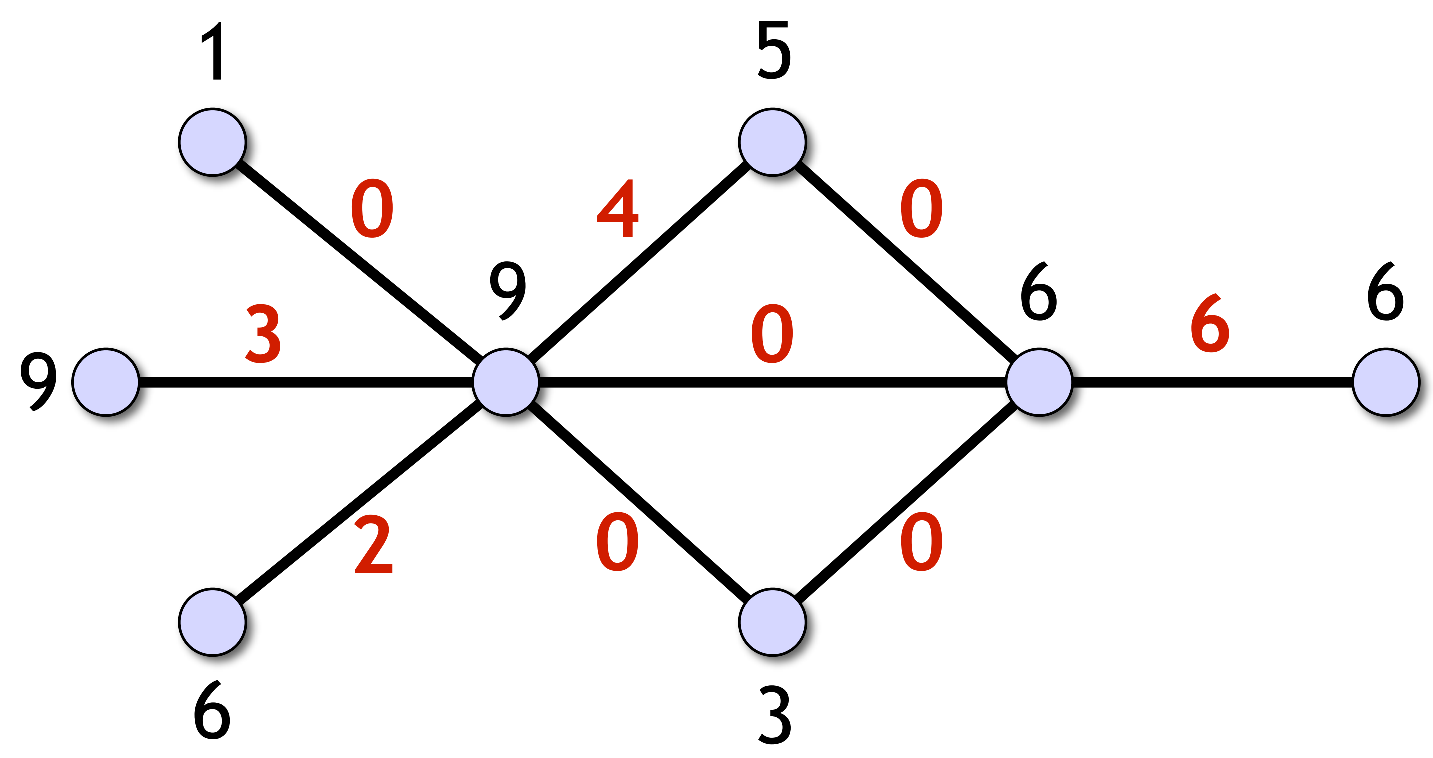
Notation:
$w(v)$ = weight of v

Edge packings and vertex covers

- **Edge packing**: weight $y(e) \geq 0$ for each edge e
  - Packing constraint: $y[v] \leq w(v)$ for each node v, where $y[v]$ = total weight of edges incident to v

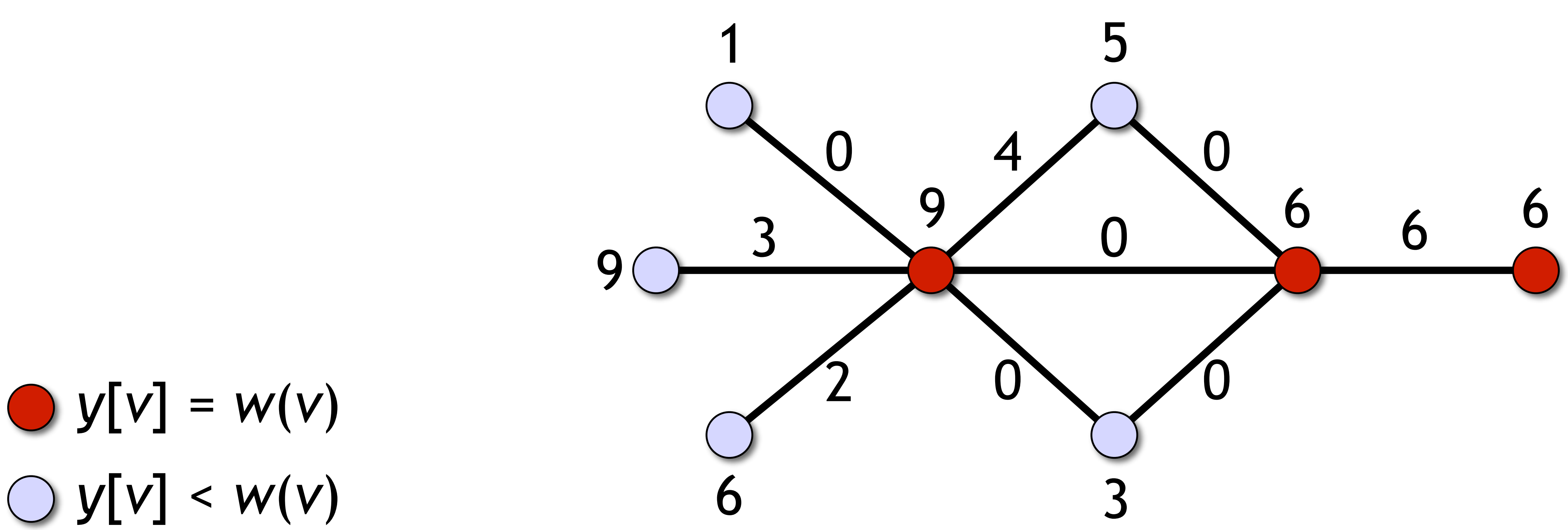
edge packing
≈ fractional matching

(LP relaxation)



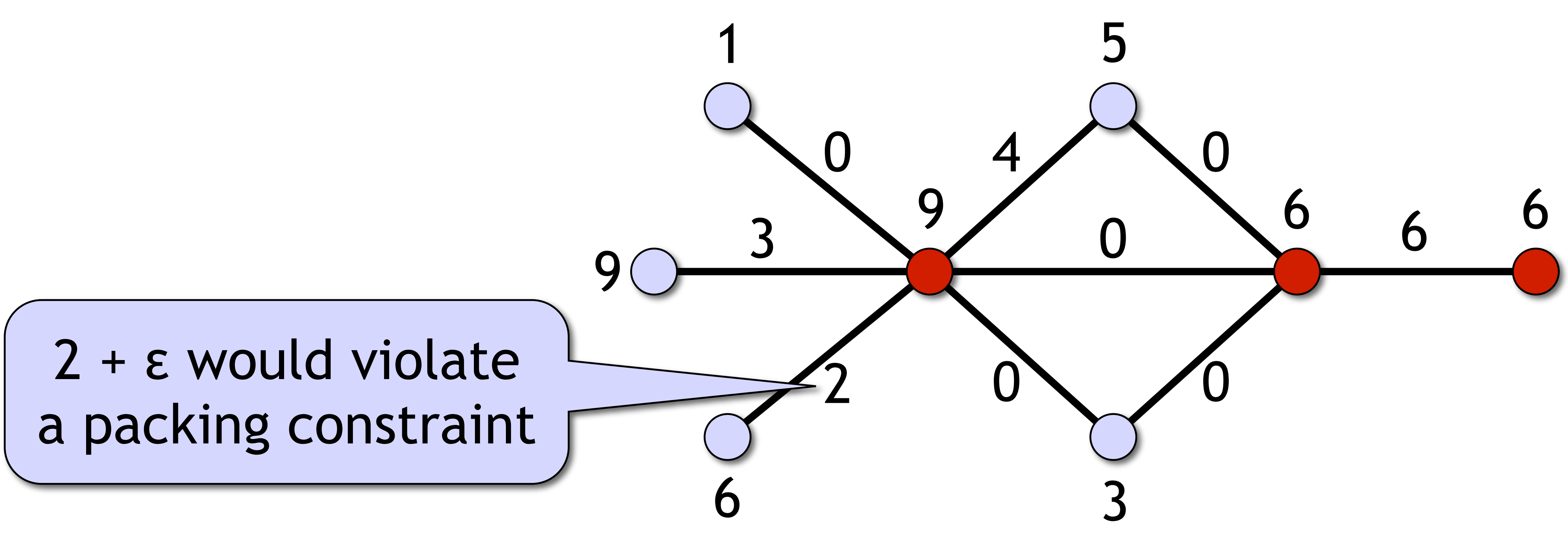
Edge packings and vertex covers

- Node v is **saturated** if $y[v] = w(v)$
  - Total weight of edges incident to v is ***equal*** to $w(v)$, i.e., the packing constraint holds with equality



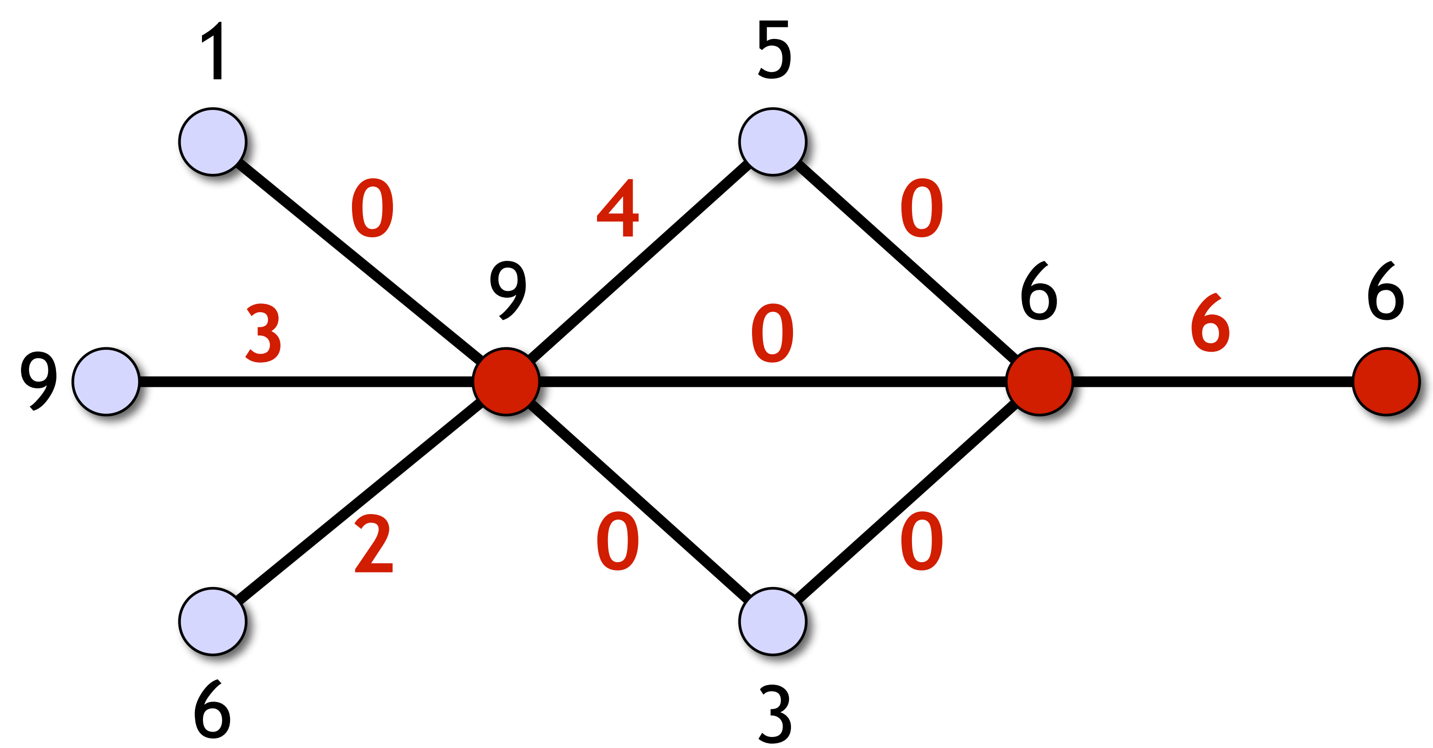
Edge packings and vertex covers

- Edge e is **saturated** if at least one endpoint of e is saturated
  - Equivalently: edge weight $y(e)$ can't be increased



Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  $\Longleftrightarrow$ none of the edge weights $y(e)$ can be increased
  $\Longleftrightarrow$ saturated nodes form a vertex cover

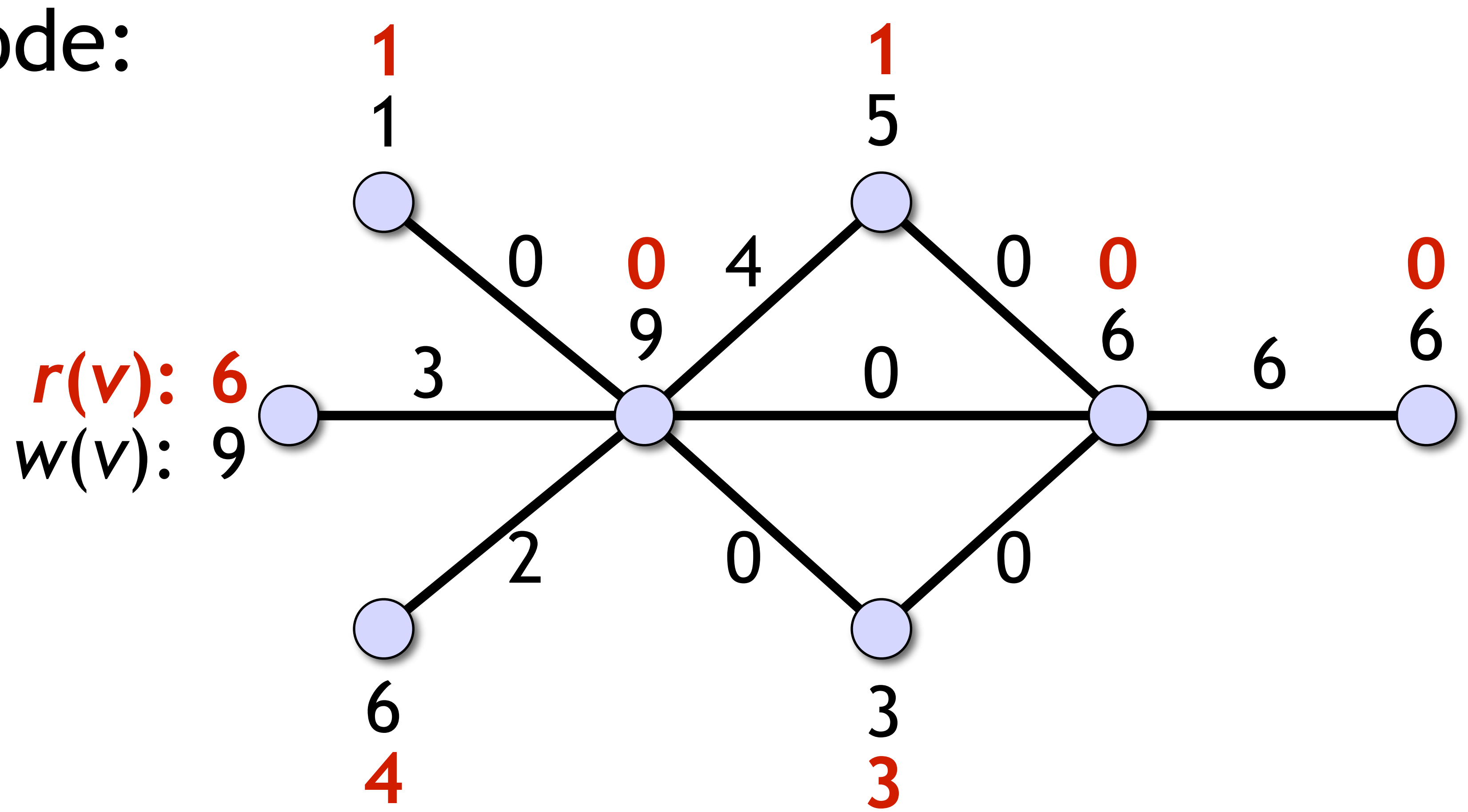


Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  $\Longleftrightarrow$ saturated nodes form a vertex cover
  - ... and saturated nodes are **2-approximation** of minimum-weight vertex cover (Bar-Yehuda & Even 1981)
- How to find a maximal edge packing...?
  - Phase I: "*greedy but safe*", cf. Khuller et al. (1994), Papadimitriou & Yannakakis (1993)
  - Phase II: if phase I fails to saturate an edge $e = \{u,v\}$, we can *break symmetry* between u and v; exploit it!

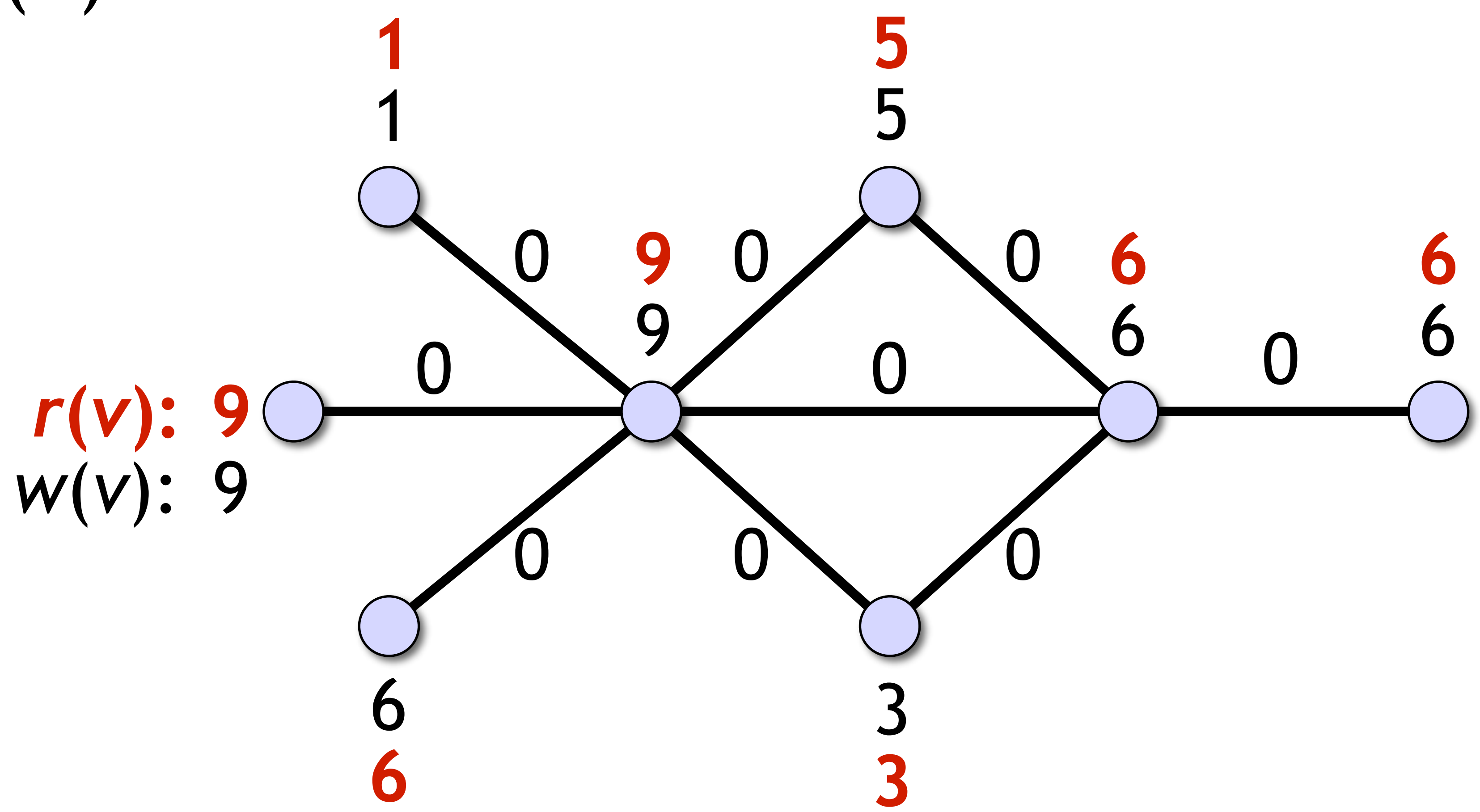
Finding a maximal edge packing: phase I

- $y[v]$ = total weight of edges incident to node v
- **Residual capacity** of node v: $r(v) = w(v) - y[v]$
- Saturated node: $r(v) = 0$



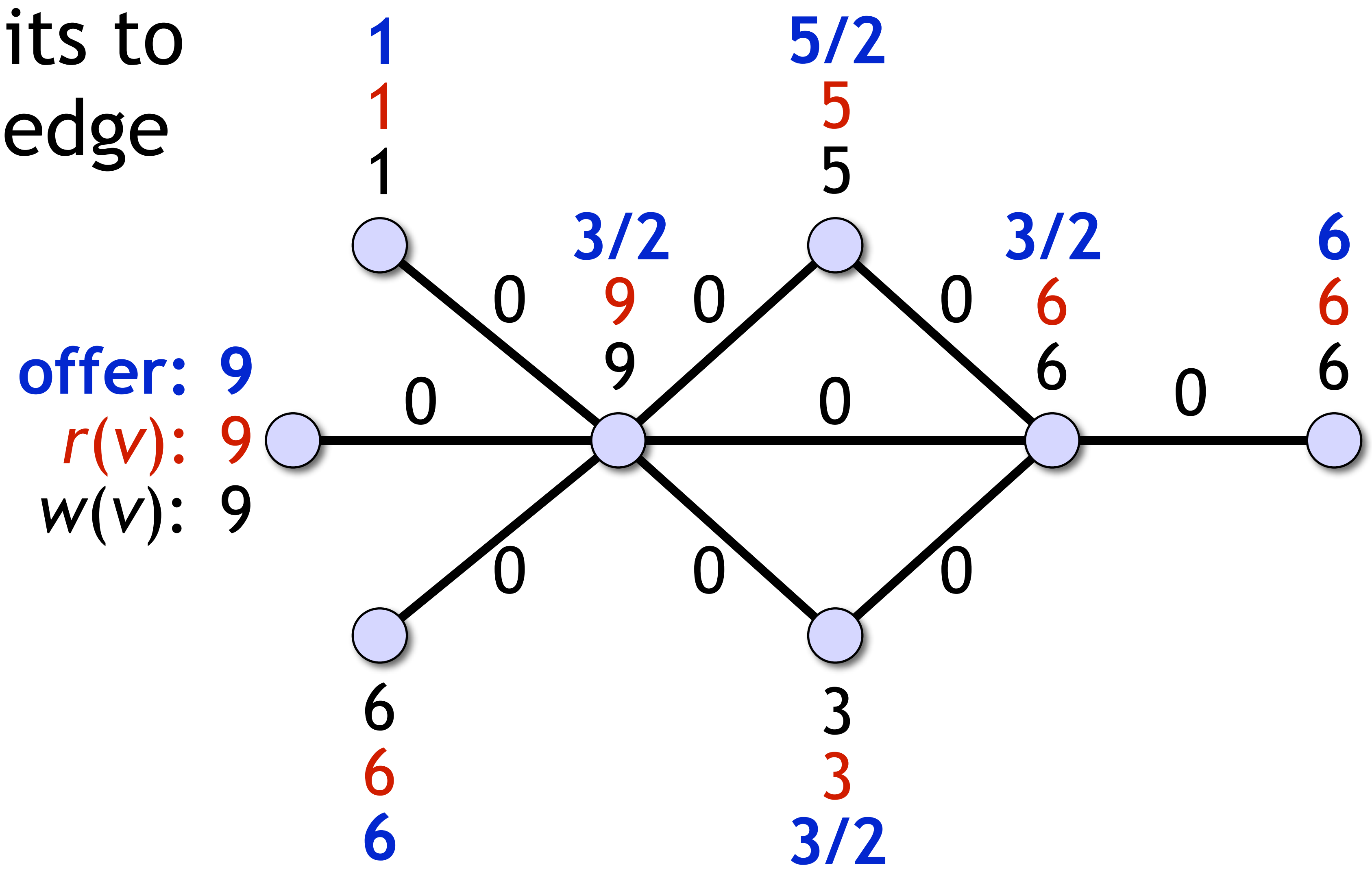
Finding a maximal edge packing: phase I

Start with a trivial edge packing $y(e) = 0$



Finding a maximal edge packing: phase I

Each node *v* **offers** *r*(*v*)/deg(*v*) units to each incident edge

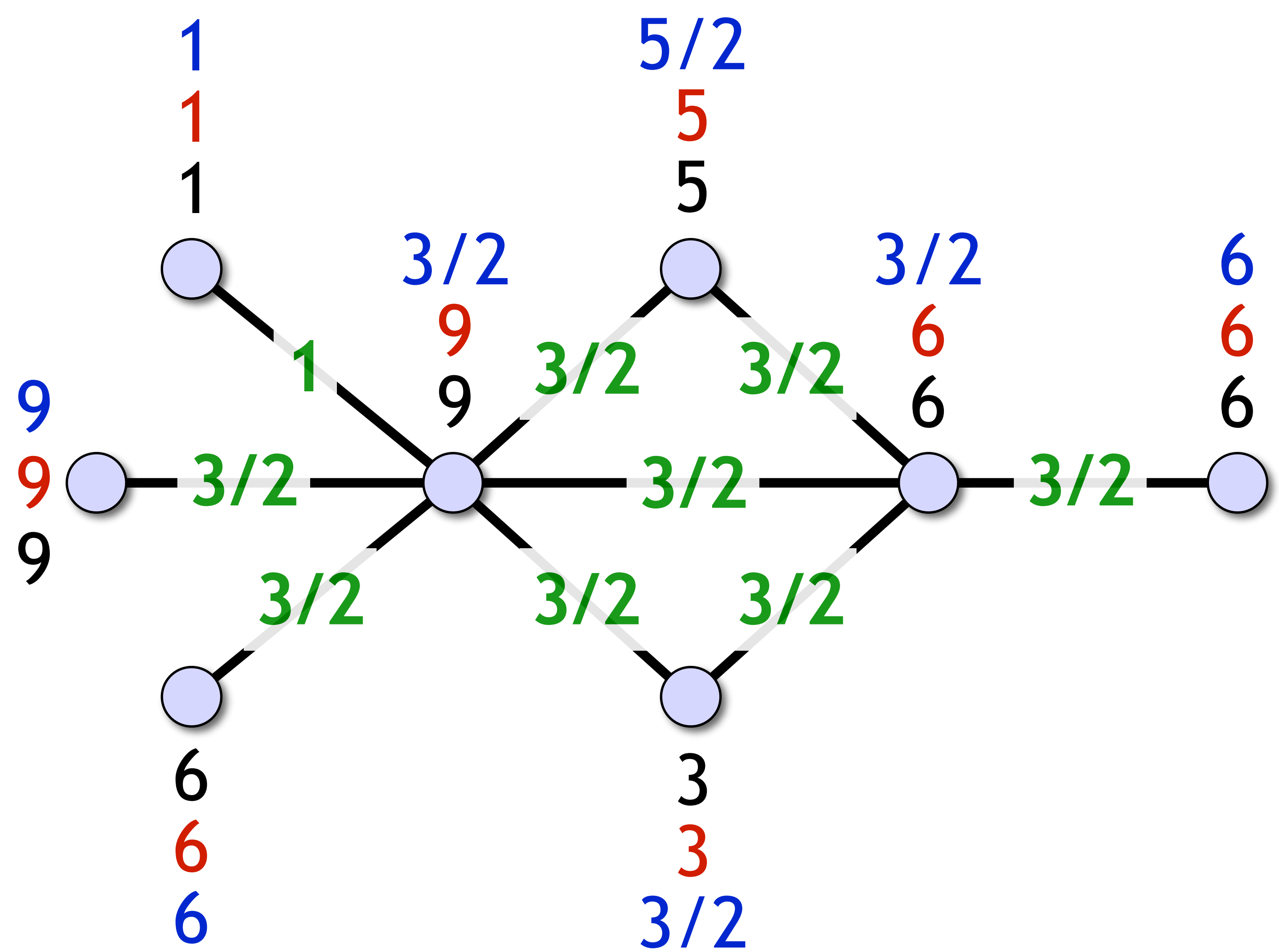


Finding a maximal edge packing: phase I

Each edge **accepts** the smallest of the 2 offers it received

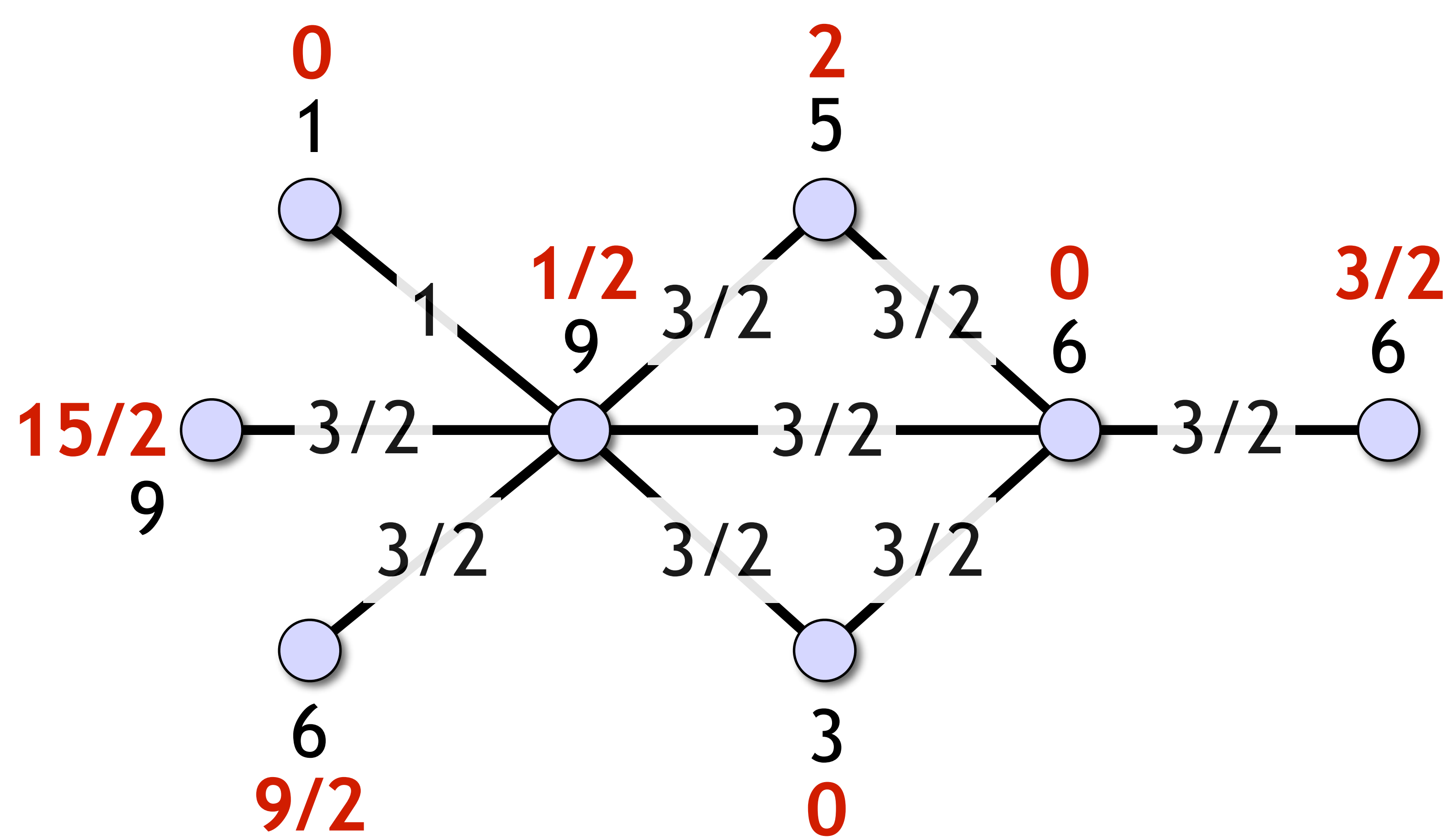
Increase $y(e)$ by this amount

- Safe, can’t violate packing constraints



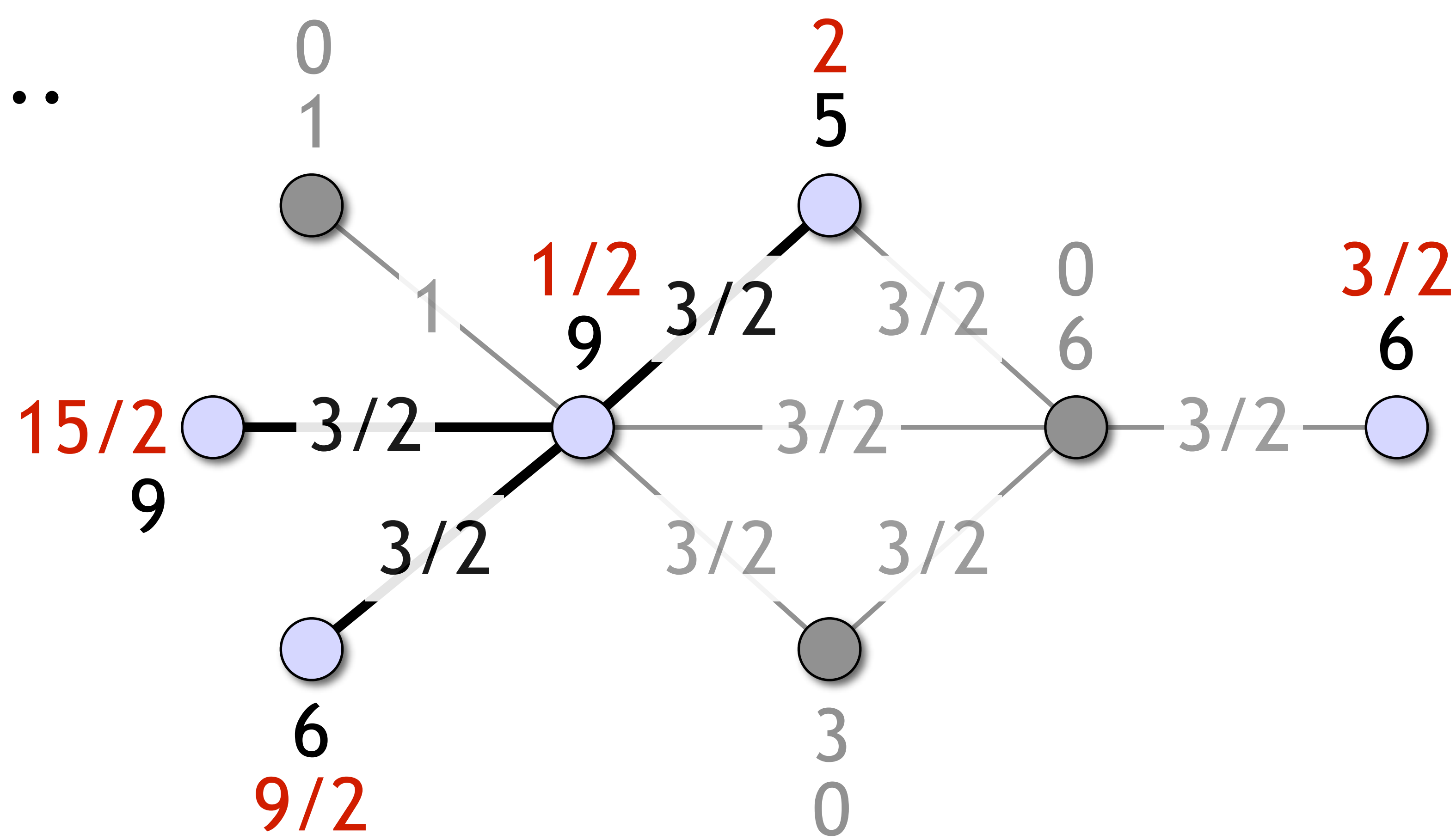
Finding a maximal edge packing: phase I

Update **residuals**...



Finding a maximal edge packing: phase I

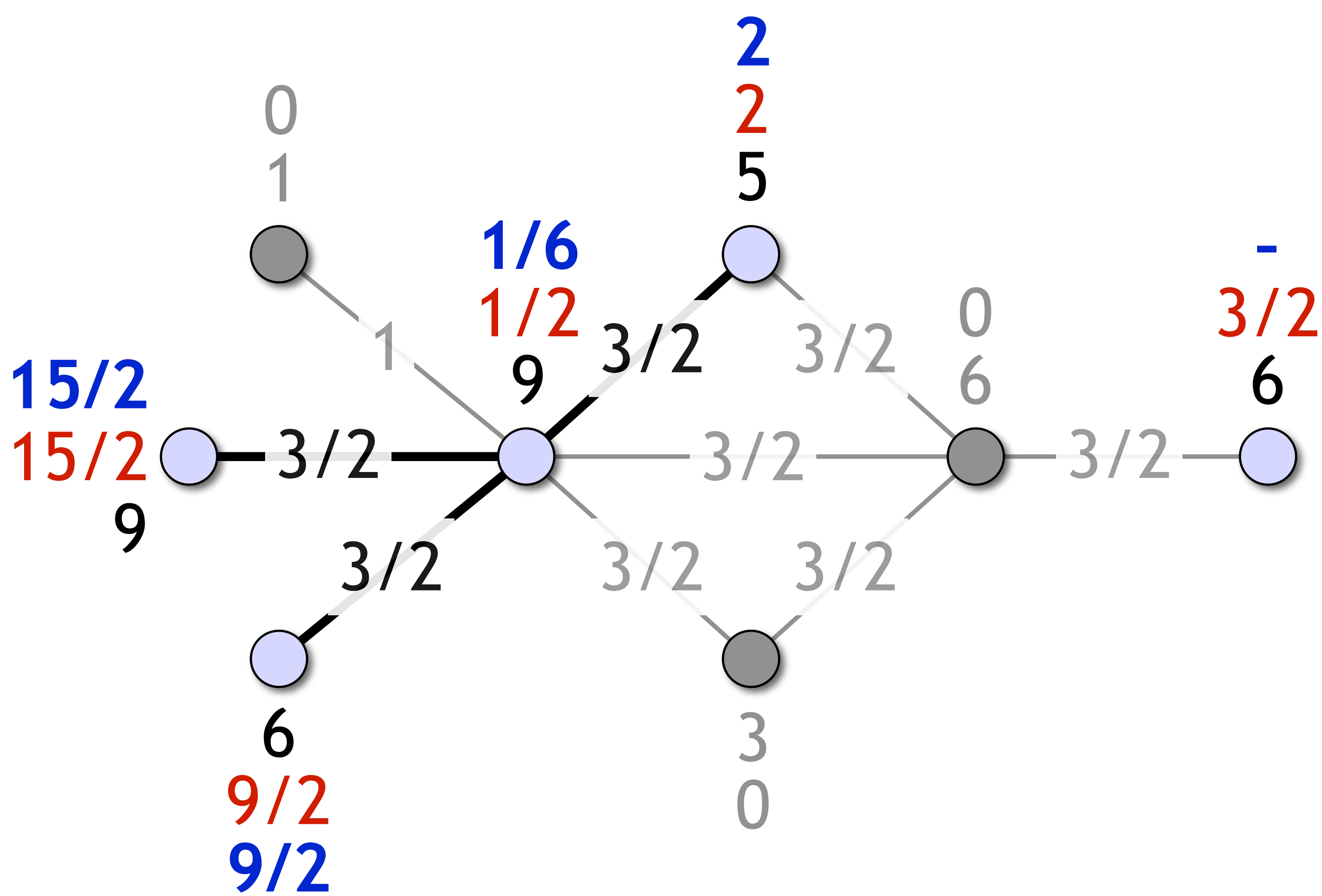
Update residuals, discard saturated nodes and edges...



Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

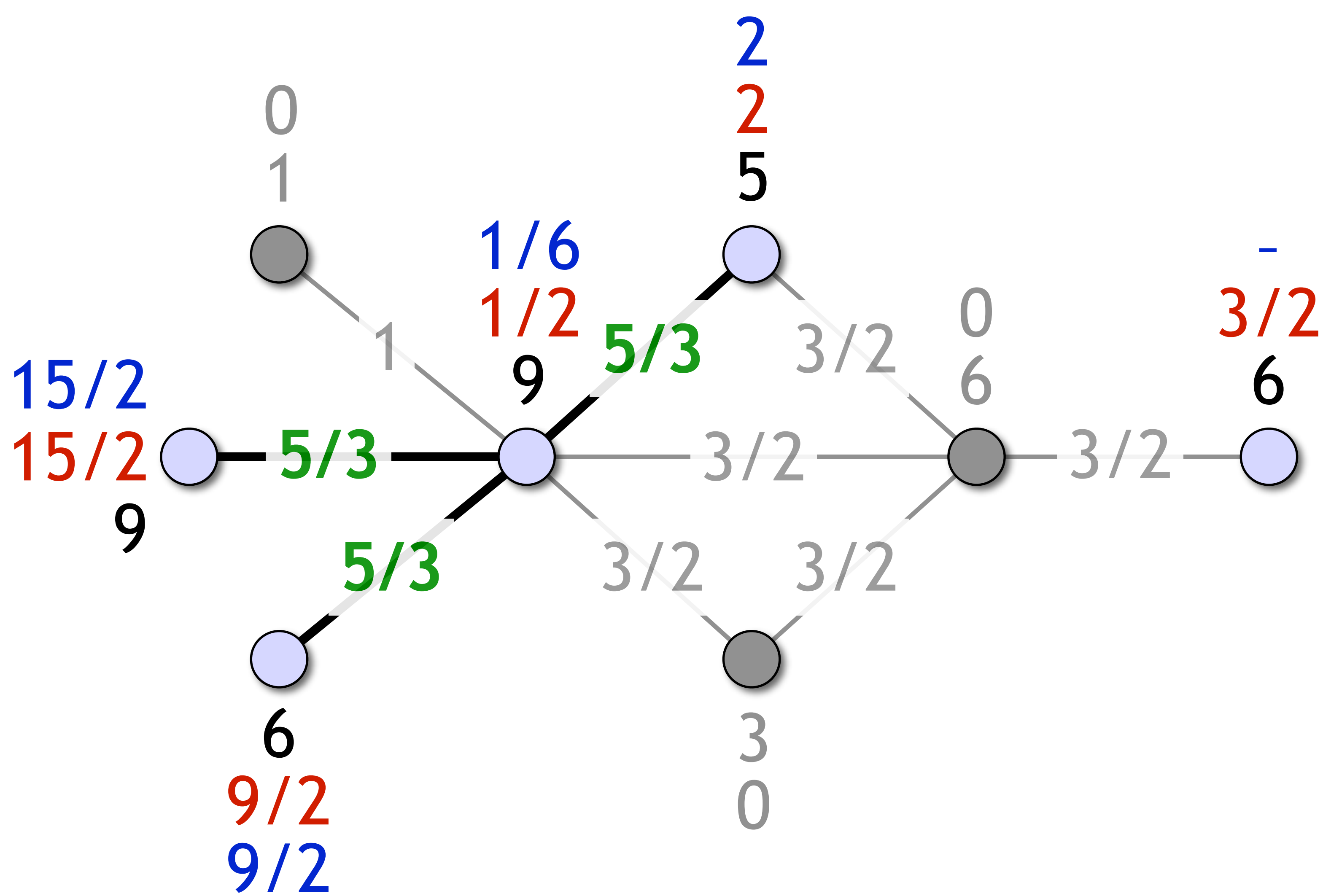


Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...



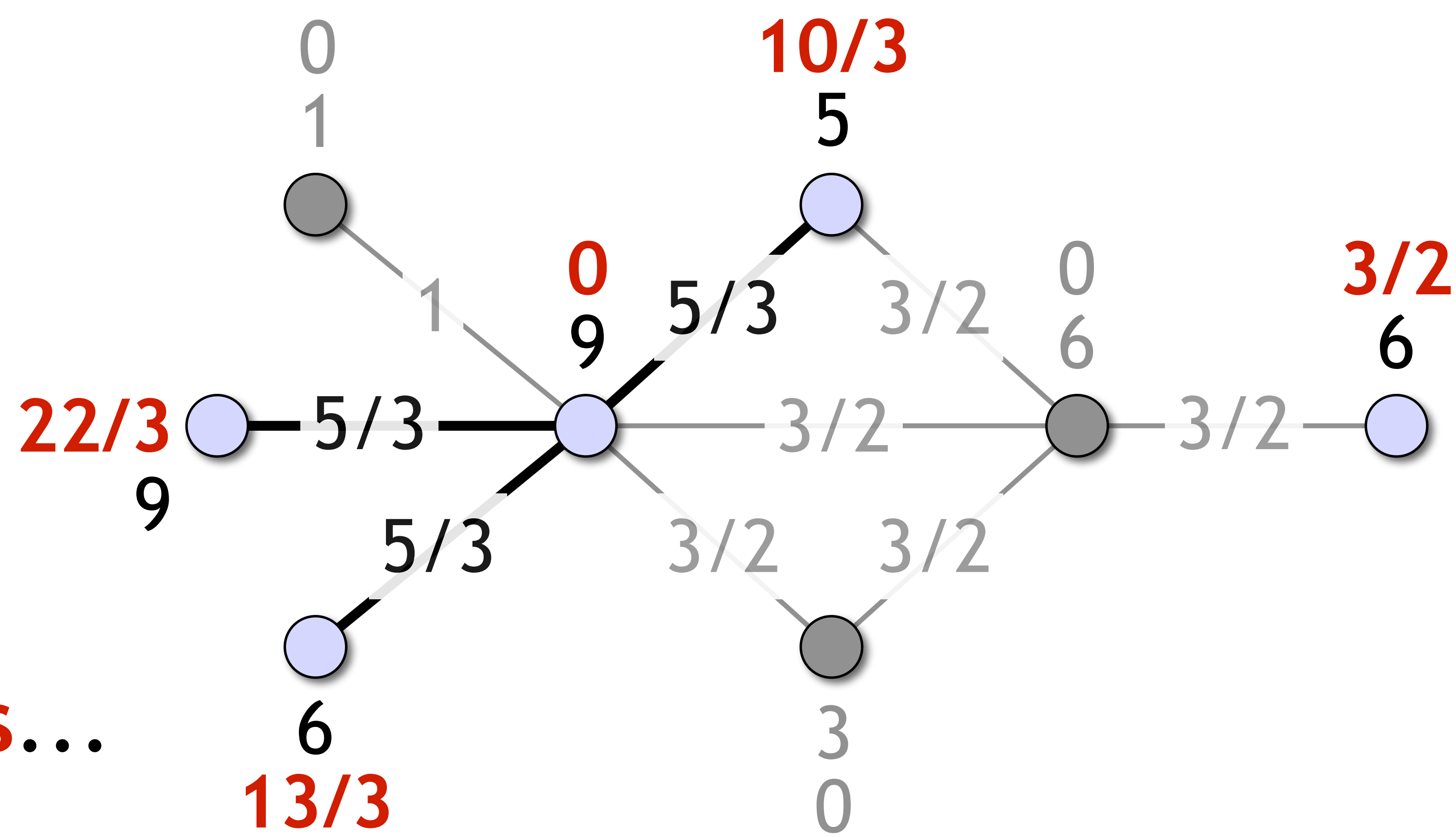
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

Update residuals...



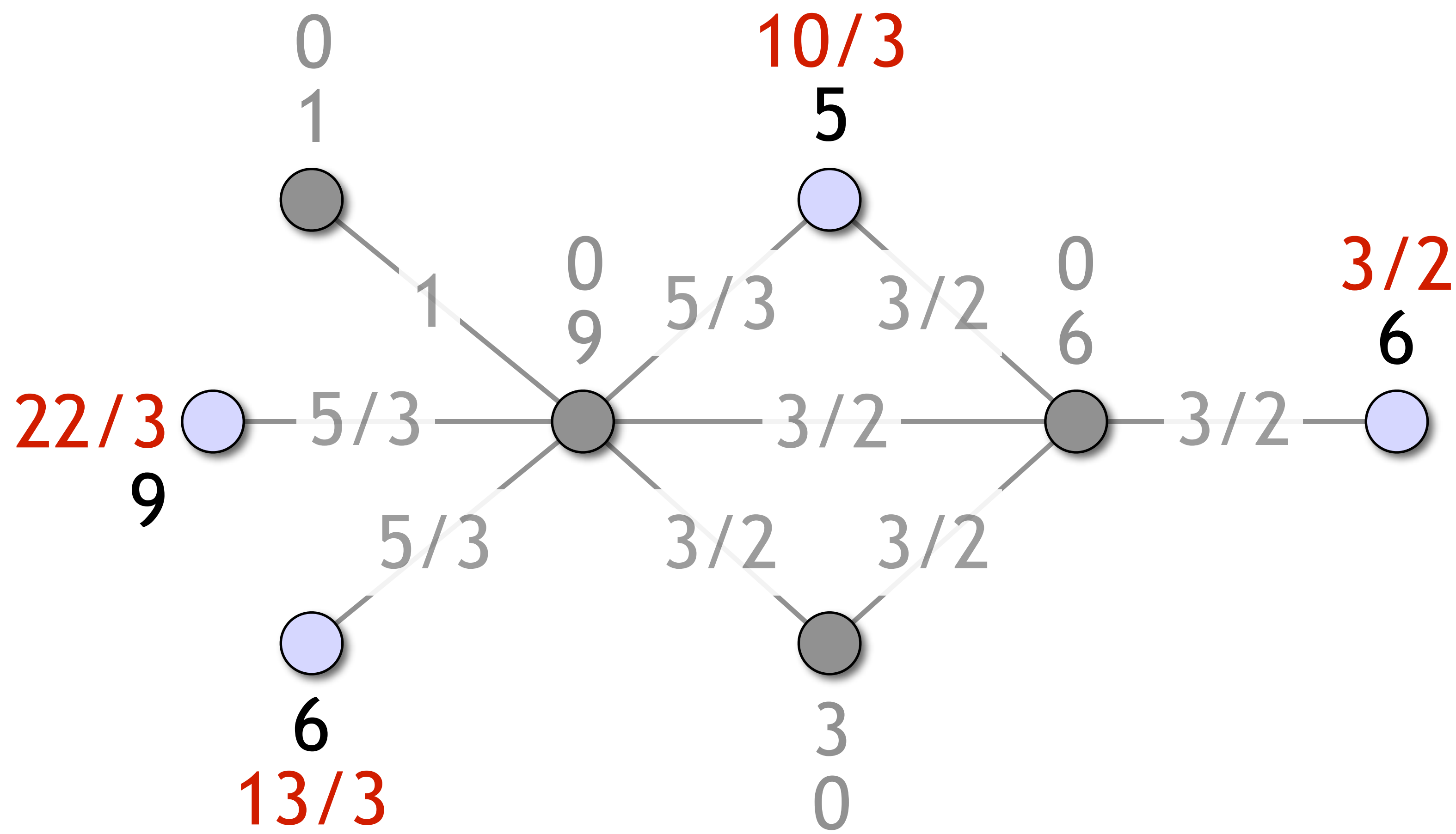
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

Update residuals and graph, etc.

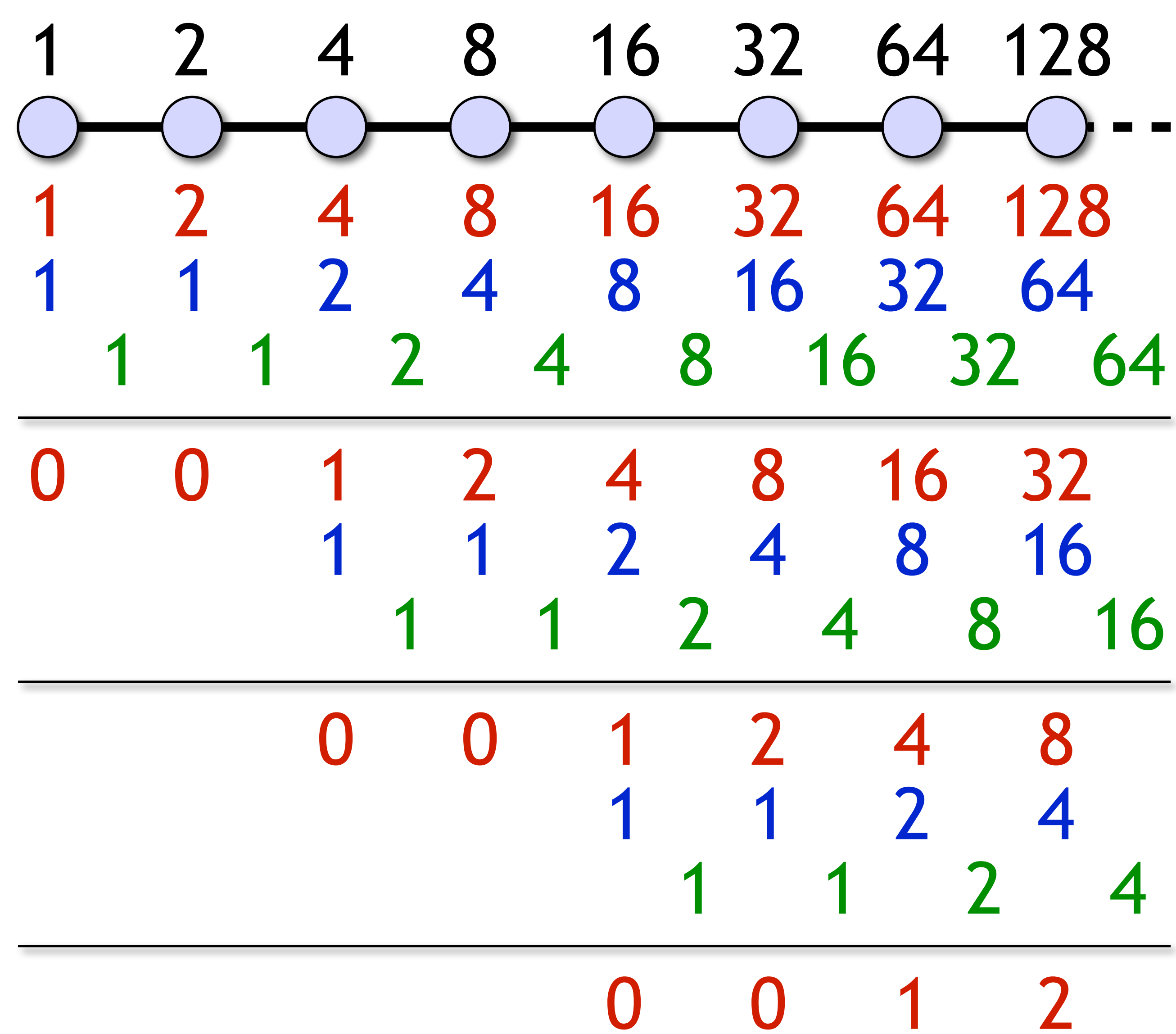


Finding a maximal edge packing: phase I

We are making some progress towards finding a maximal edge packing...

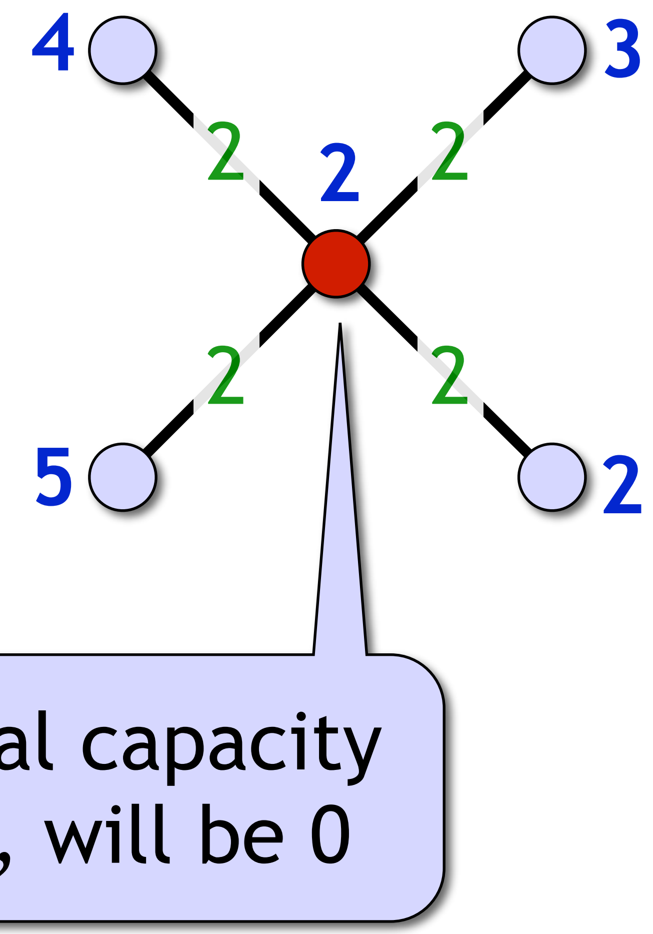
But this is **too slow**!

How to make it faster?



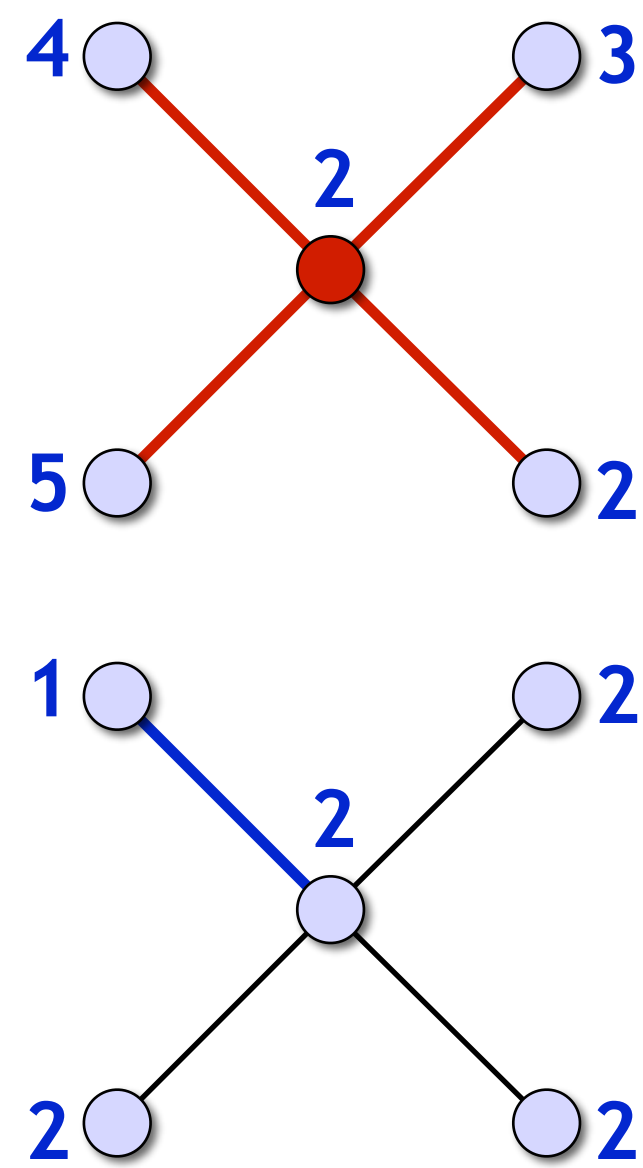
Finding a maximal edge packing: colouring trick

- Offer is a local minimum:
  - Node will be **saturated**
  - And all edges incident to it will be saturated as well



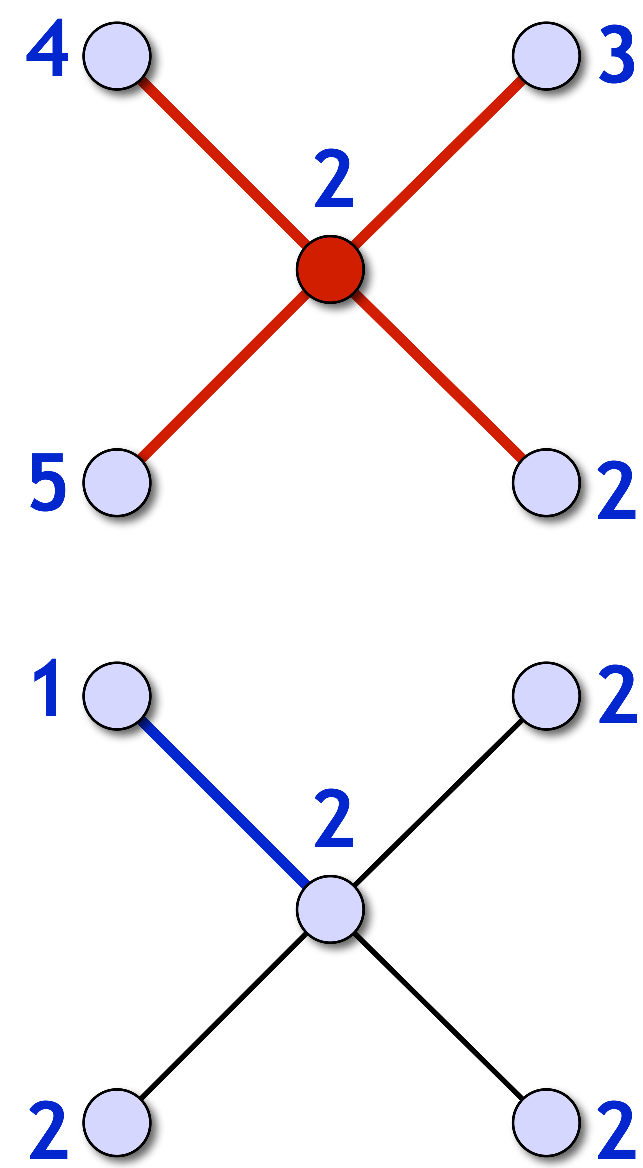
Finding a maximal edge packing: colouring trick

- Offer is a local minimum:
  - Node will be **saturated**
- Otherwise there is a neighbour with a different offer:
  - Interpret the offer sequences as colours
  - Nodes *u* and *v* have different colours: {*u*, *v*} is **multicoloured**



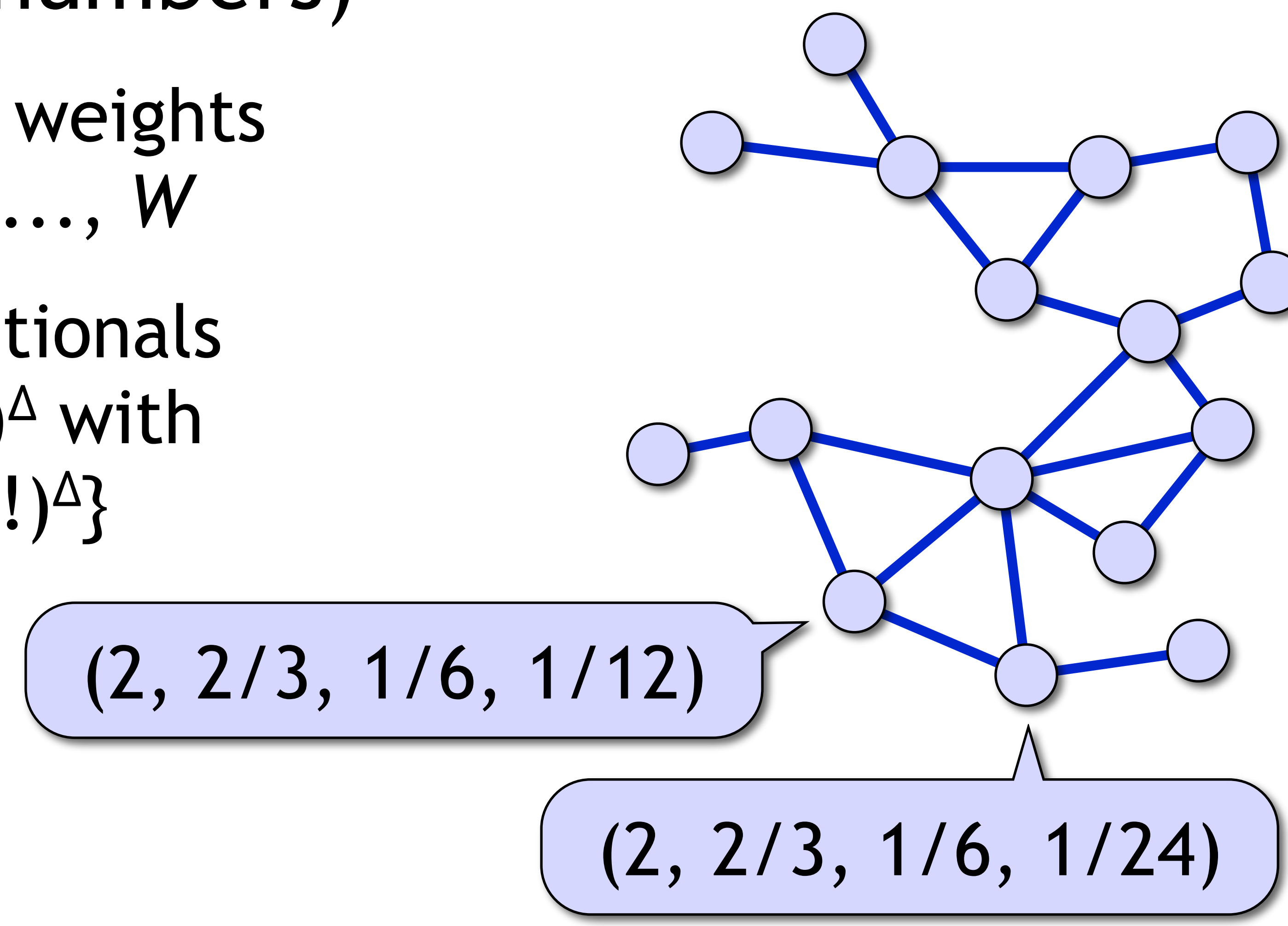
Finding a maximal edge packing: colouring trick

- Progress guaranteed:
  - On each iteration, for each node, at least one incident edge becomes **saturated** or **multicoloured**
  - Such edges are be discarded in phase I; maximum degree Δ decreases by at least one
  - Hence in Δ rounds all edges are saturated or multicoloured



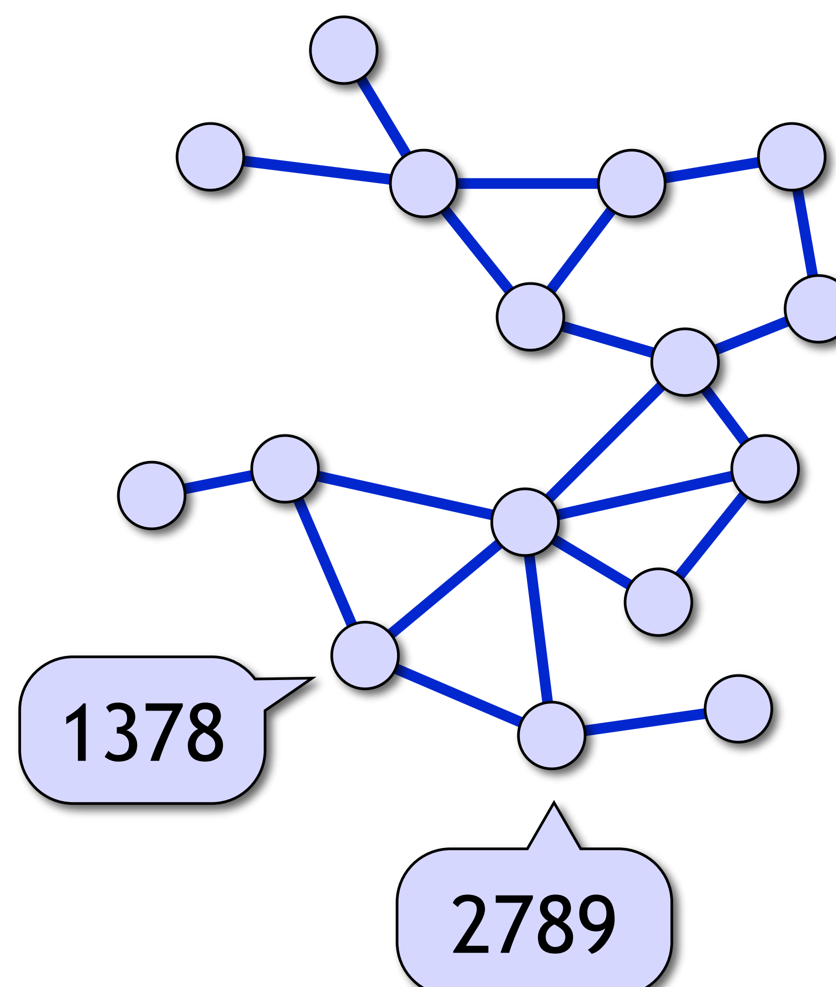
Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ offers (rational numbers)
  - Assume that node weights are integers 1, 2, ..., W
  - Then offers are rationals of the form $q/(\Delta!)^\Delta$ with $q \in \{1, 2, \ldots, W(\Delta!)^\Delta\}$



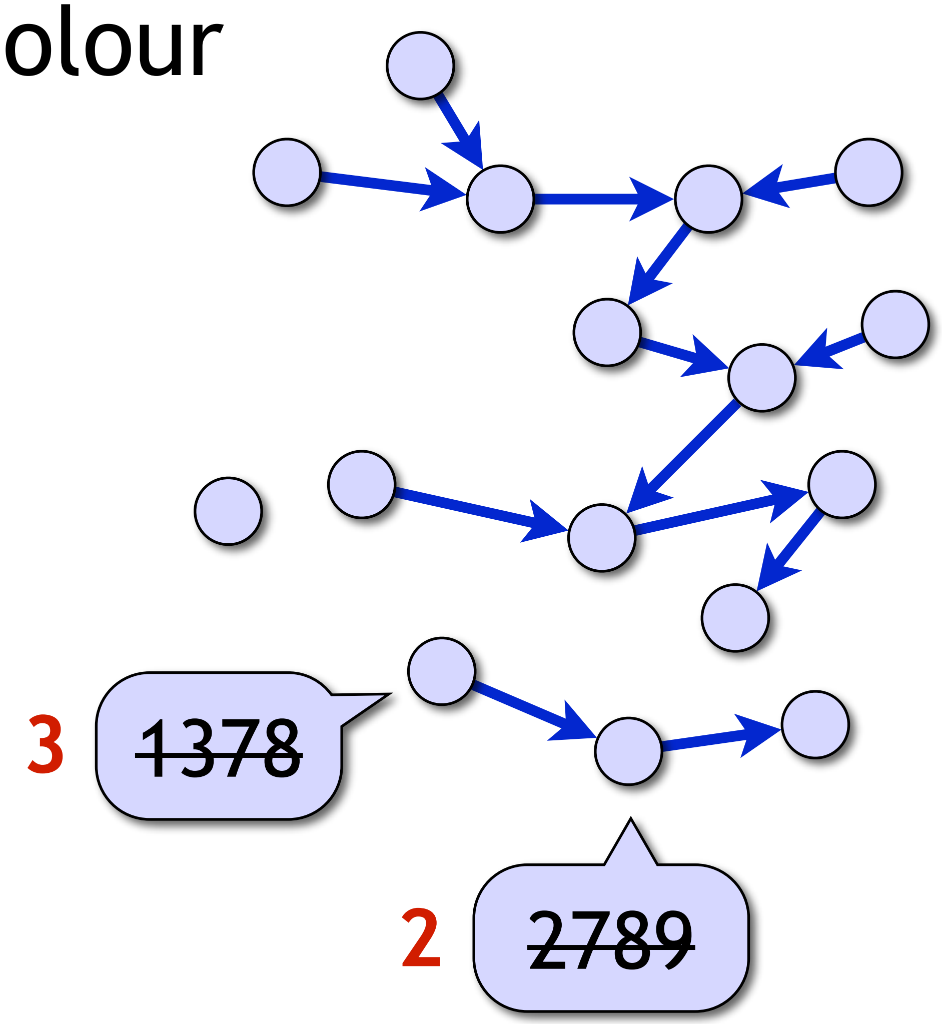
Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ offers (rational numbers)
  - Assume that node weights are integers 1, 2, ..., W
  - Then offers are rationals of the form $q/(\Delta!)^\Delta$ with $q \in \{1, 2, \ldots, W(\Delta!)^\Delta\}$
  - $k = (W(\Delta!)^\Delta)^\Delta$ possible colours, replace with integers 1, 2, ..., k



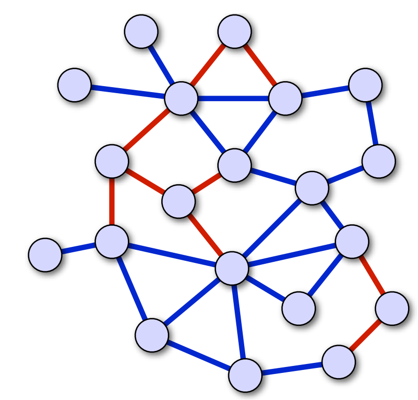
Finding a maximal edge packing: phase II

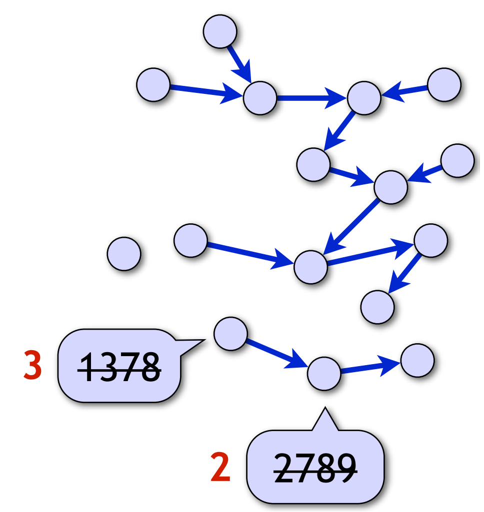
- Proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour
- Partition in Δ forests
  - Use Cole–Vishkin (1986) style colour reduction algorithm
- Use colour classes to saturate edges
- $O(\Delta + \log^* W)$ rounds

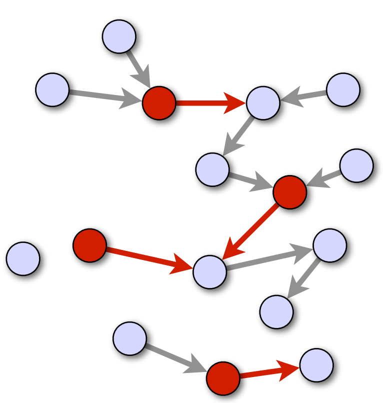


Finding a maximal edge packing: summary

- Maximal edge packing and 2-approximation of vertex cover in time $O(\Delta + \log^* W)$
  - W = maximum node weight
- Unweighted graphs: running time simply $O(\Delta)$, independent of n
- Everything can be implemented in the **port-numbering model**







Vertex cover and set cover in anonymous networks: summary

- 2-approximation of vertex cover in time $O(\Delta)$ in the **port-numbering model**
  - Idea: consider a more general problem, minimum-***weight*** vertex cover
- 2-approximation of vertex cover in time poly(Δ) in the **broadcast model**?
  - Idea: consider a more general problem, minimum-weight ***set cover***!

Take-home messages

- Algorithms that we saw today are **strictly local**
  - Running time independent of the number of nodes
  - Output of a node depends only on its local neighbourhood
  - Very efficient, can be used in arbitrarily large networks
  - Deterministic, highly fault-tolerant
- There are non-trivial graph problems that can be solved with strictly local algorithms!
  - More: www.cs.helsinki.fi/jukka.suomela/local-survey