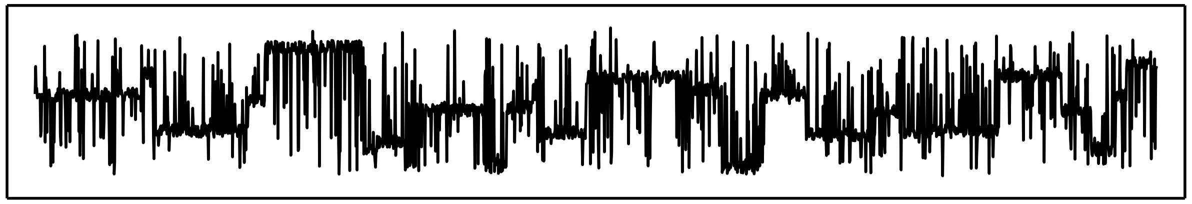
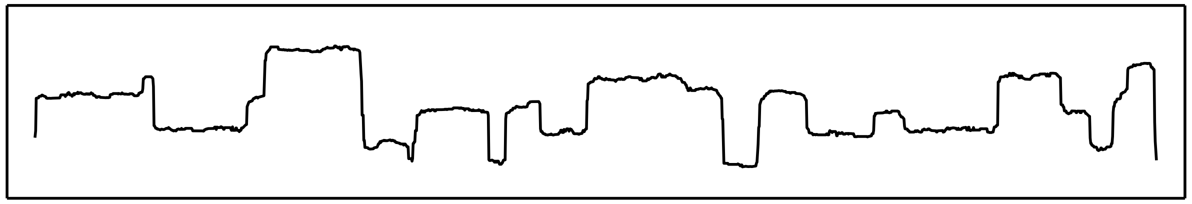
Median Filtering is Equivalent to Sorting

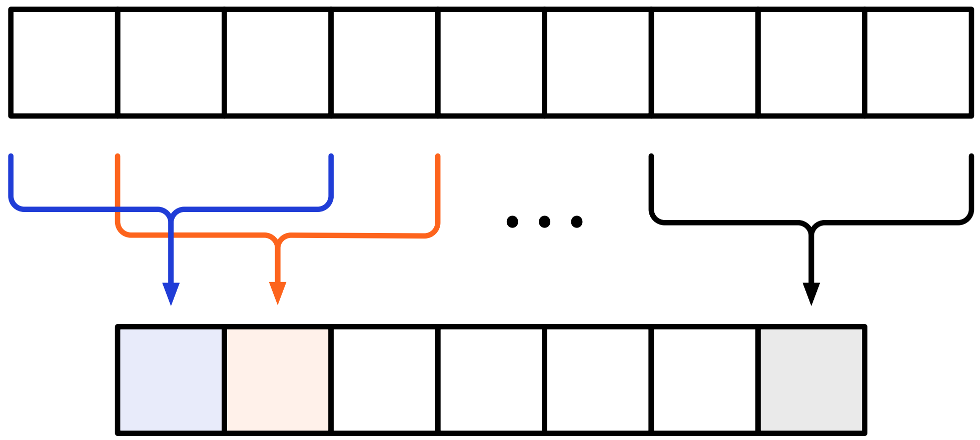




Jukka Suomela · Aalto University

TU Berlin · 6 August 2014

Median filter



input: n elements

window size: k

output: $n-k+1$ medians

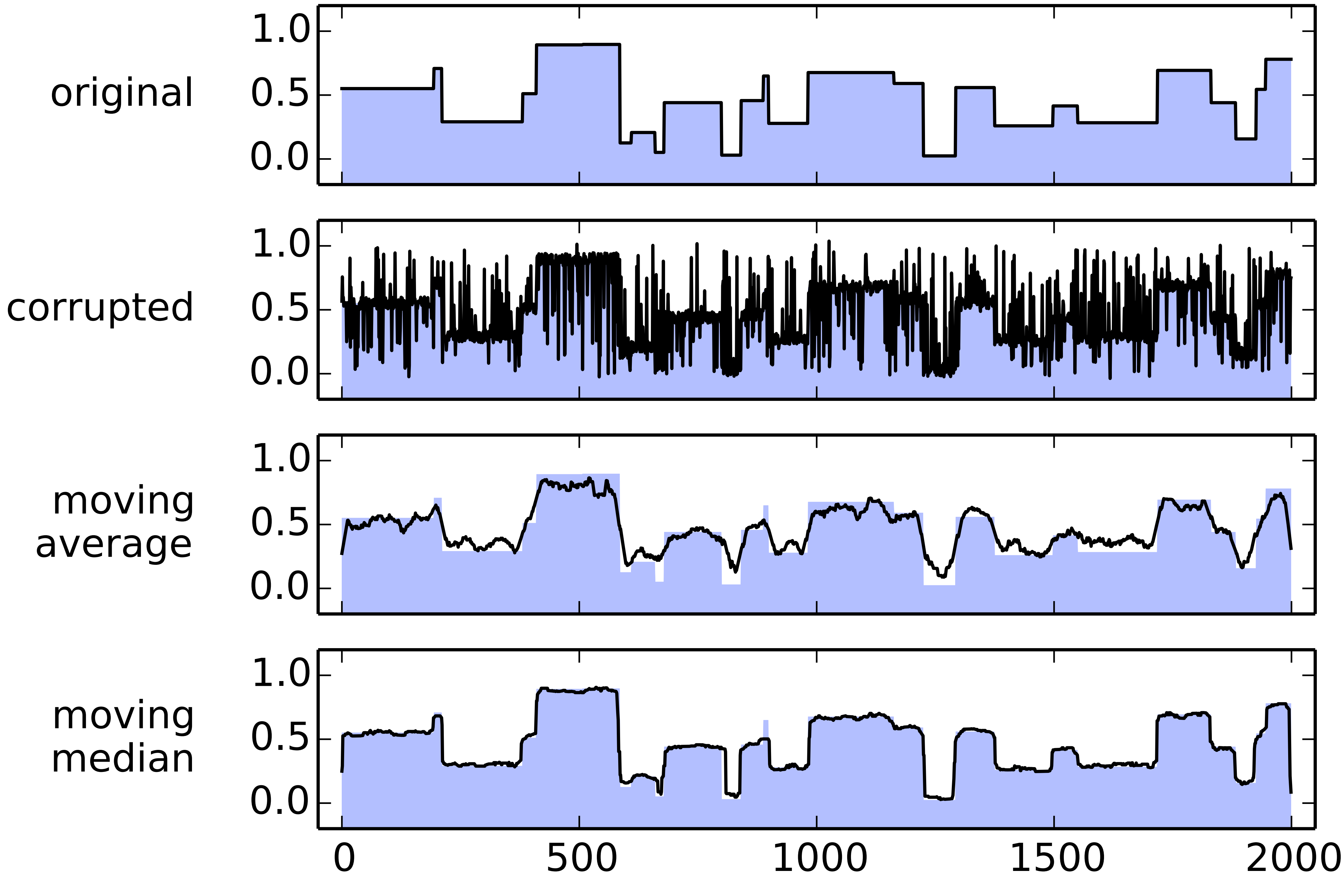
a.k.a. sliding window median,
moving median, running median,
rolling median, median smoothing

Median filter

- **In numerous scientific computing systems:**
  - *R*: “runmed”
  - *Mathematica*: “MedianFilter”
  - *Matlab*: “medfilt1”
  - *Octave*: “medfilt1” (signal package)
  - *SciPy*: “medfilt1” (scipy.signal module)

Median filter

- **In numerous scientific computing systems:**
  - *R*, *Mathematica*, *Matlab*, *Octave*, *SciPy* …
- **2D version in image processing:**
  - *Photoshop*: “Median” filter
  - *Gimp*: “Despeckle” filter



original
corrupted
moving average
moving median
1.0
0.5
0.0
0
500
1000
1500
2000

n: input size
k: window size

Prior work

- **Trivial:**
  - compute each median separately
  - $O(nk)$
- **“Streaming approach”:**
  - maintain a sliding window
  - $O(n \log k)$

n: input size
k: window size

Prior work

- **“Streaming approach”**
- **Sliding window data structure, supports operations:**
  - “find median”
  - “remove oldest and add new element”

n: input size
k: window size

Prior work

- **Sliding window data structures for *B*-bit integers:**
  - histogram with 2^B buckets
  - with linear scanning: $O(n2^B)$
  - with binary trees: $O(nB)$
  - with van Emde Boas trees: $O(n \log B)$

n: input size
k: window size

Prior work

- **General sliding window data structures:**
  - maxheap-minheap pair: $O(n \log k)$
  - binary search trees: $O(n \log k)$
  - finger trees: $O(n \log k)$
  - doubly-linked lists: $O(nk)$
  - sorted arrays: $O(nk)$

n: input size
k: window size

Prior work

- **Maxheap-minheap pair**
  - Astola–Campbell (1989)
    Juhola et al. (1991)
    Härdle–Steiger (1995) …
- **Fast in practice**
- **Fast in theory, $O(n \log k)$ comparisons**

n: input size
k: window size

Lower bounds

- **For comparison-based algorithms: $O(n \log k)$ is optimal**
  - Juhola et al. (1991)
    Krizanc et al. (2005) …
- **Reduction from sorting**

n: input size
k: window size

State of the art

- **$O(n \log k)$ comparisons is optimal**
  - known since 1990s
  - nothing more to do here,
    case closed, problem solved

n: input size
k: window size

State of the art

- **$O(n \log k)$ comparisons is optimal**
- **But we also know that $O(n \log n)$ comparisons is optimal for sorting in the worst case, yet this is not the full story!**
  - integer sorting, adaptive sorting, cache-efficient sorting, GPU sorting …

n: input size
k: window size

State of the art

- **And what about implementations...**
  - *R*: $\approx O(n \log k)$
  - *Mathematica*: $\approx O(nk)$
  - *Matlab*: $\approx O(nk)$
  - *Octave*: $\approx O(nk)$
  - *SciPy*: $\approx O(nk)$

why?!

didn't we do better already in 1980s?

Key idea

- **Prior work:**
  - "*median filtering is as hard as sorting*"
- **Could we prove a matching upper bound:**
  - "*median filtering is as easy as sorting*" ??

Key idea

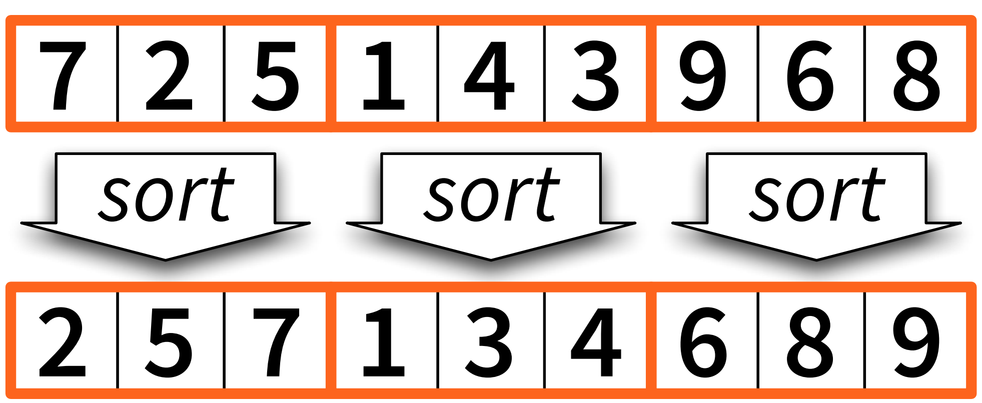
- **If we could show that:**
  - "*median filtering is equivalent to sorting*"
- **Then we could apply everything that we know about sorting here!**
  - adaptive sorting → adaptive median filter
  - integer sorting → integer median filter …

Key idea

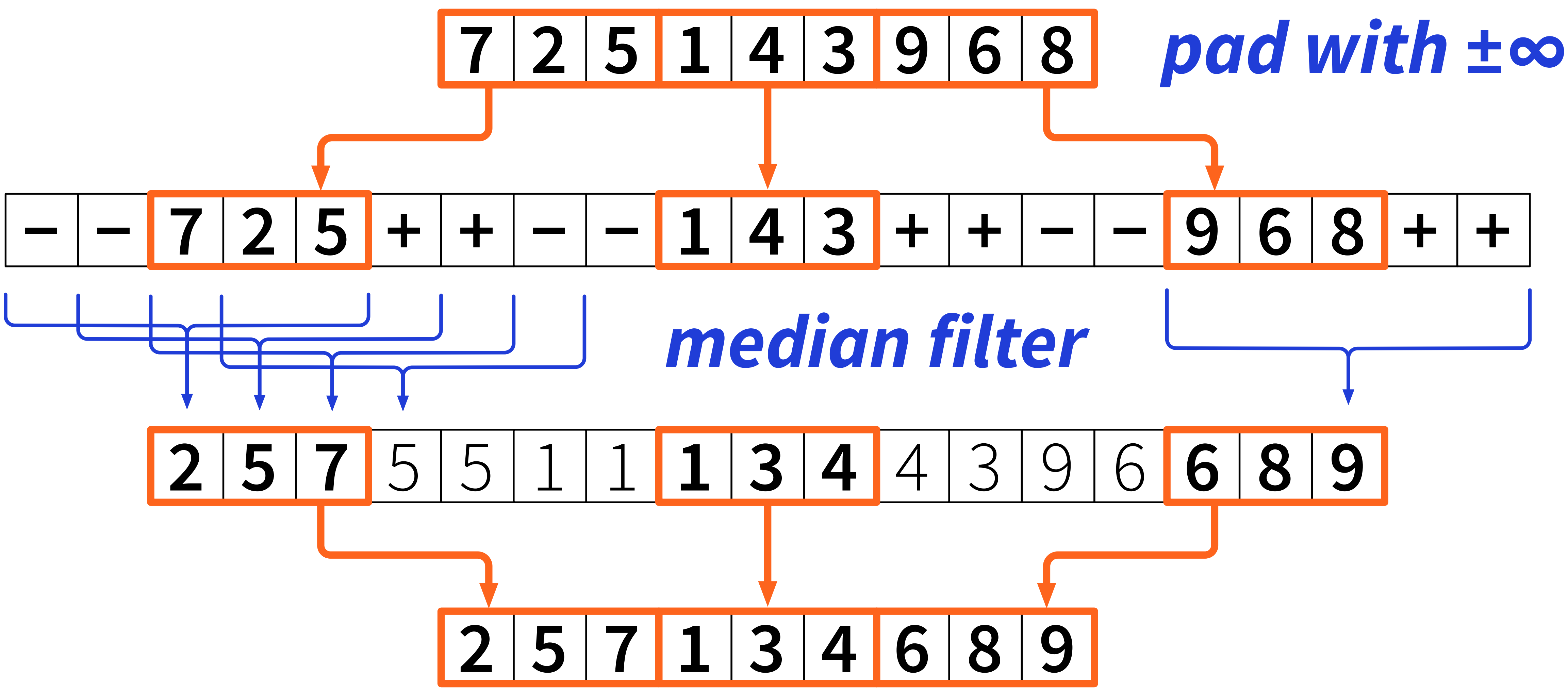
- **If we could show that:**
  - “*median filtering is equivalent to sorting*”
- **Then we could apply everything that we know about sorting here!**
  - all scientific computing packages know how to sort efficiently

Sorting-based lower bound

- **Piecewise sorting: sort *n*/*k* blocks of size *k***
  - with comparison sort: $O(n \log k)$ optimal



Sorting-based lower bound



Sorting-based lower bound

n: input size
k: window size

- **Piecewise sorting: sort n/k blocks of size k**
  - with comparison sort: $O(n \log k)$ optimal
- **Can be solved with $O(1)$ median filter operations**
  - and some preprocessing & postprocessing

Sorting-based median filter

n: input size
k: window size

- **Piecewise sorting: sort n/k blocks of size k**
- **Prior work:**
  - median filter ≈ as hard as piecewise sorting
- **This work:**
  - median filter ≈ as easy as piecewise sorting

Sorting-based median filter

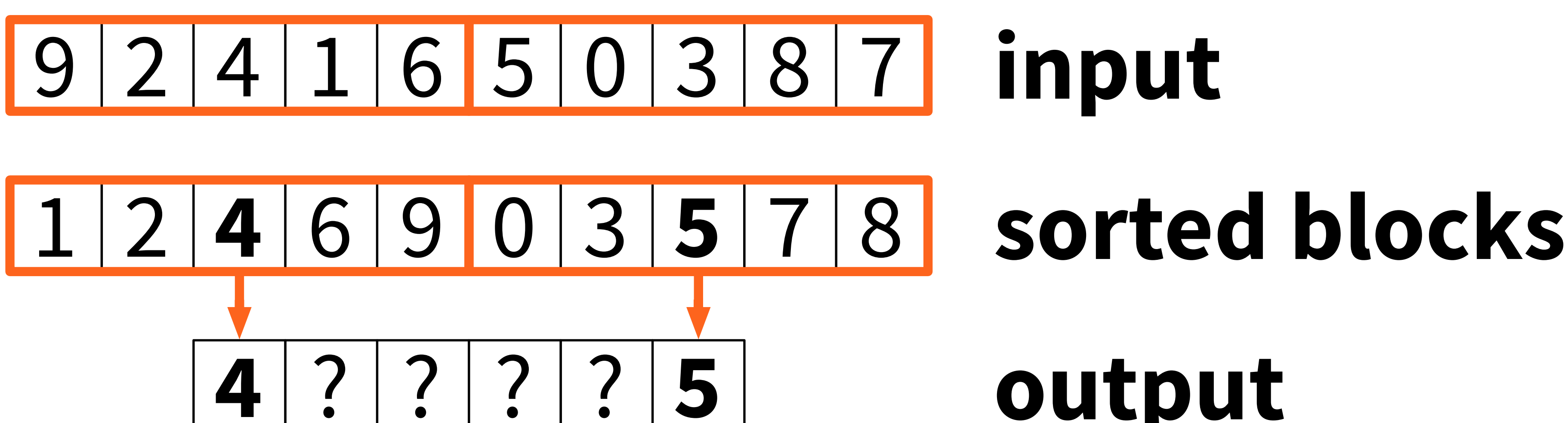
- **High-level idea:**
  - preprocessing = piecewise sorting
  - median filtering now possible in linear time!
- **Simple and efficient**
  - works very well also in practice

Sorting-based median filter

- **Prior work:**
  - median filtering ≈ data structure problem
  - how to maintain sliding window efficiently?
- **This work:**
  - median filtering ≈ algorithm problem
  - how to preprocess data?

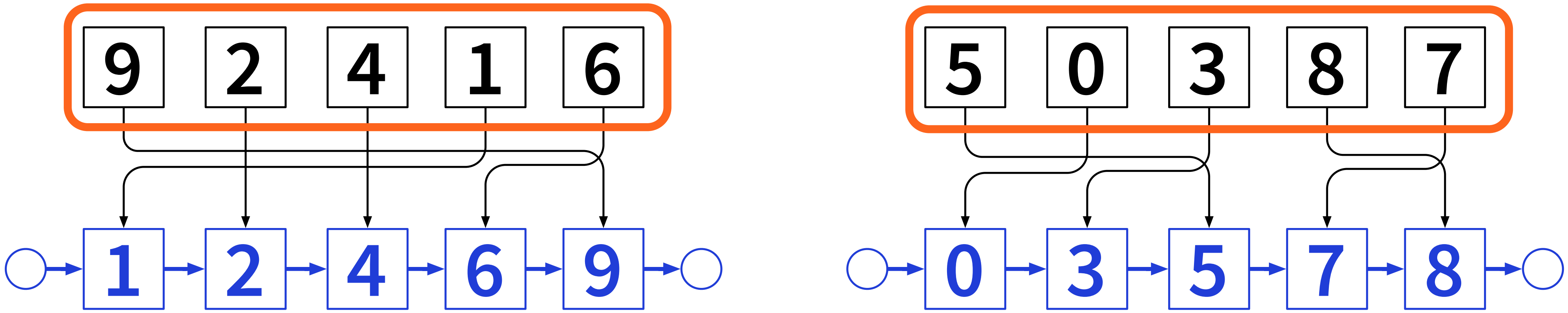
Sorting-based median filter

- **How does piecewise sorting help?**
  We only know one median per block...



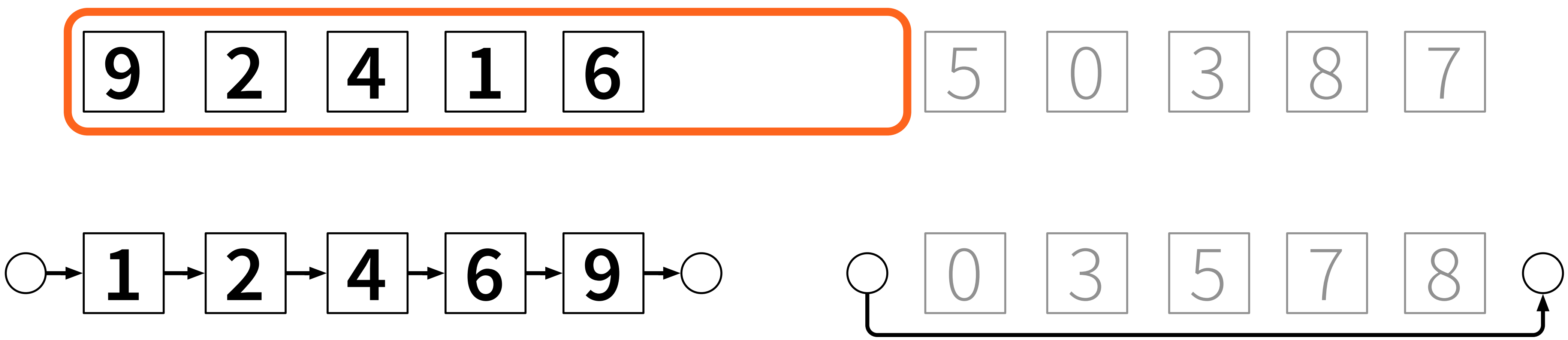
Sorting-based median filter

- **Basic idea: maintain *sorted doubly-linked lists* for each *block***



Sorting-based median filter

- Sliding window = two sorted linked lists



Sorting-based median filter

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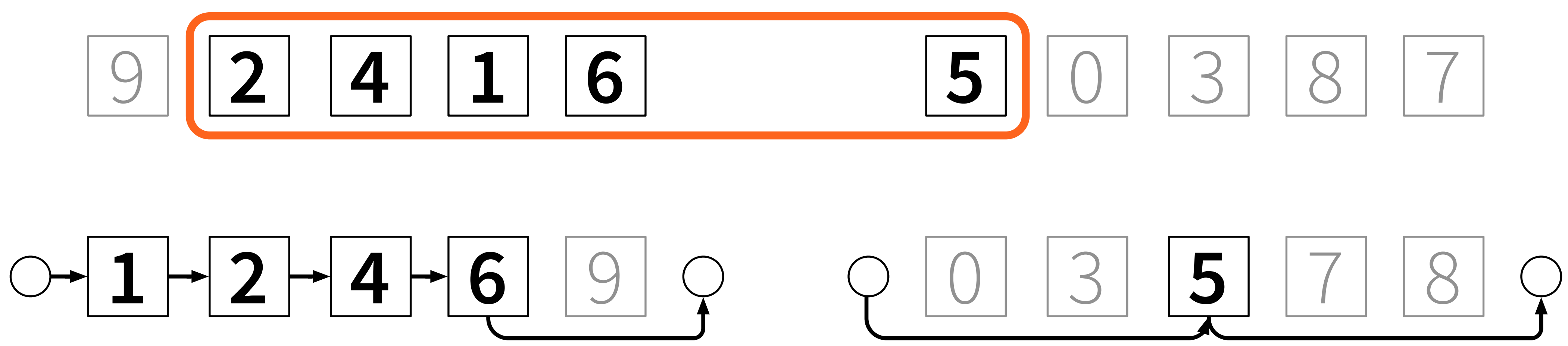
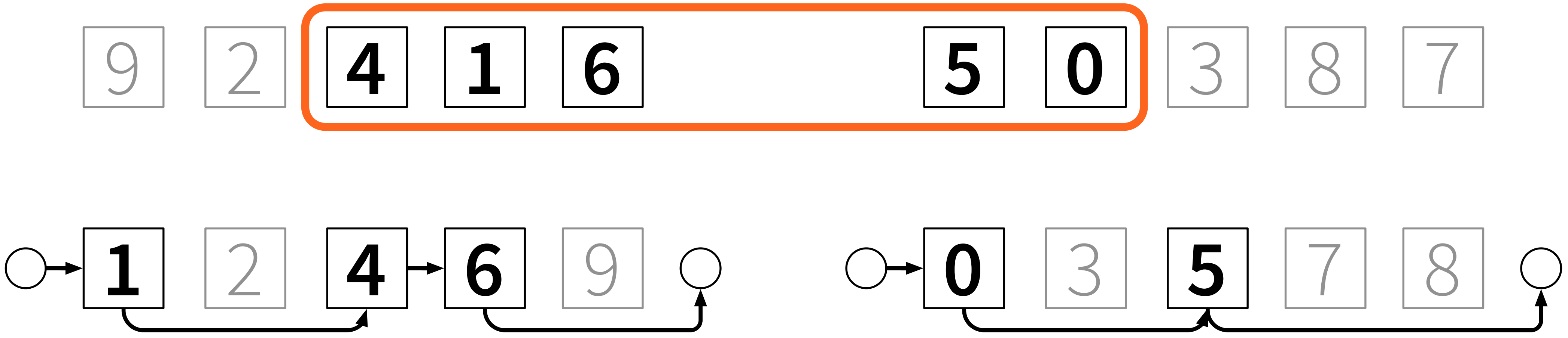


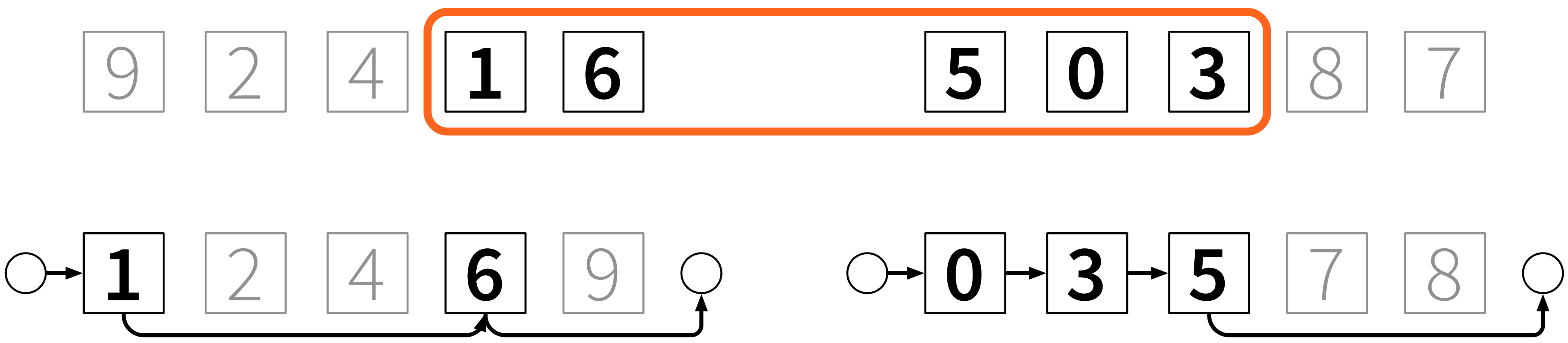
Sorting-based median filter

- **Sliding window** = **two sorted linked lists**



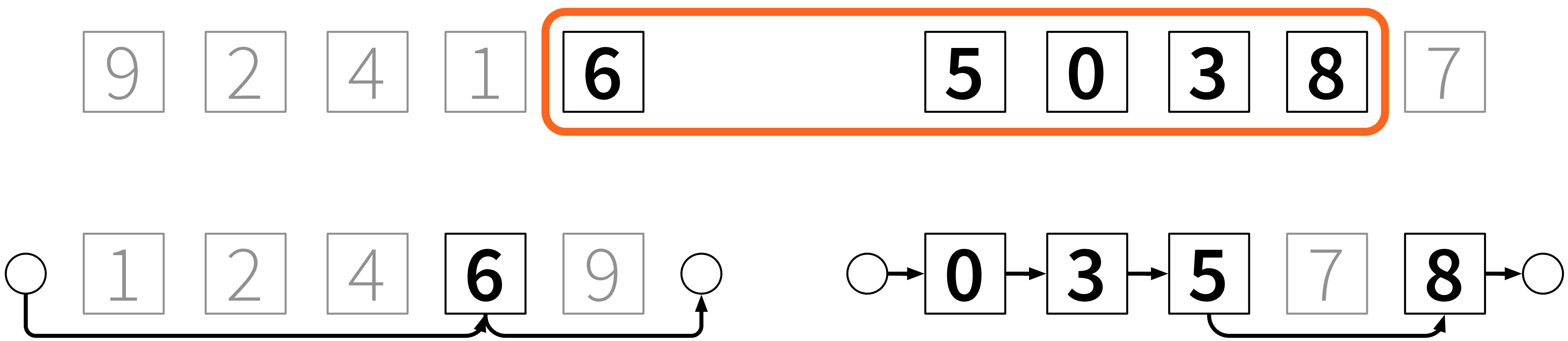
Sorting-based median filter

- **Sliding window = two sorted linked lists**



Sorting-based median filter

- **Sliding window** = **two sorted linked lists**



Sorting-based median filter

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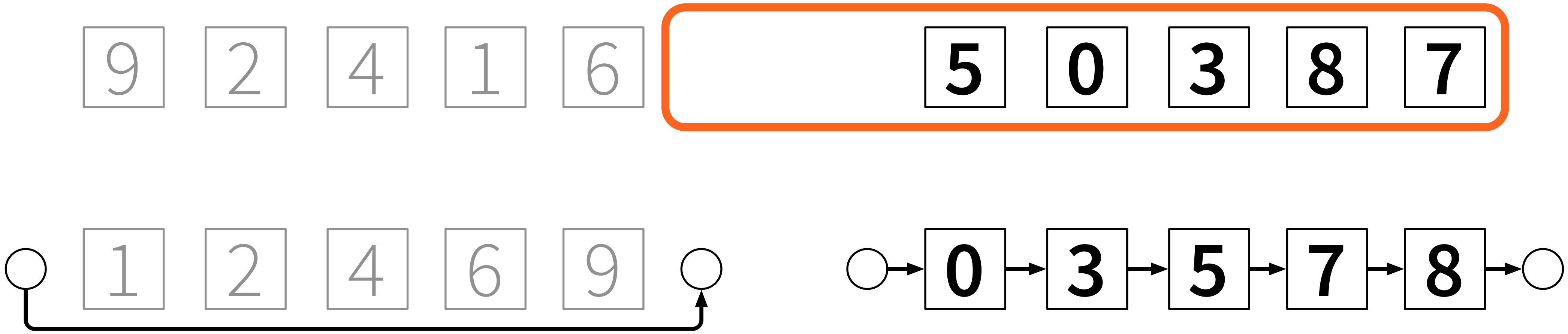
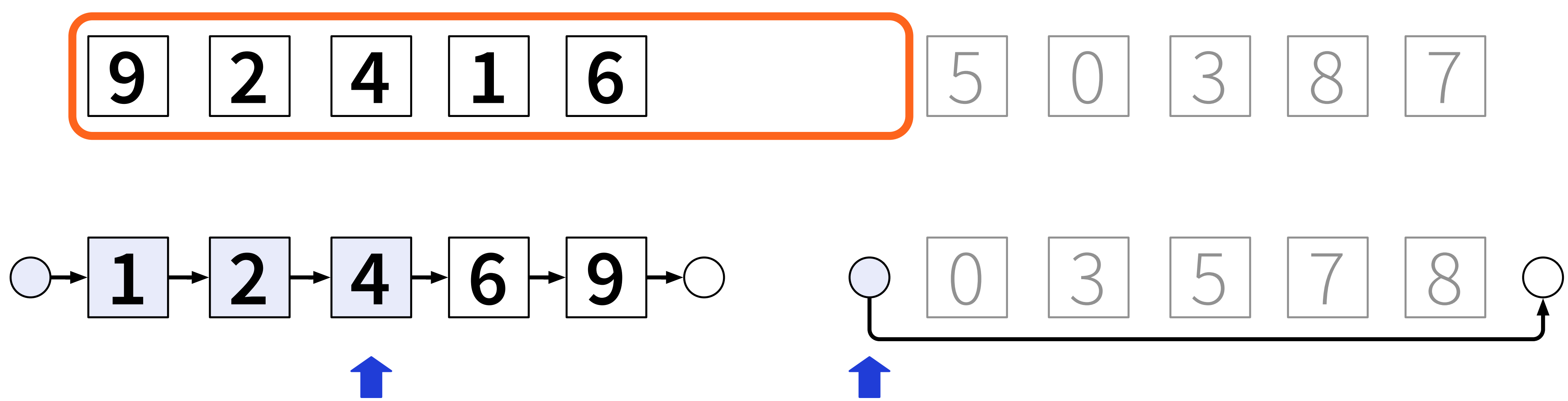


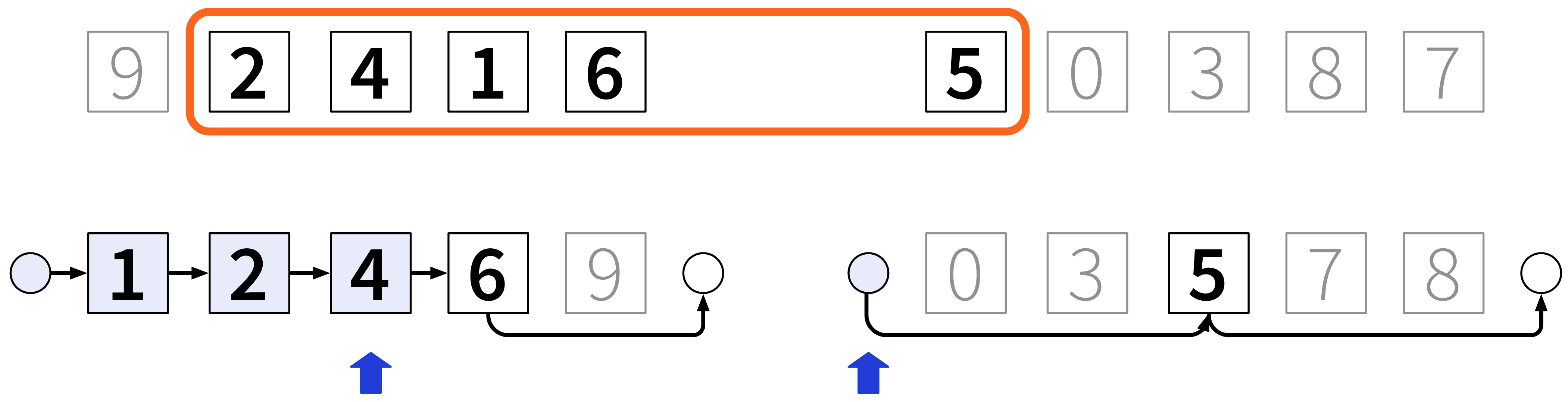
Sorting-based median filter

- **Maintain "*median pointers*" for each list (one of these is the median)**



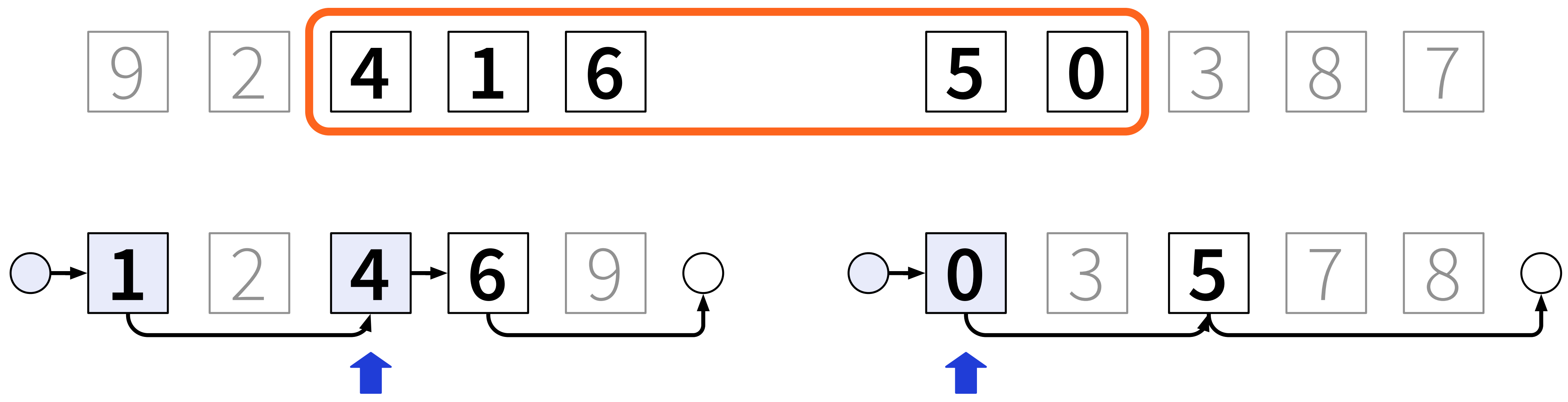
Sorting-based median filter

- Maintain "*median pointers*" for each list (one of these is the median)



Sorting-based median filter

- **Maintain “*median pointers*” for each list (one of these is the median)**



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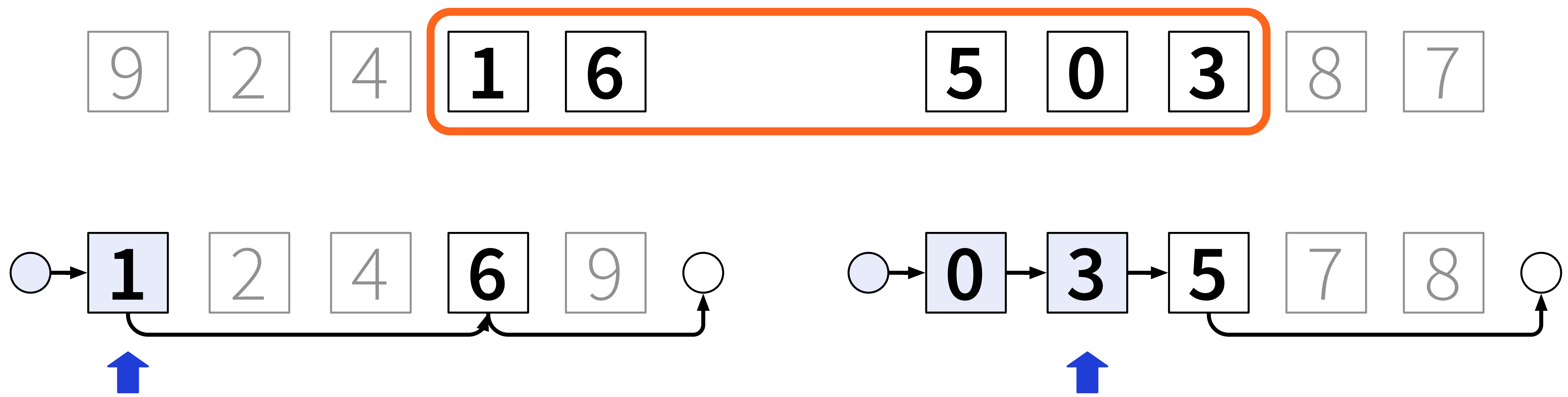

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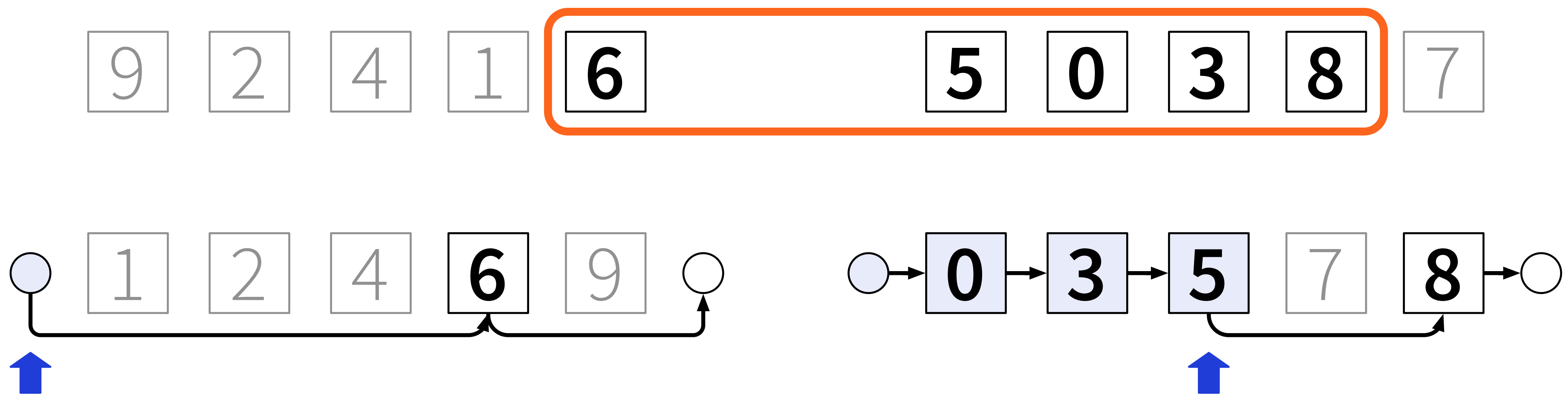

Sorting-based median filter

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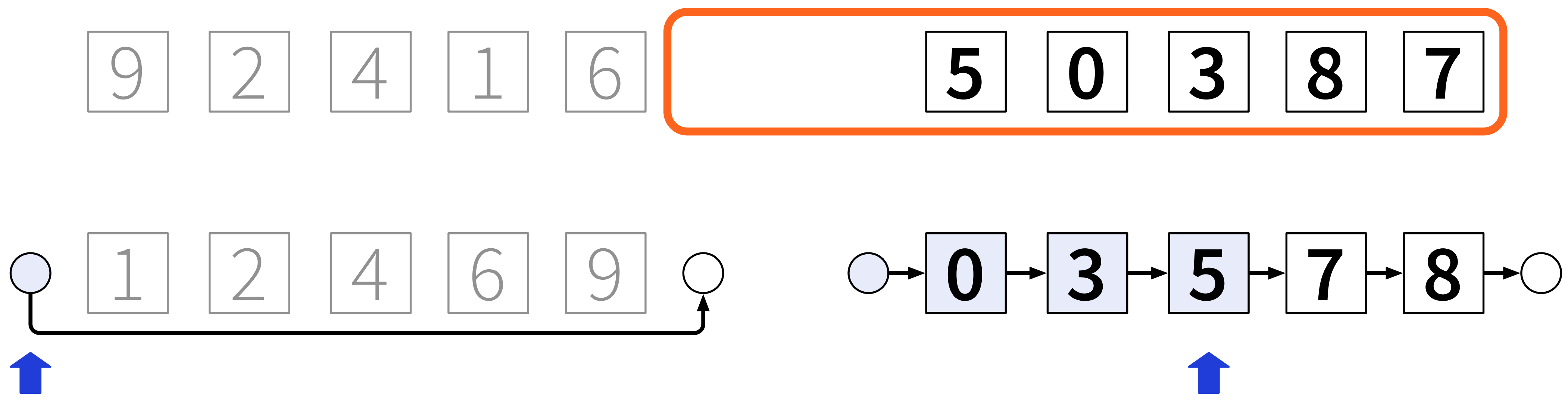
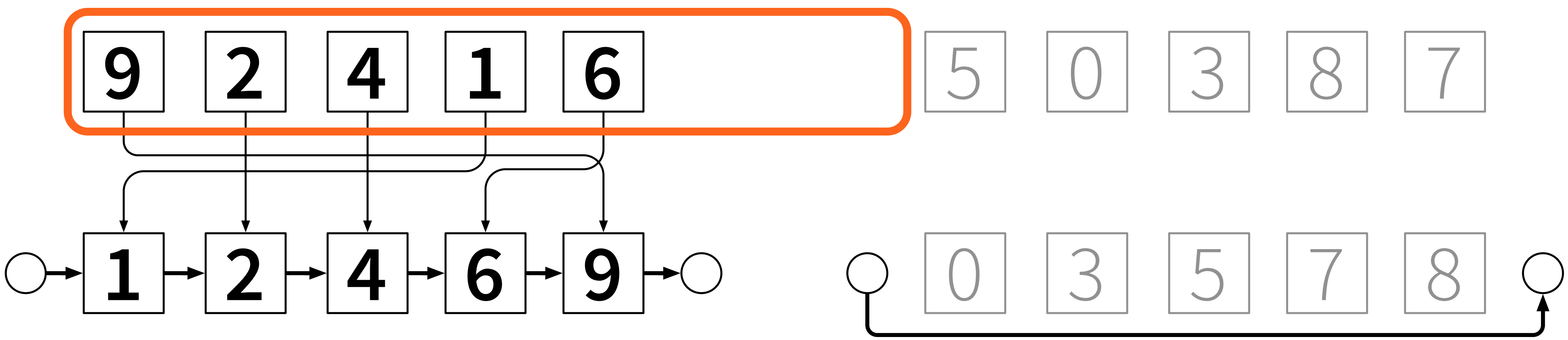


Sorting-based median filter

- **Median pointers:**
  - straightforward in $O(1)$ time per element
  - cf. merge sort
- **Sorted linked lists:**
  - how to insert & delete in $O(1)$ time?

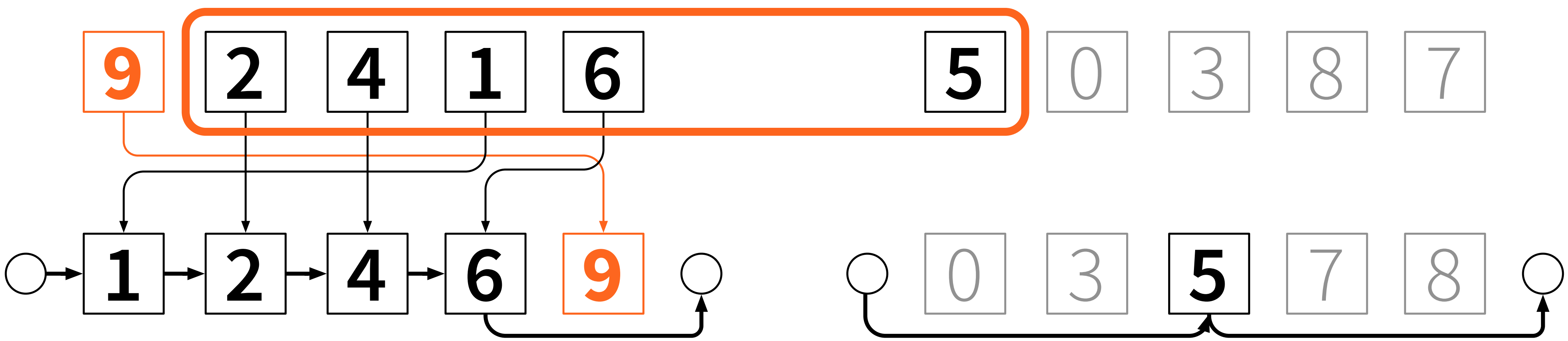
Sorting-based median filter

- *Deletions* are easy if we know what to delete: start with a sorted list + pointers to it



Sorting-based median filter

- ***Deletions*** **are easy if we know what to delete: start with a sorted list + pointers to it**



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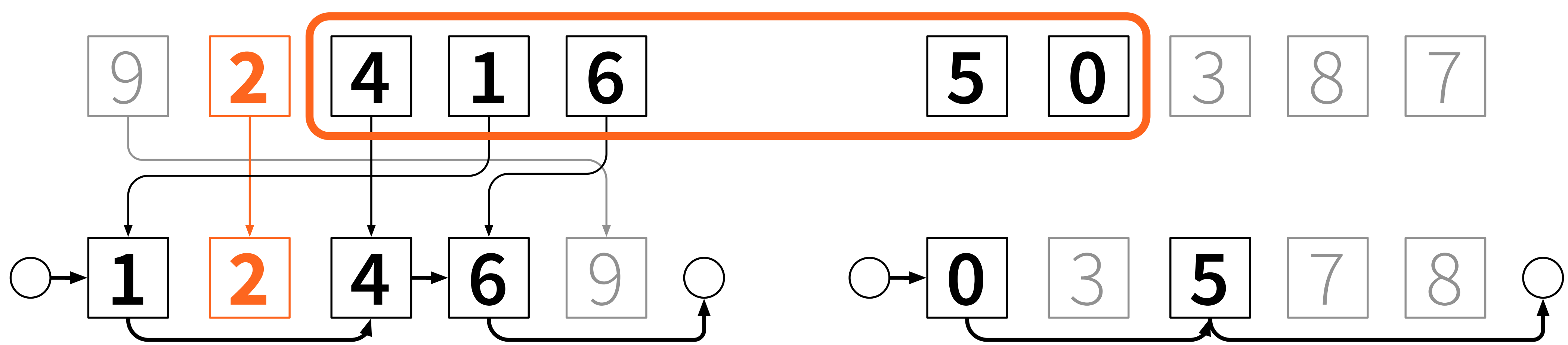

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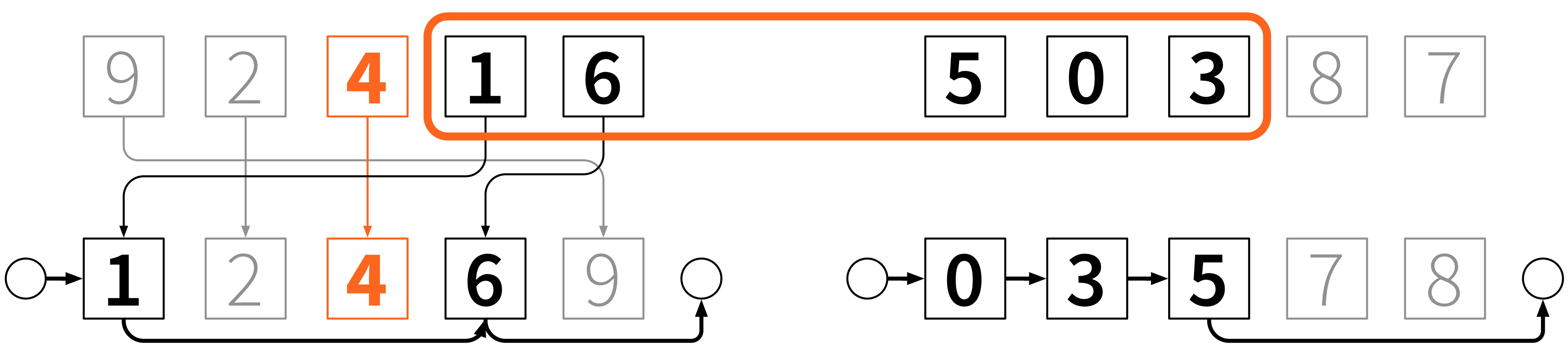

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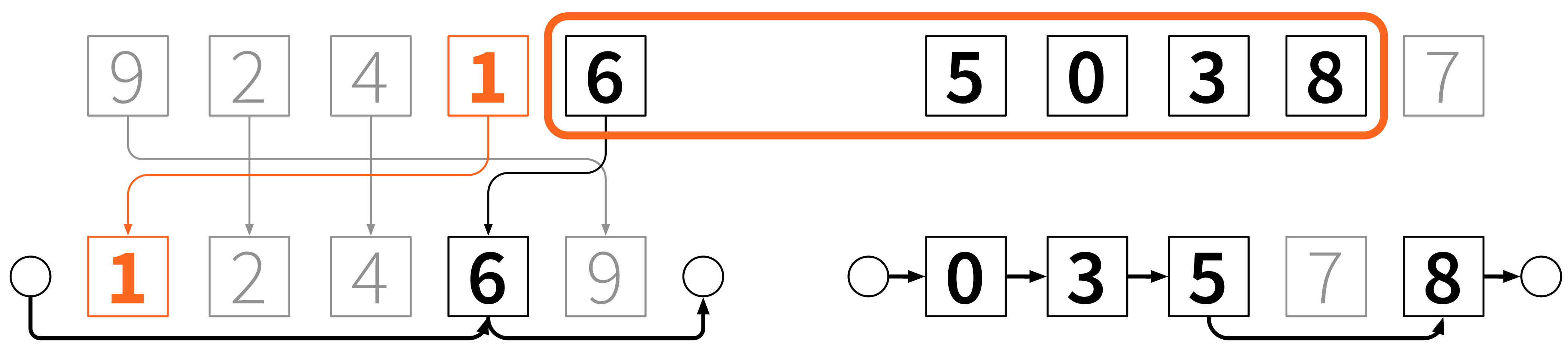

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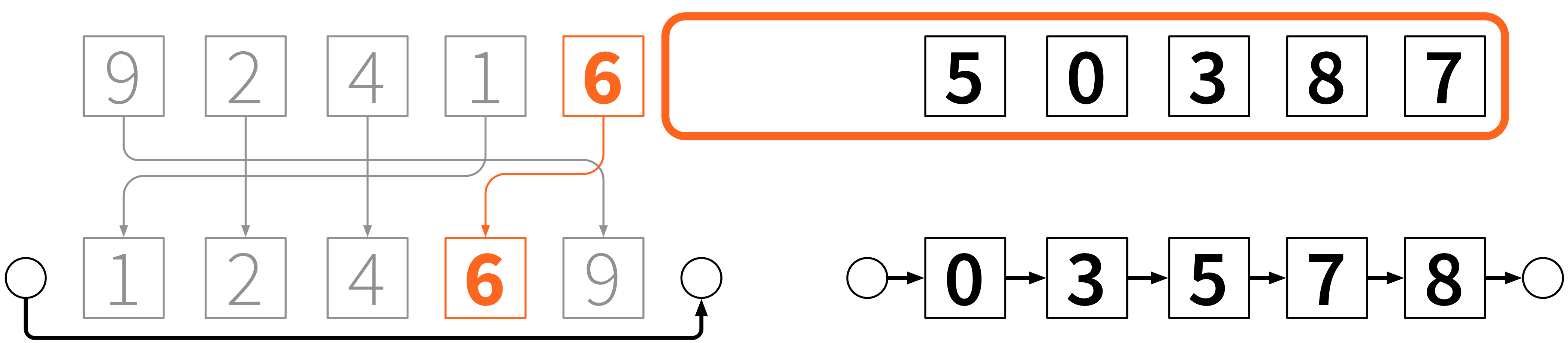
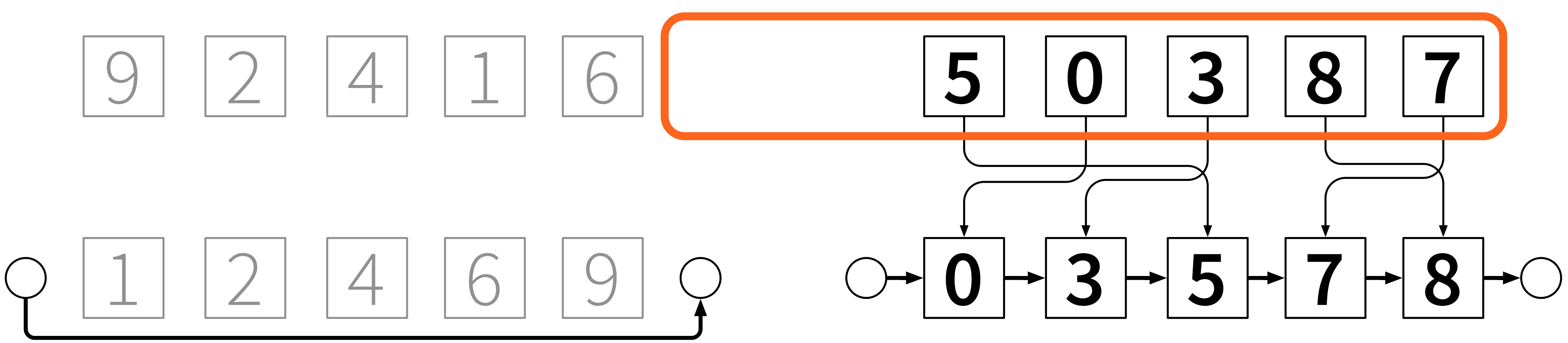


Sorting-based median filter

- **Asymmetry:**
  - deletions from sorted linked lists easy
  - insertions to sorted linked lists hard
- **Reverse time!**
  - insertions become deletions, easy

Sorting-based median filter

- Reverse time: insertions become deletions, easy to do if we start with a sorted list



Sorting-based median filter

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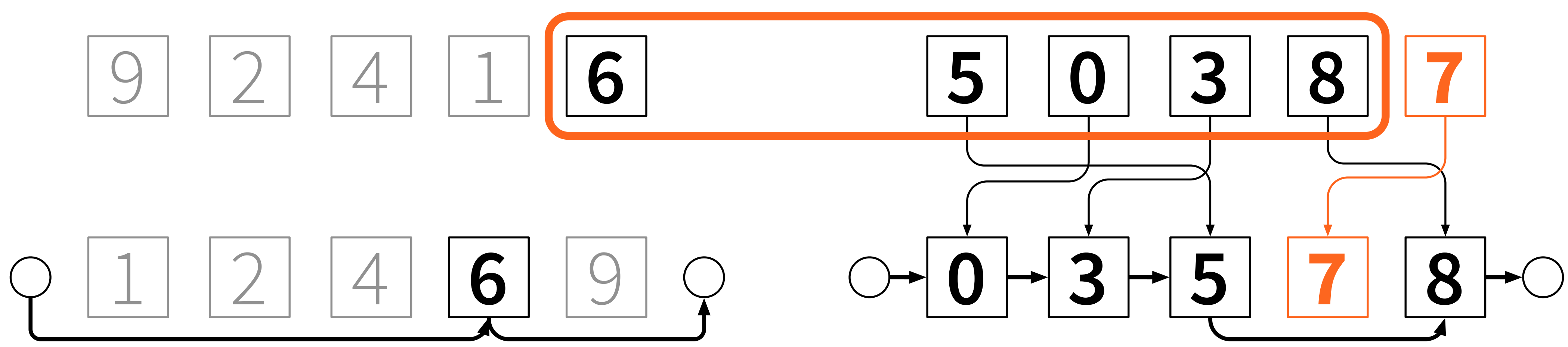

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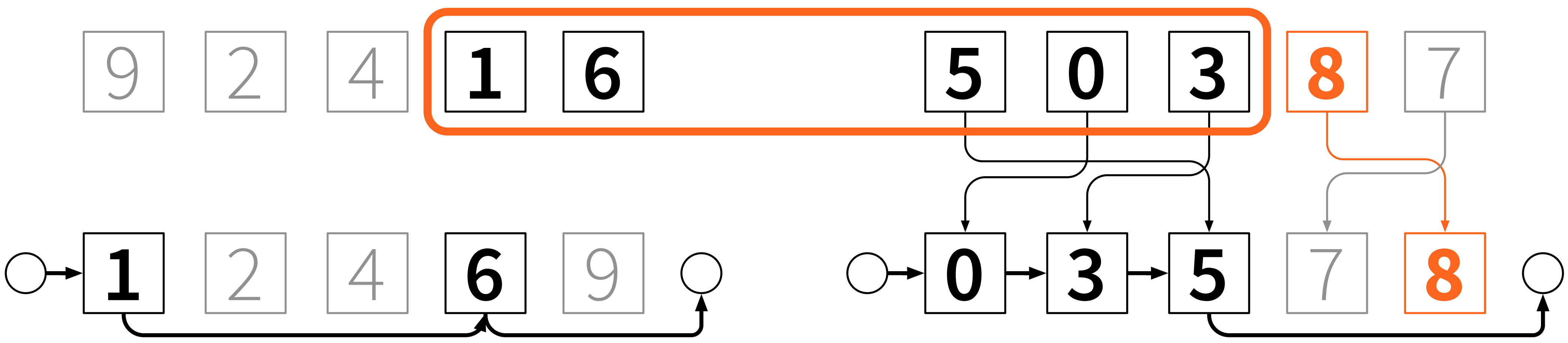

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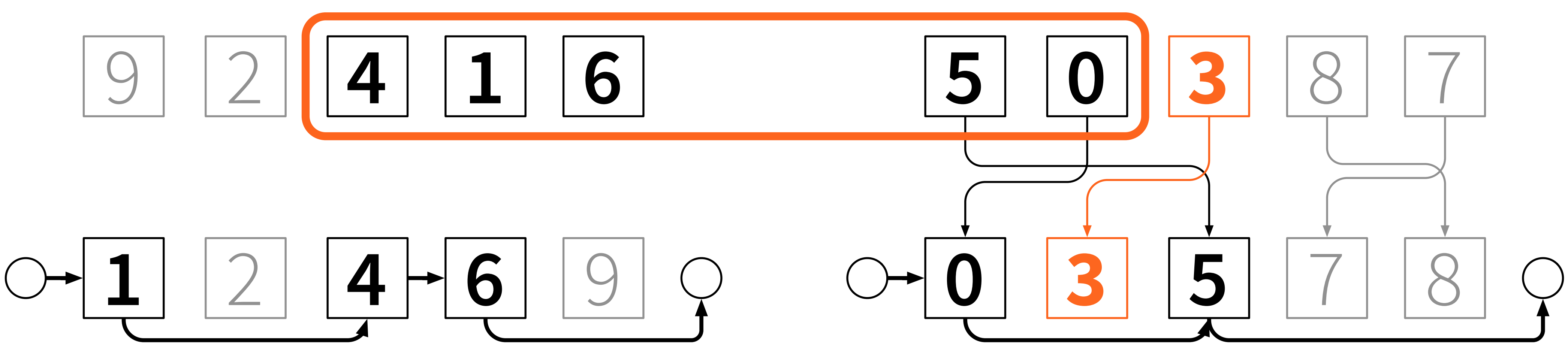

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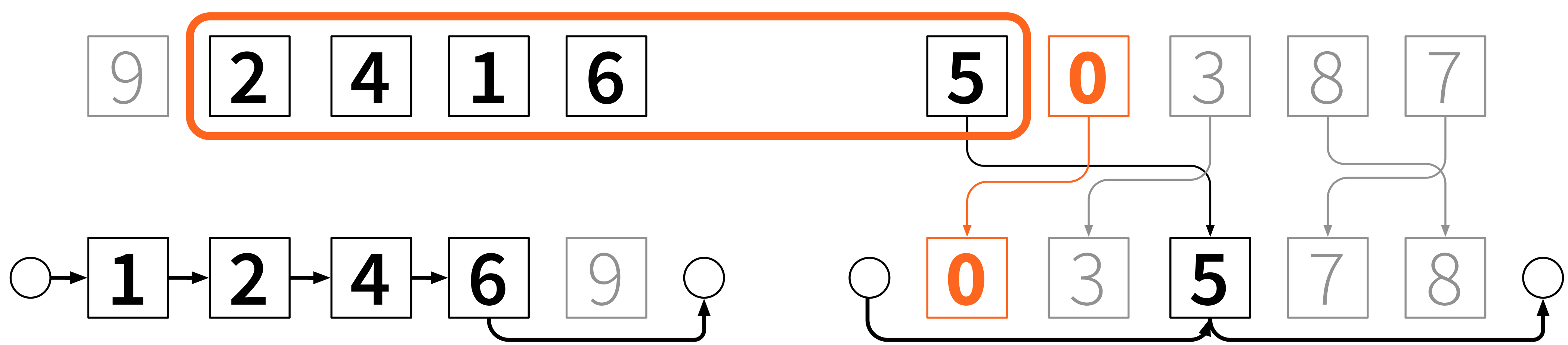

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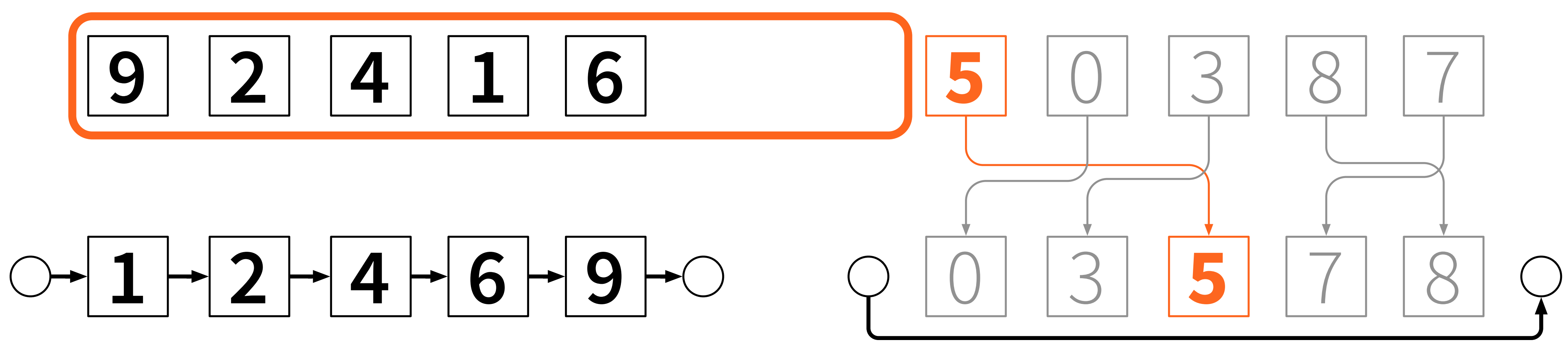
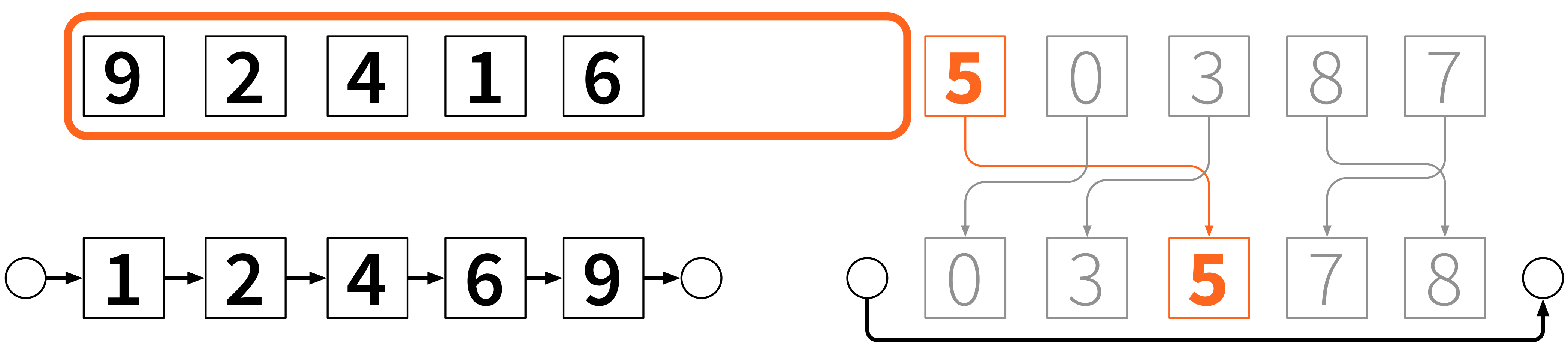


Sorting-based median filter

- **Reverse time**
- **How does this help?**
  - insertions become deletions, nice
  - deletions become insertions, bad
- **Solution:** ***reverse time again***

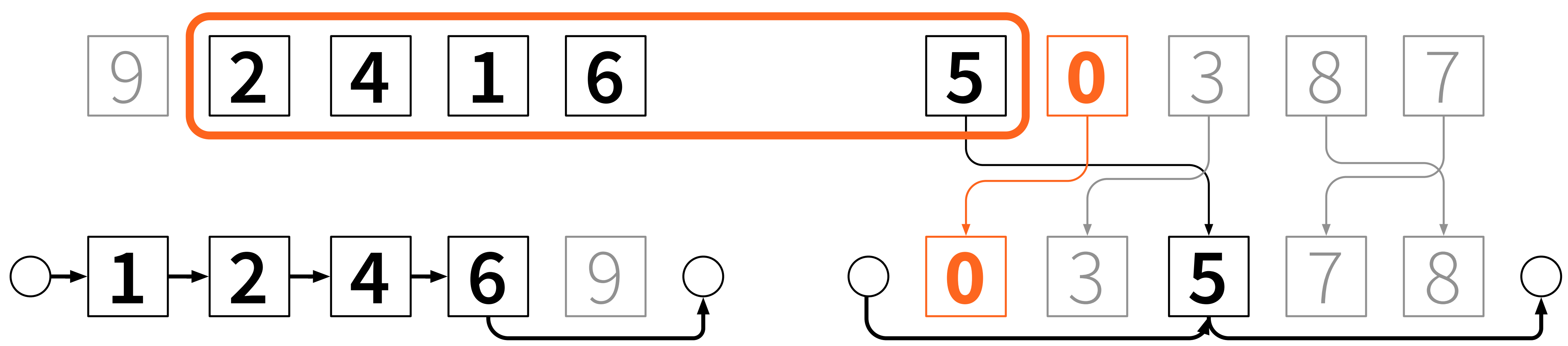
Sorting-based median filter

- **Reverse time again: insert = *undo deletion***



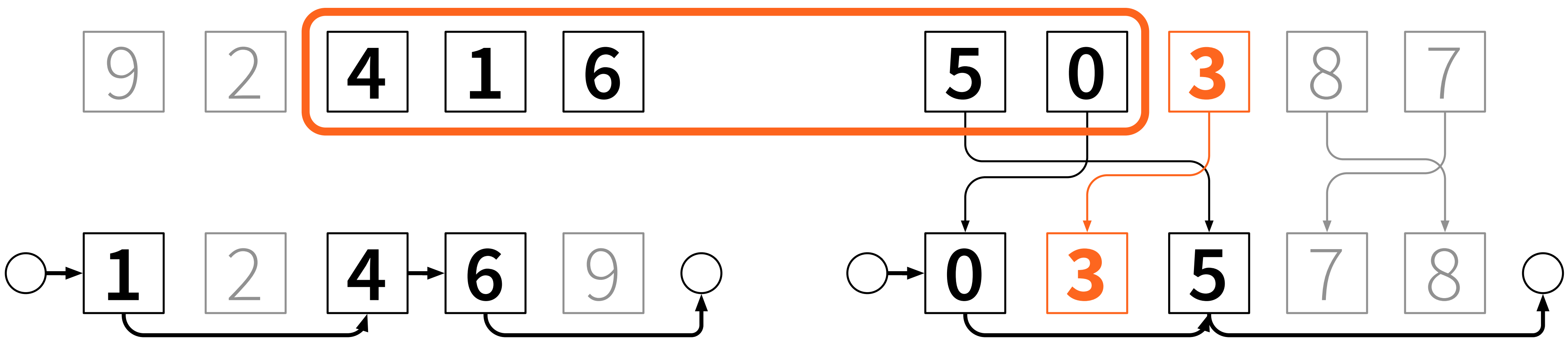
Sorting-based median filter

- **Reverse time again: insert = *undo deletion***



Sorting-based median filter

- **Reverse time again:**
  insert = *undo deletion*



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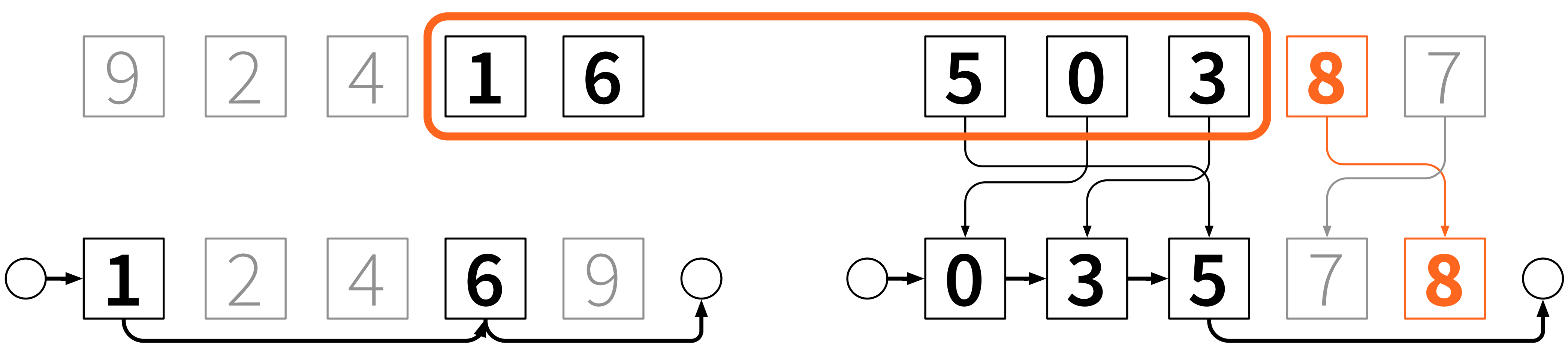
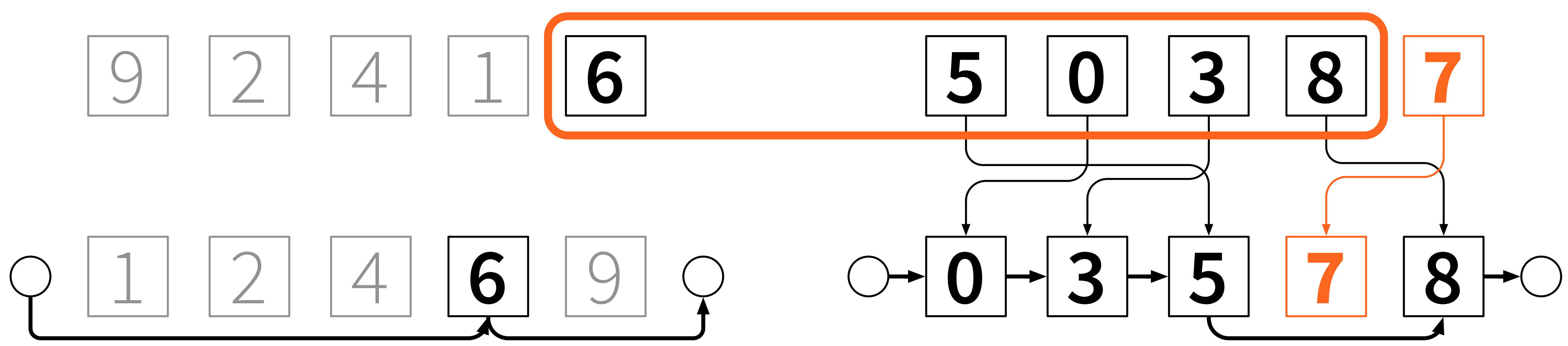


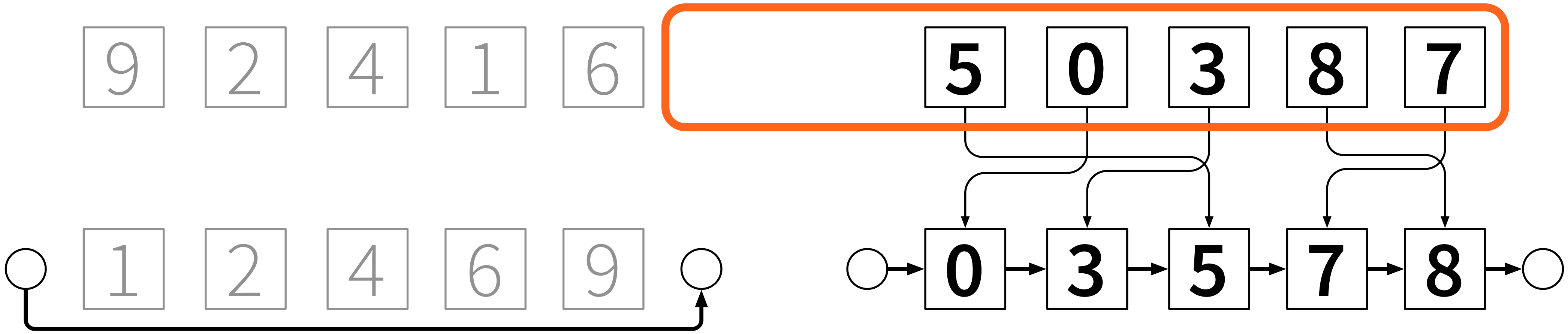
Sorting-based median filter

- **Reverse time again:**
  insert = *undo deletion*



Sorting-based median filter

- **Reverse time again:**
  insert = *undo deletion*

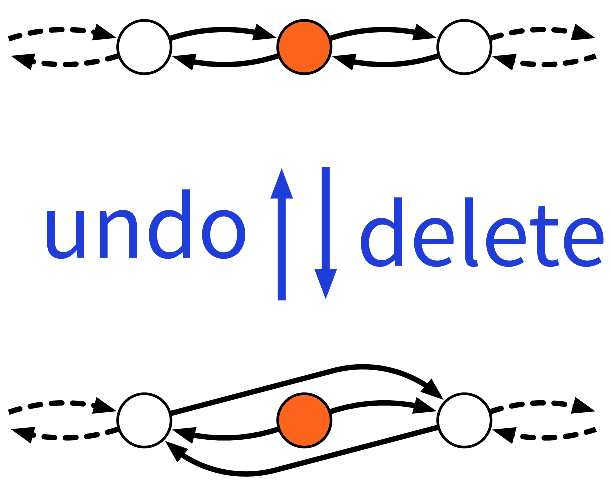


Sorting-based median filter

- **Shrinking list: start with a sorted list**
  - process one element = *one deletion*
- **Growing list: start with a sorted list**
  - first *delete* each element in reverse order
  - process one element = *undo one deletion*

Undo deletions from doubly-linked lists

- **Knuth (2000): "*dancing links*"**
- **Delete:** $\text{prev}[\text{next}[i]] \leftarrow \text{prev}[i]$
  $\text{next}[\text{prev}[i]] \leftarrow \text{next}[i]$
- **Undo:** $\text{prev}[\text{next}[i]] \leftarrow i$
  $\text{next}[\text{prev}[i]] \leftarrow i$



Sorting-based median filter

- **Preprocessing: piecewise sorting**
- **Sliding window = sorted doubly-linked lists**
    - shrinking list: easy
    - growing list: reverse time twice
    - insert = undo deletion,
      easy with dancing links

Sorting-based median filter

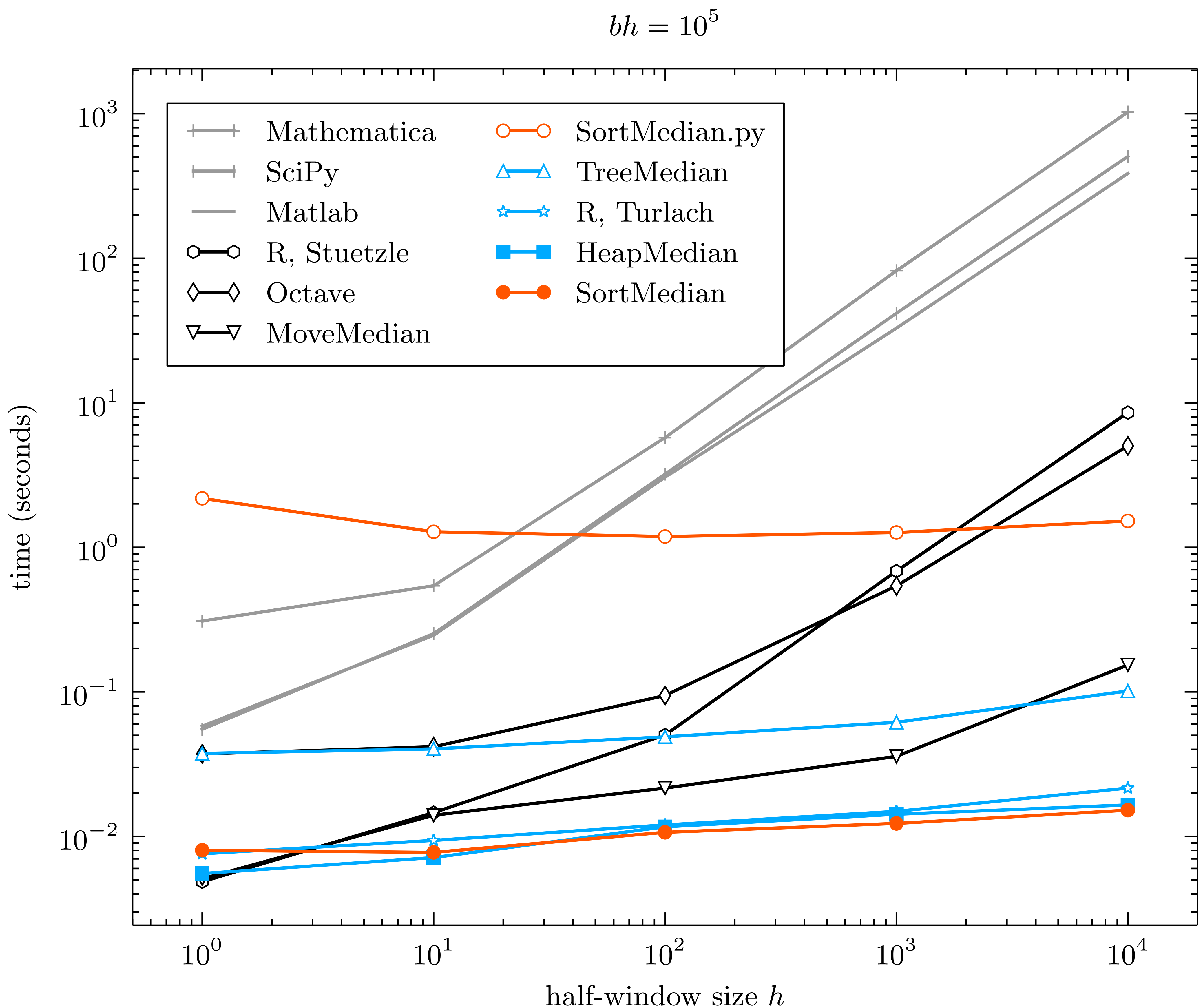
- **Optimal algorithm for any input distribution, for almost any model of computing**
  - just use optimal sorting algorithm for this setting
  - then $O(n)$ time postprocessing suffices
- **Matching lower bound**

Sorting-based median filter

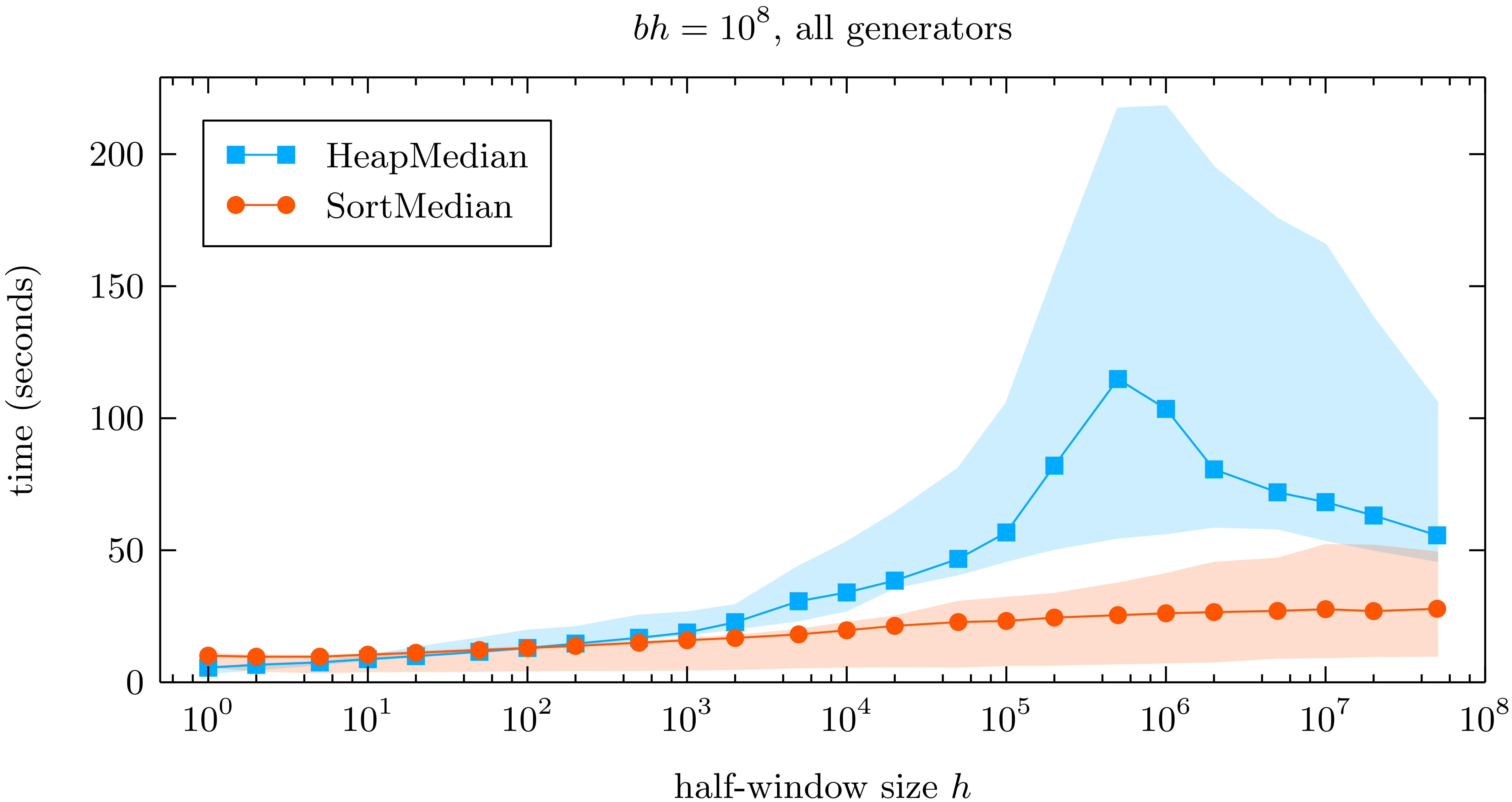
- Easy to implement
- Very fast

```python
def create_array(n):
    return [None] * n

def sort_block(alpha):
    pairs = [(alpha[i], i) for i in range(len(alpha))]
    return [i for v,i in sorted(pairs)]

class Block:
    def __init__(self, h, alpha):
        self.k = len(alpha)
        self.alpha = alpha
        self.pi = sort_block(alpha)
        self.prev = create_array(self.k + 1)
        self.next = create_array(self.k + 1)
        self.tail = self.k
        self.init_links()
        self.m = self.pi[h]
        self.s = h

    def init_links(self):
        p = self.tail
        for i in range(self.k):
            q = self.pi[i]
            self.next[p] = q
            self.prev[q] = p
            p = q
        self.next[p] = self.tail
        self.prev[self.tail] = p

    def unwind(self):
        for i in range(self.k-1, -1, -1):
            self.next[self.prev[i]] = self.next[i]
            self.prev[self.next[i]] = self.prev[i]
        self.m = self.tail
        self.s = 0

    def delete(self, i):
        self.next[self.prev[i]] = self.next[i]
        self.prev[self.next[i]] = self.prev[i]
        if self.is_small(i):
            self.s -= 1
        else:
            if self.m == i:
                self.m = self.next[self.m]
            if self.s > 0:
                self.m = self.prev[self.m]
                self.s -= 1

    def undelete(self, i):
        self.next[self.prev[i]] = i
        self.prev[self.next[i]] = i
        if self.is_small(i):
            self.m = self.prev[self.m]

    def advance(self):
        self.m = self.next[self.m]
        self.s += 1

    def at_end(self):
        return self.m == self.tail

    def peek(self):
        return float('Inf') if self.at_end() \
        else self.alpha[self.m]

    def get_pair(self, i):
        return (self.alpha[i], i)

    def is_small(self, i):
        return self.at_end() or \
        self.get_pair(i) < self.get_pair(self.m)

def sort_median(h, b, x):
    k = 2 * h + 1
    B = Block(h, x[0:k])
    y = []
    y.append(B.peek())
    for j in range(1, b):
        A = B
        B = Block(h, x[j*k:(j+1)*k])
        B.unwind()
        for i in range(k):
            A.delete(i)
            B.undelete(i)
            if A.s + B.s < h:
                if A.peek() <= B.peek():
                    A.advance()
                else:
                    B.advance()
            y.append(min(A.peek(), B.peek()))
    return y
```

complete Python implementation



$bh = 10^5$
time (seconds)
half-window size h
Mathematica
SciPy
Matlab
R, Stuetzle
Octave
MoveMedian
SortMedian.py
TreeMedian
R, Turlach
HeapMedian
SortMedian
10^{-2}
10^{-1}
10^0
10^1
10^2
10^3
10^0
10^1
10^2
10^3
10^4


$bh = 10^8$, all generators
HeapMedian
SortMedian
time (seconds)
0
50
100
150
200
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
half-window size h

Conclusions

- **Median filtering ≈ *piecewise sorting***
- **In theory and in practice**
- **arXiv:1406.1717**