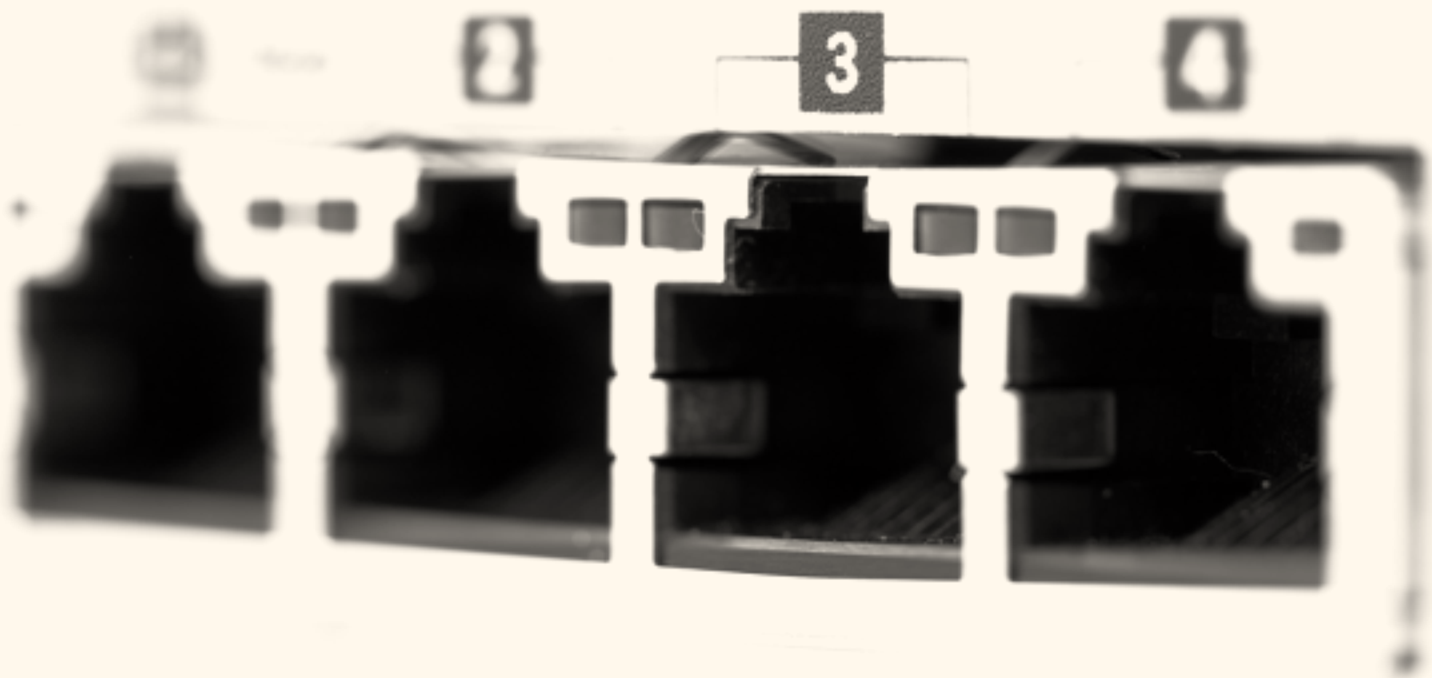
Port-Numbering Model



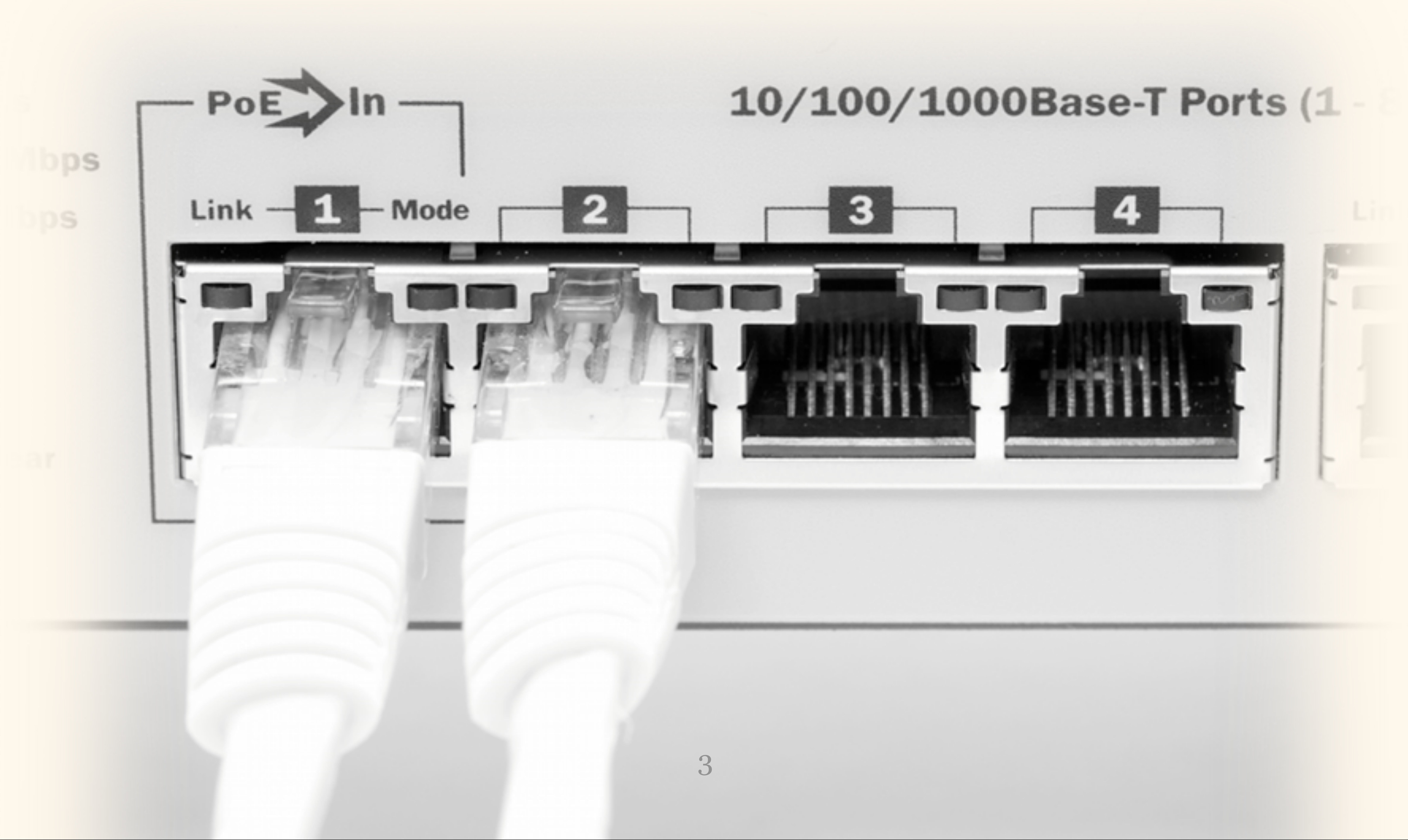
DDA Course

week 2

Distributed Systems

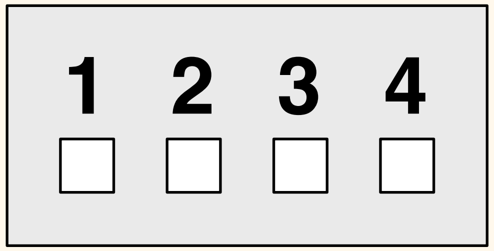
- Intuition:
  - distributed system
    ≈ communication network
    ≈ network equipment + communication links
  - distributed algorithm
    ≈ computer program
- Precisely how are we going to model this?

Port Numbering



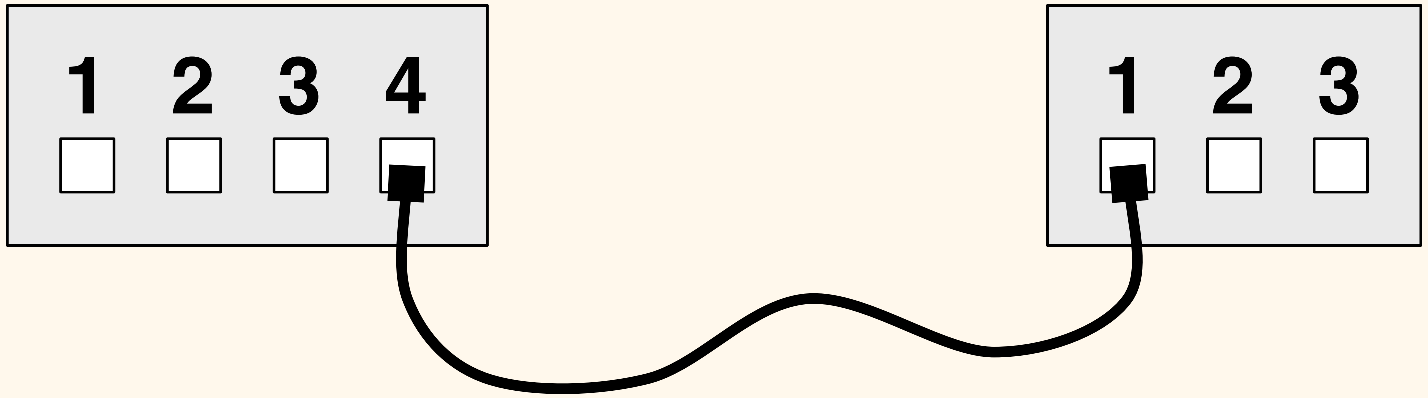
Port Numbering

- Network device = state machine with *communication ports*
- Ports are *numbered*: 1, 2, 3, ...



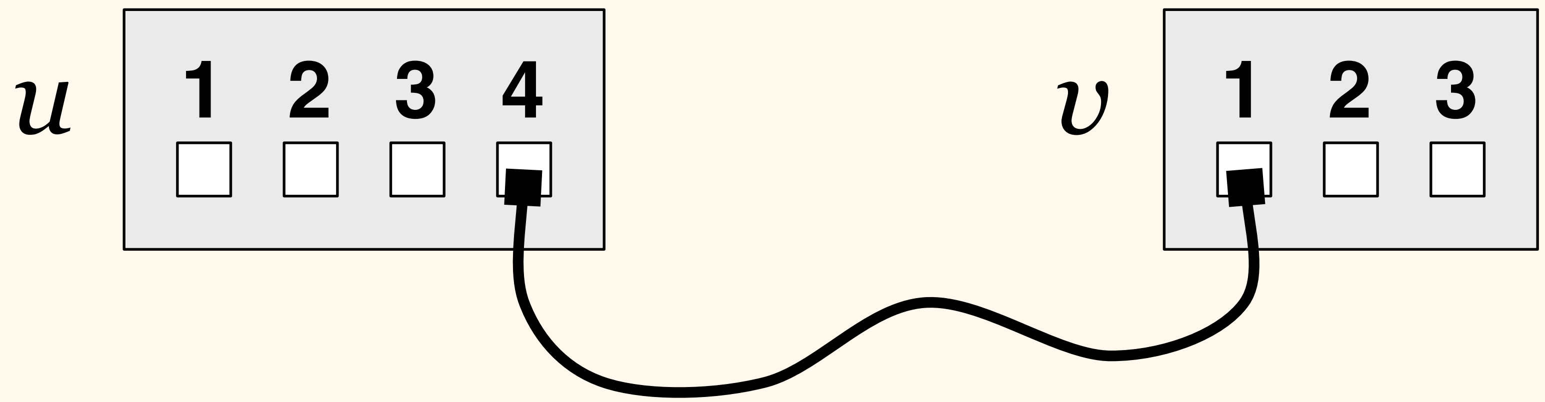
Port-Numbered Network

- Network = several devices, *connections* between ports
    - we will formalise it as a triple $N = (V, P, p)$



Port-Numbered Network

- nodes $V = \{u, v, \ldots\}$
- ports $P = \{(u, 1), (u, 2), (u, 3), (u, 4), (v, 1), (v, 2), (v, 3), \ldots\}$
- connections $p(u, 4) = (v, 1)$, $p(v, 1) = (u, 4)$, ...



Port-Numbered Network

- nodes $V = \{u, v, \ldots\}$
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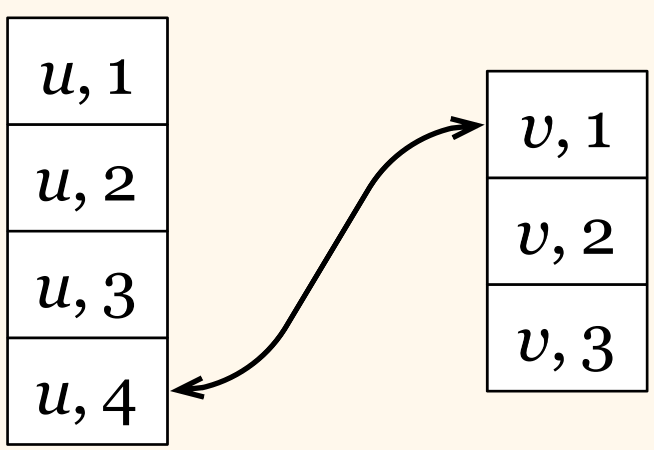
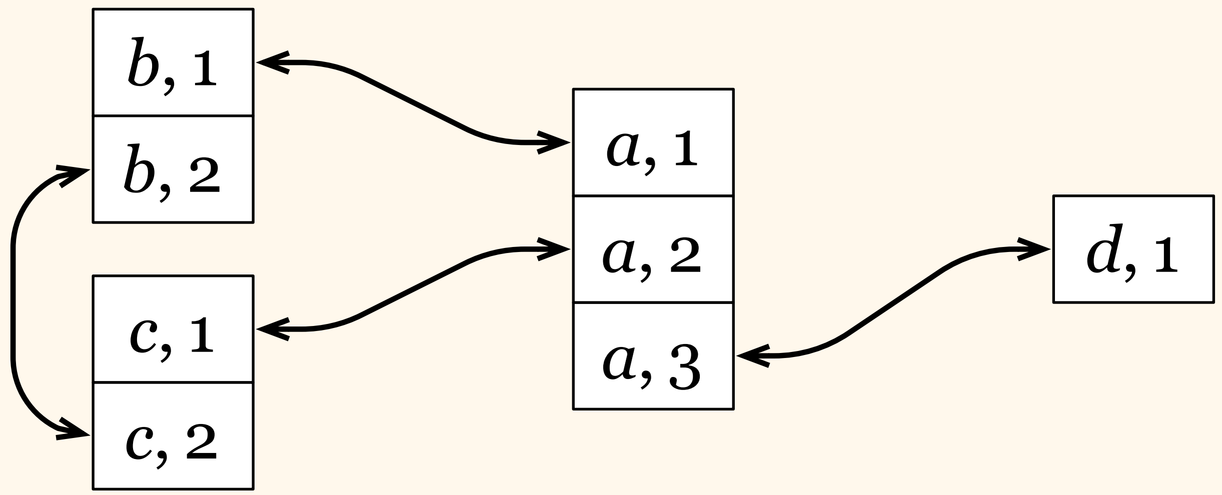


not a complete example, some ports not connected!

Port-Numbered Network

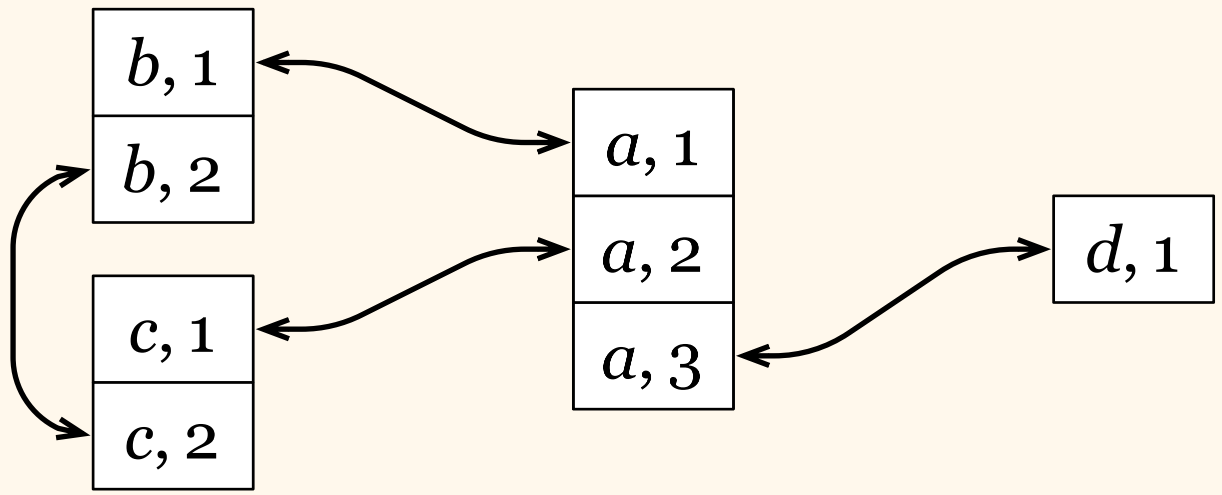
- nodes $V = \{a, b, c, d\}$
- ports $P = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (c, 1), (c, 2), (d, 1)\}$
- connections $p(a, 1) = (b, 1)$, $p(b, 1) = (a, 1)$, …



all ports connected

Port-Numbered Network

- nodes V = a finite set
- ports P = a finite set of (node, number) pairs
- connections p = an involution $P \to P$



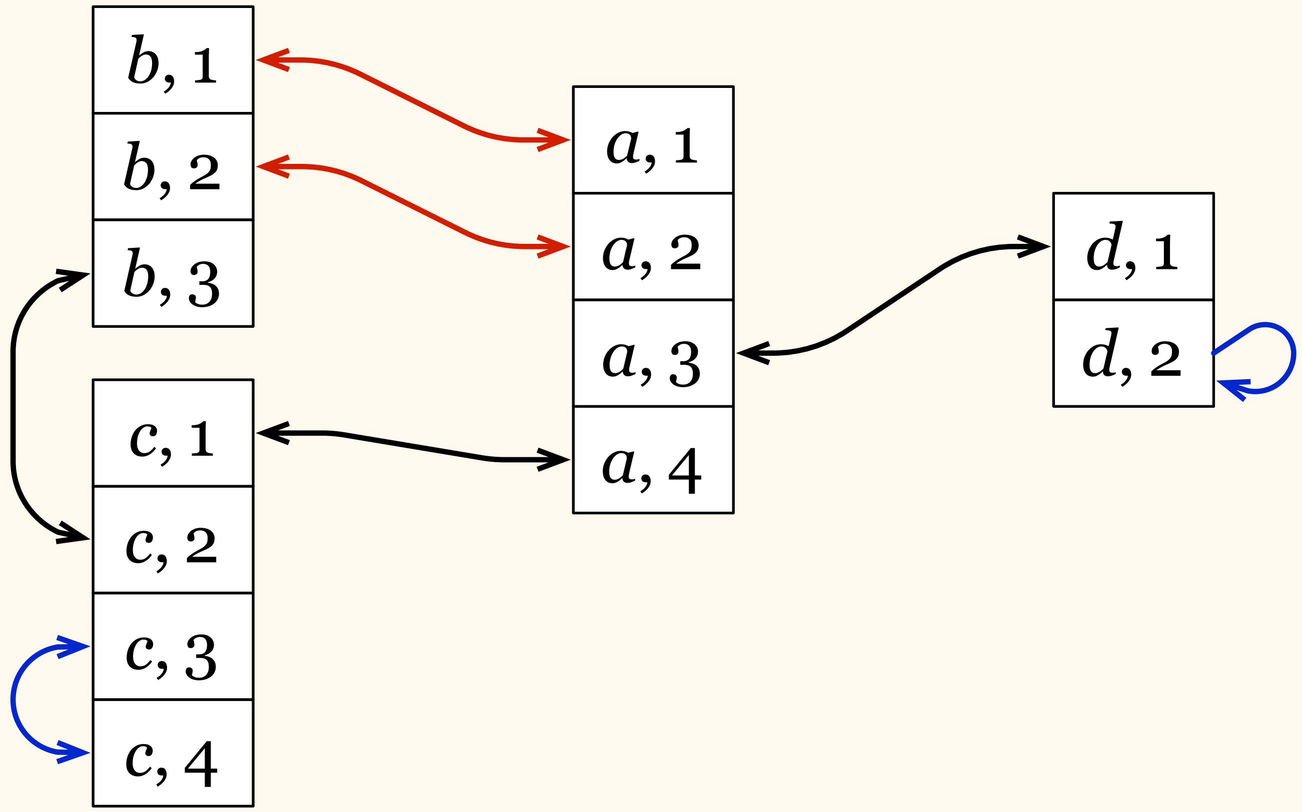
involution:

$p^{-1} = p$

$p(p(x)) = x$

Port-Numbered Network

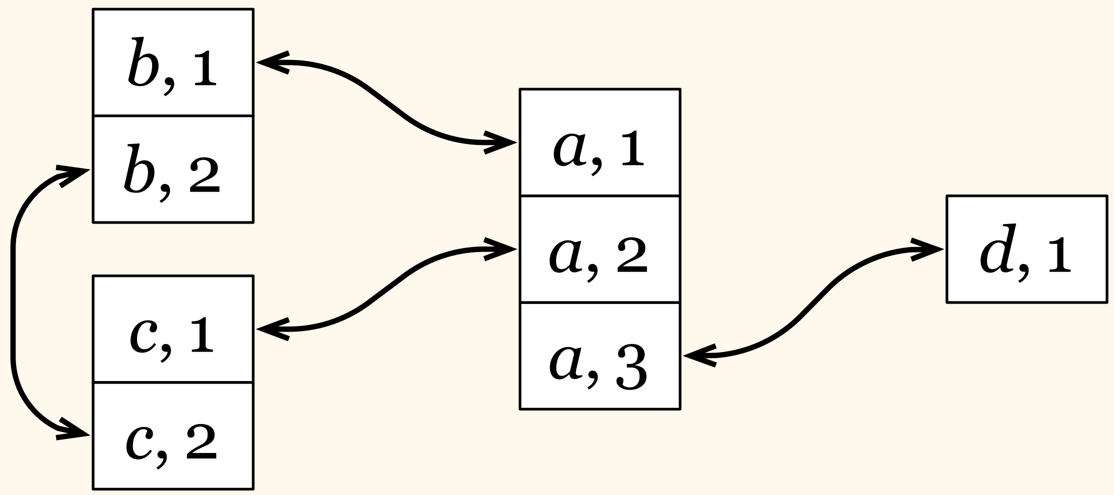
- We may have *multiple connections* or *loops*



$$p(c, 3) = (c, 4)$$
$$p(c, 4) = (c, 3)$$
$$p(d, 2) = (d, 2)$$

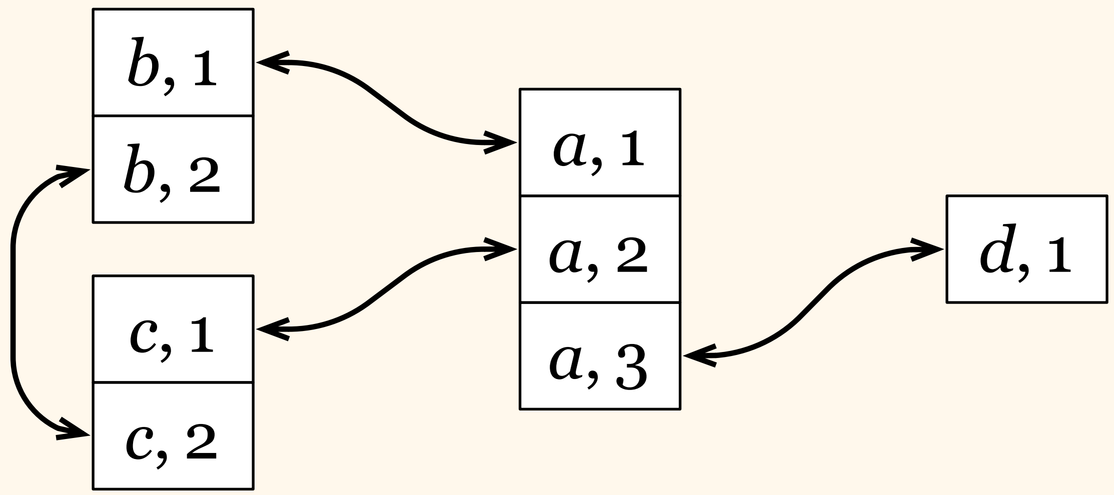
Port-Numbered Network

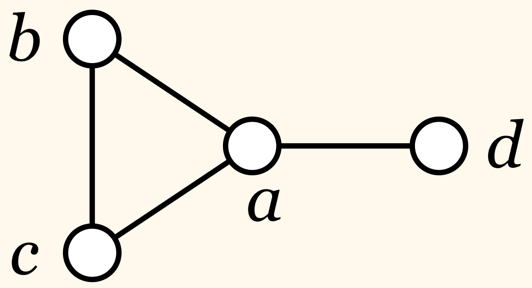
- *Simple* port-numbered network: no multiple connections, no loops



Port-Numbered Network

- *Underlying graph* of
  a simple port-numbered network





Distributed Algorithms

Distributed Algorithm

- State machine, x = current state:
  - $x \leftarrow$ **init**(z): initial state for local input z
  - **send**(x): construct *outgoing messages*
    - send(x) = vector, one element per port
  - $x \leftarrow$ **receive**(x, m): process *incoming messages*
    - m = vector, one element per port

Execution

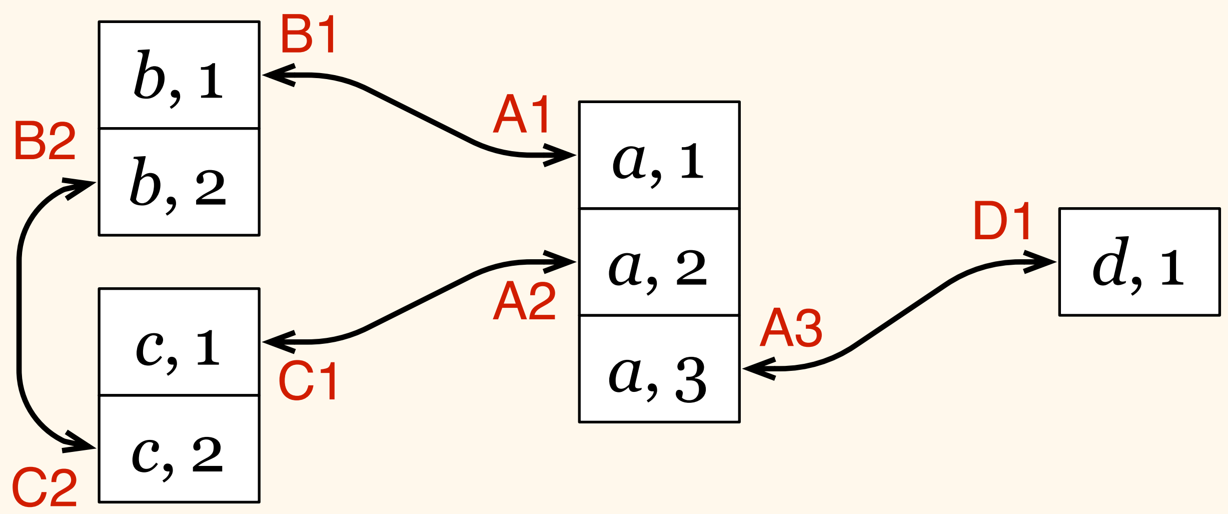
- "Execution of algorithm *A* in network *N*"
- All nodes of *N* are *identical copies* of the same state machine *A*
  - functions **init**, **send**, and **receive** may depend on node degree (number of ports)
  - in all other aspects the nodes are identical

Execution

- All nodes are initialised
- Time step (*communication round*):
  - all nodes construct outgoing messages
  - messages are propagated
  - all nodes process incoming messages
- Continue until all nodes have stopped

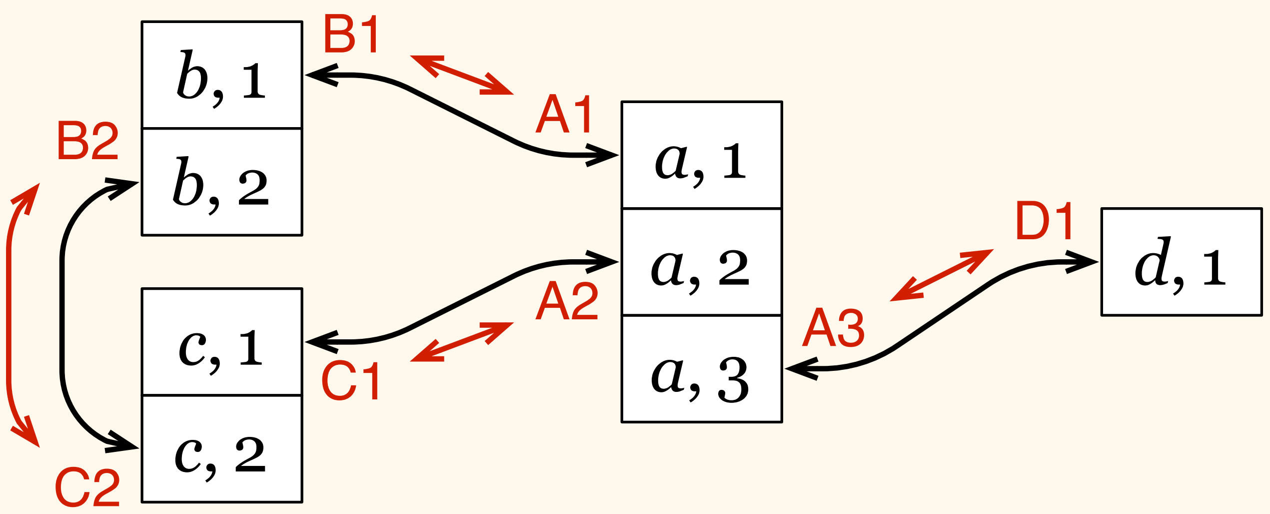
Communication Round

- Construct *outgoing messages*



Communication Round

- Construct outgoing messages
- Exchange messages along communication links



Communication Round

- Construct outgoing messages
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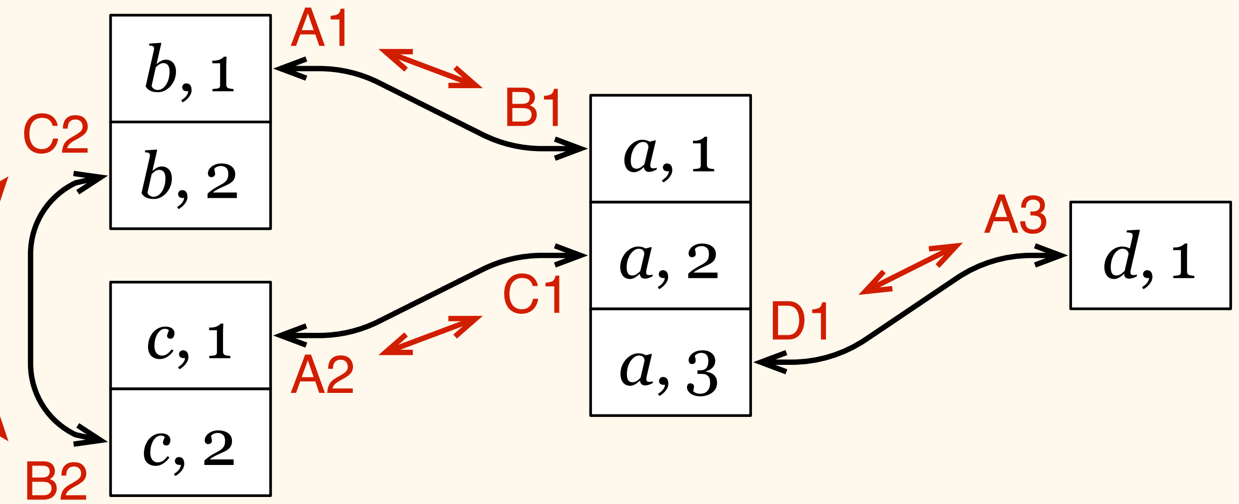
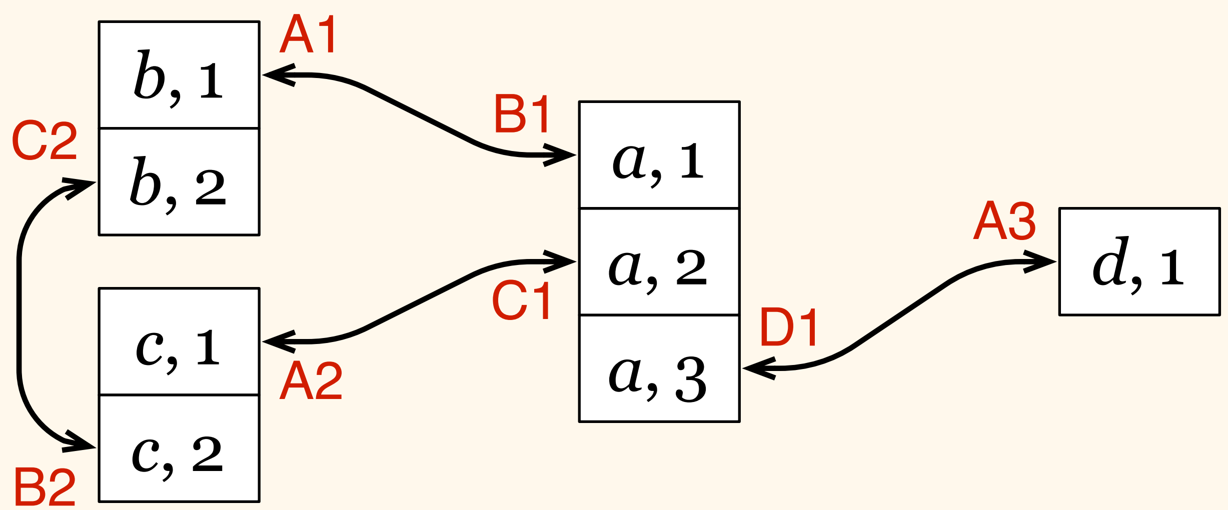


Communication Round

- Construct outgoing messages
- Exchange messages along communication links
- Process *incoming messages*



Communication Round

- Construct outgoing messages
- Exchange messages along communication links
- Process incoming messages

- Communication rounds are *synchronous*
- Each step happens synchronously in parallel for all nodes
- Everything is *deterministic*

Distributed Algorithm

- Algorithm designed chooses:
  - how to initialise nodes
  - how to construct outgoing messages
  - how to process incoming messages
- Network structure determines:
  - how messages are propagated between ports

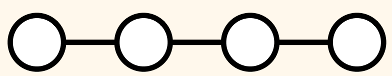
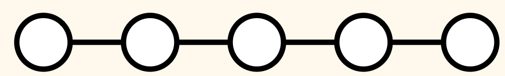
Distributed Algorithm

- "Algorithm A solves graph problem Π on graph family $\mathcal{F}$":
  - for any graph $G \in \mathcal{F}$,
  - for *any simple port-numbered network* N that has G as underlying graph,
  - execution of A on N stops and produces a valid solution of Π

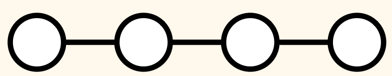
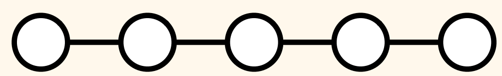
Distributed Algorithm

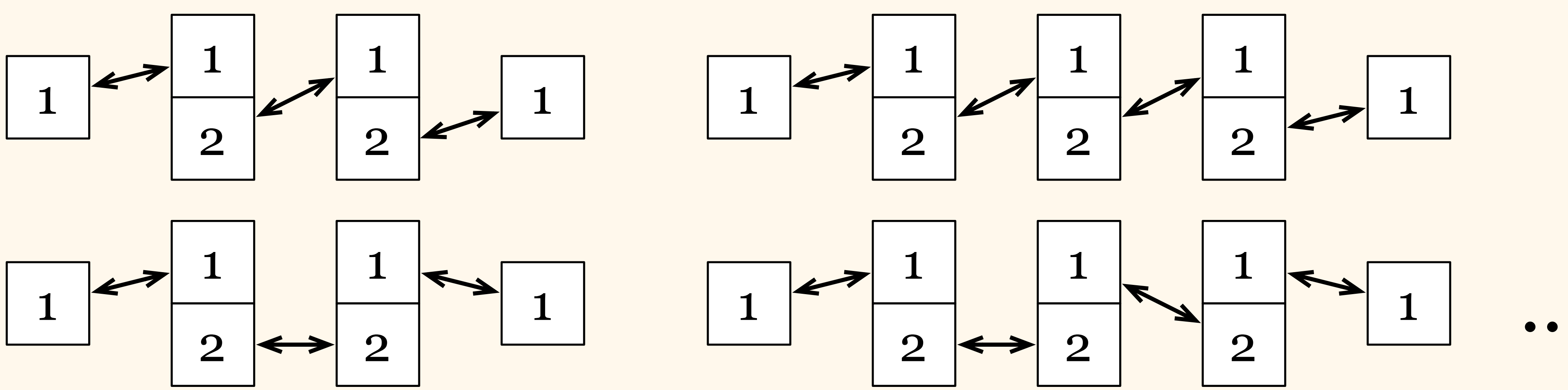
- "Algorithm *A* finds a minimum vertex cover in any regular graph":
  - for *any simple port-numbered network* *N* that has a regular graph as underlying graph,
  - execution of *A* on *N* stops,
  - the stopping states of the nodes are "**0**" and "**1**",
  - nodes in state "**1**" form a minimum vertex cover

Example

- Design a distributed algorithm that finds a *minimum vertex cover* in $\mathcal{F} = \{$ ,  $\}$

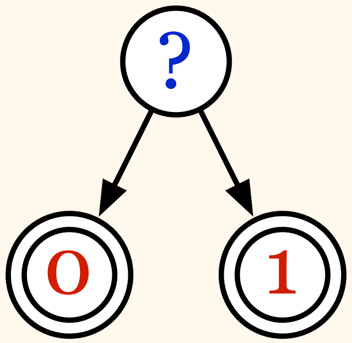
Example

- Design a distributed algorithm that finds a *minimum vertex cover* in $\mathcal{F} = \{$  ,  $\}$

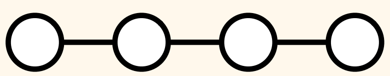
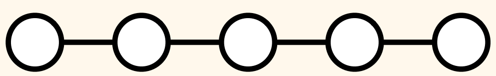


Example

- Nodes of degree 1:
  - $\text{init}_1 = ?$, $\text{send}_1(?) = (A)$
  - $\text{receive}_1(?, A) = 0$, $\text{receive}_1(?, B) = 0$
- Nodes of degree 2:
  - $\text{init}_2 = ?$, $\text{send}_2(?) = (B, B)$
  - $\text{receive}_2(?, A, A) = 1$, $\text{receive}_2(?, A, B) = 1$,
    $\text{receive}_2(?, B, A) = 1$, $\text{receive}_2(?, B, B) = 0$



Example

- Design a distributed algorithm that finds a *minimum vertex cover* in $\mathcal{F} = \{$ ,  $\}$

- Solved!

- Running time: 1 communication round

General Principles

General Principles

- Synchronous execution
  - “worst case”
  - synchronisers exist

General Principles

- Synchronous execution
- Deterministic algorithms
    - cf. the name of this course
    - nodes do not have any source of randomness

General Principles

- Synchronous execution
- Deterministic algorithms
- Anonymous networks
  - identical nodes (except for their degree)
  - Chapters 5–6: what happens if each node has a unique name

General Principles

- Synchronous execution
- Deterministic algorithms
- Anonymous networks
- Time = number of communication rounds
  - focus on communication, not computation...

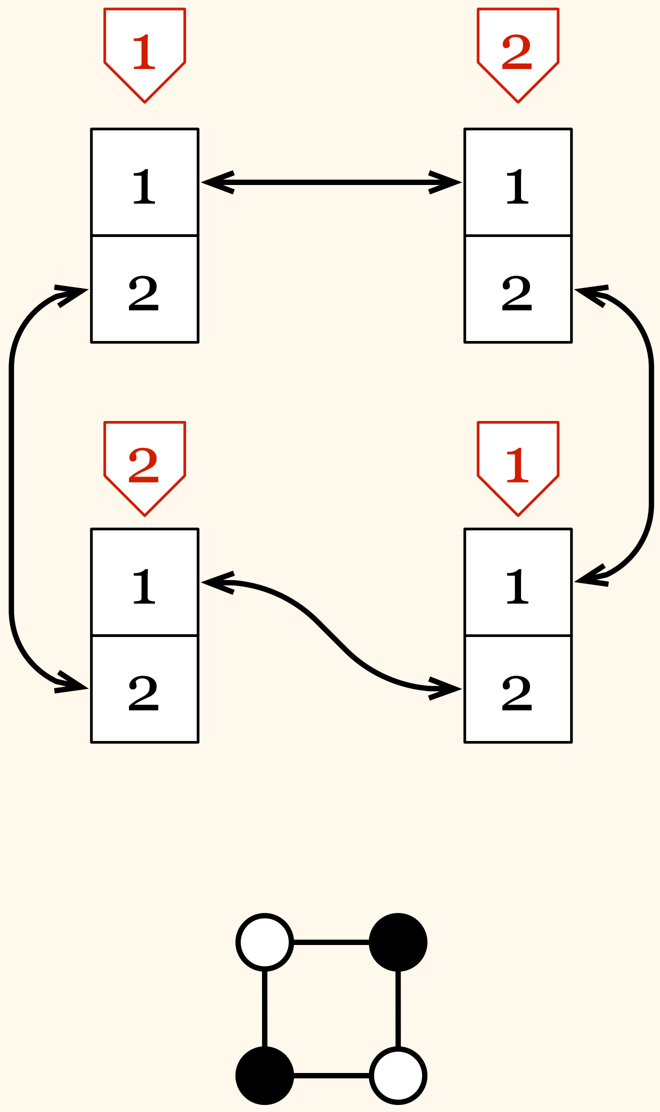
Examples

Maximal Matching

- We will design distributed algorithm BMM that finds a *maximal matching* in any *2-coloured graph*
    - we assume that we are given a proper 2-colouring of the underlying graph as input
    - algorithm will output a maximal matching

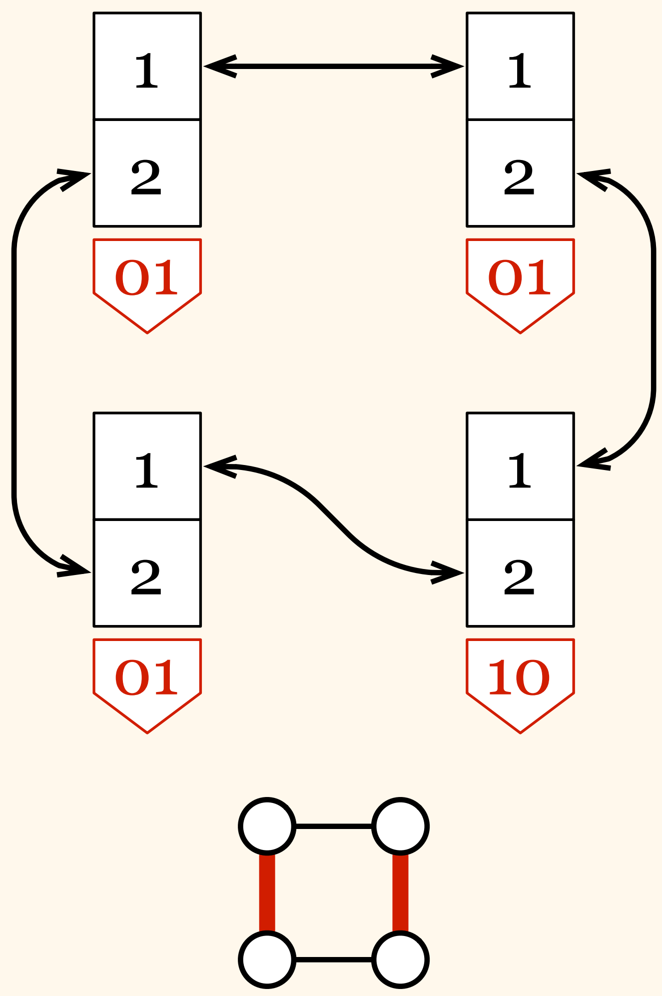
Given

encoding of 2-colouring



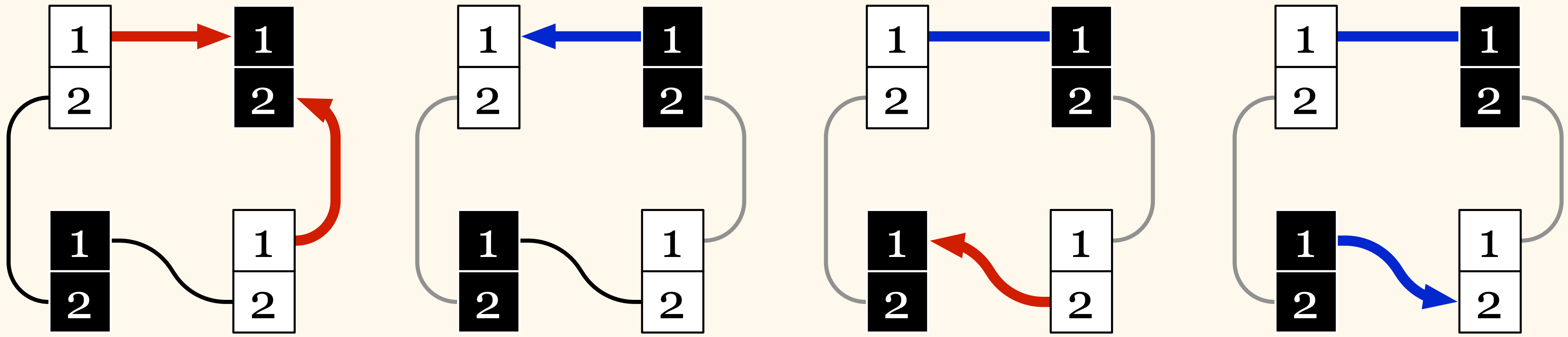
Find

encoding of maximal matching



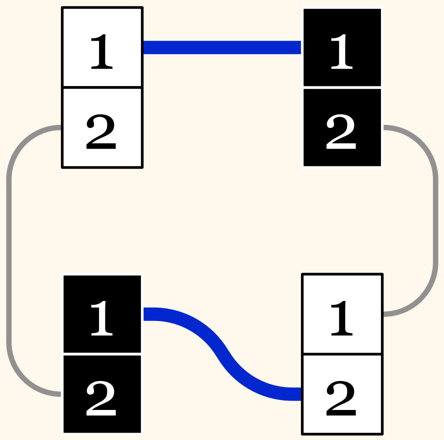
Maximal Matching

- Algorithm idea:
  - white nodes send *proposals* to their ports, one by one
  - black nodes *accept* the first proposal that they get



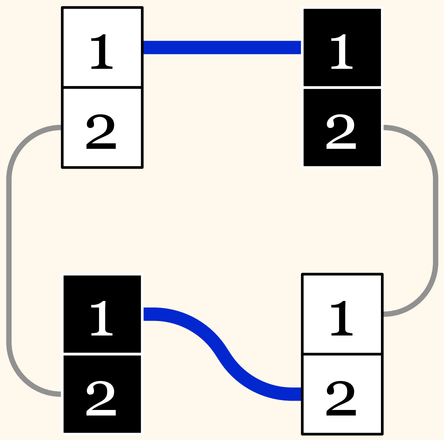
Maximal Matching

- Algorithm idea:
  - white nodes send *proposals* to their ports, one by one
  - black nodes *accept* the first proposal that they get
  - proposal–accept pair = edge in matching
- Running time: $O(\Delta)$
  - Δ = maximum degree



Maximal Matching

- We can find a maximal matching if we are given a 2-colouring
  - some auxiliary information is necessary, as we will see in Chapter 3
- Application: vertex cover approximation
  - works correctly in any network, no need to have 2-colouring!

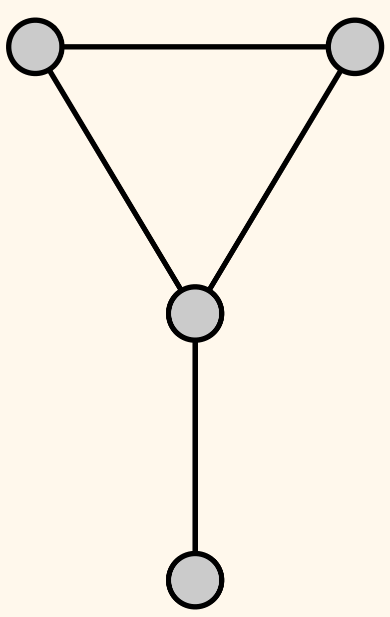


Vertex Cover

- We will design distributed algorithm VC3 that finds a *3-approximation of minimum vertex cover* in any graph
  - each node stops and outputs “0” or “1”
  - nodes that output “1” form a 3-approximation of a minimum vertex cover for the underlying graph

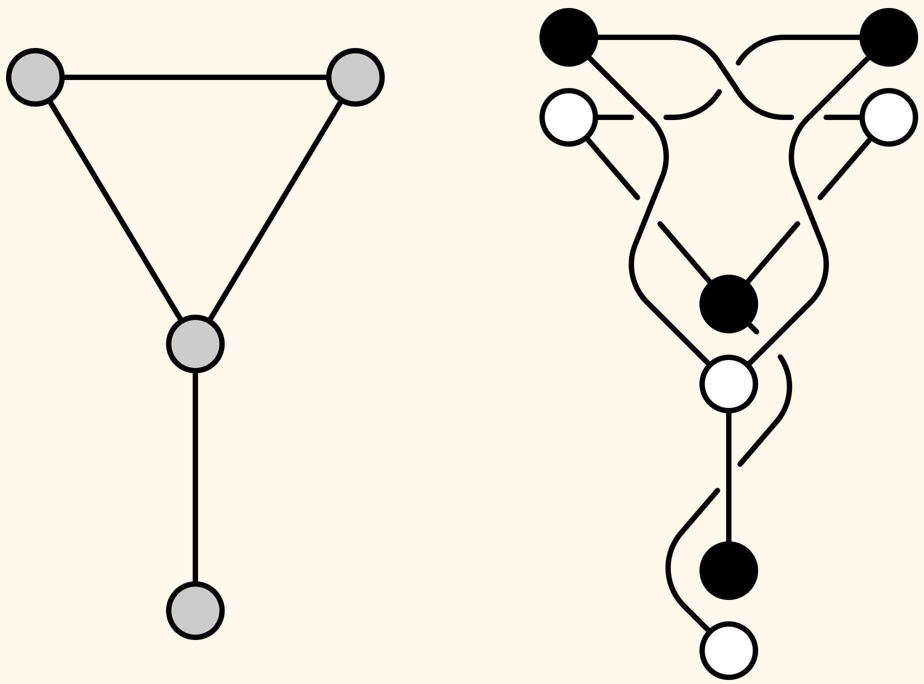
Vertex Cover

- Given: a port-numbered network
  - drawing here just the underlying graph…



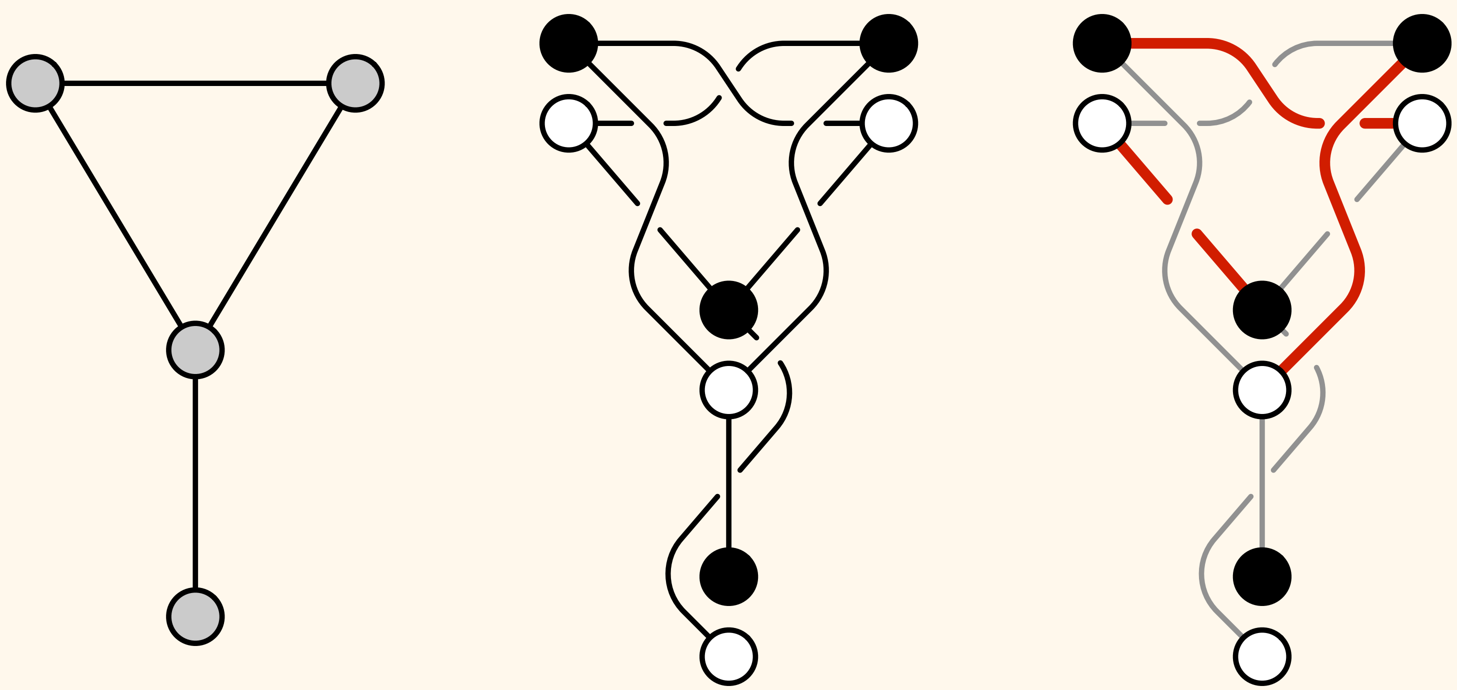
Vertex Cover

- Construct the *bipartite double cover*: two copies of each node, edges across



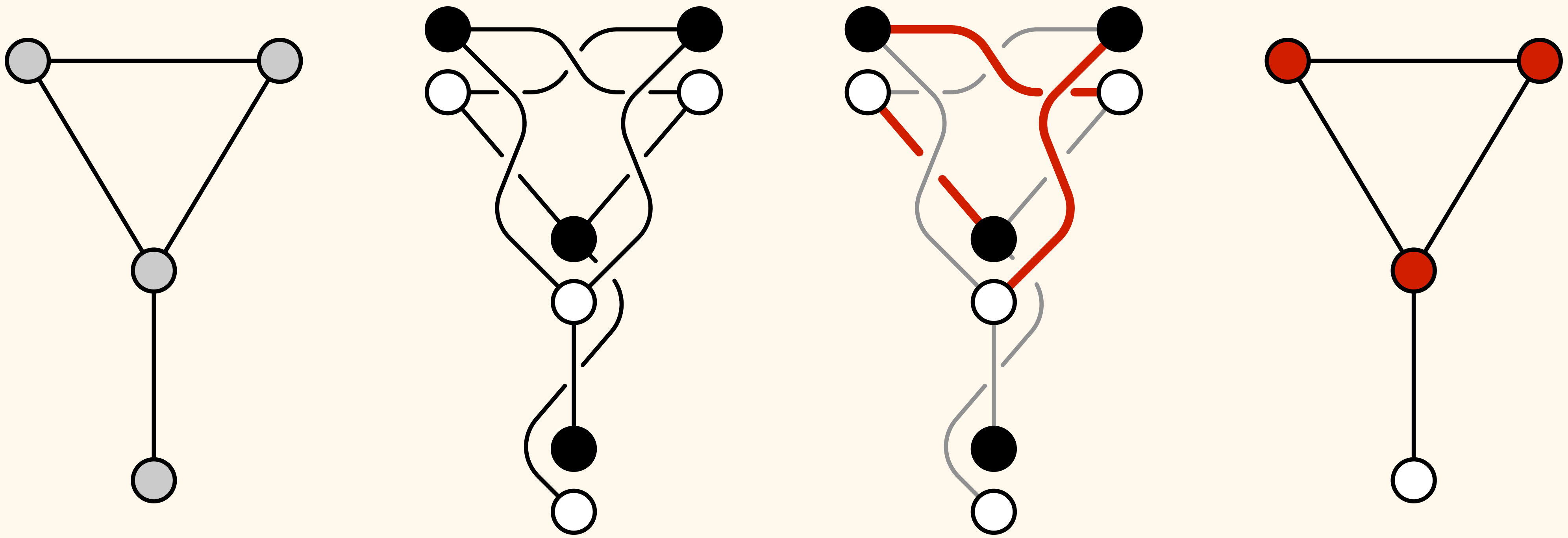
Vertex Cover

- Simulate algorithm BMM, outputs a *maximal matching M’*



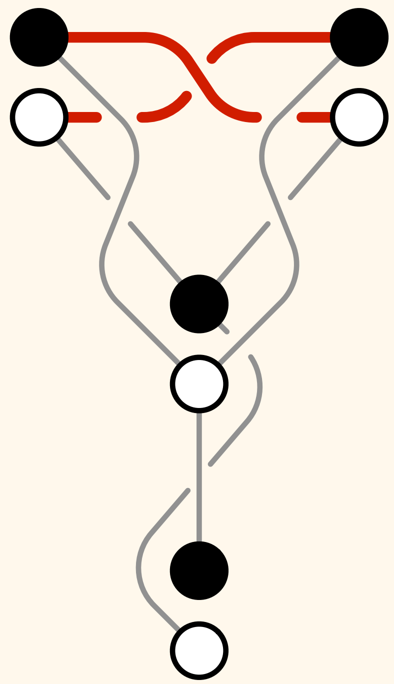
Vertex Cover

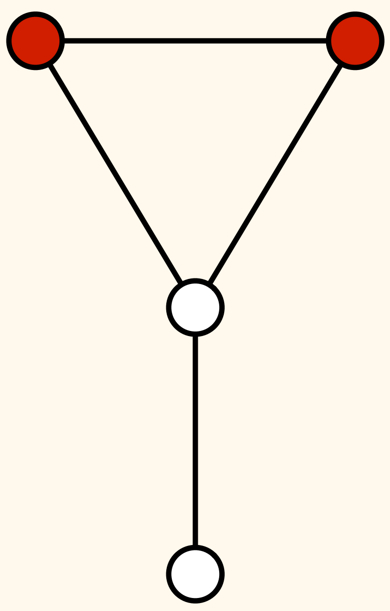
- C = nodes with at least one copy matched: 3-approximation of minimum vertex cover!



Vertex Cover

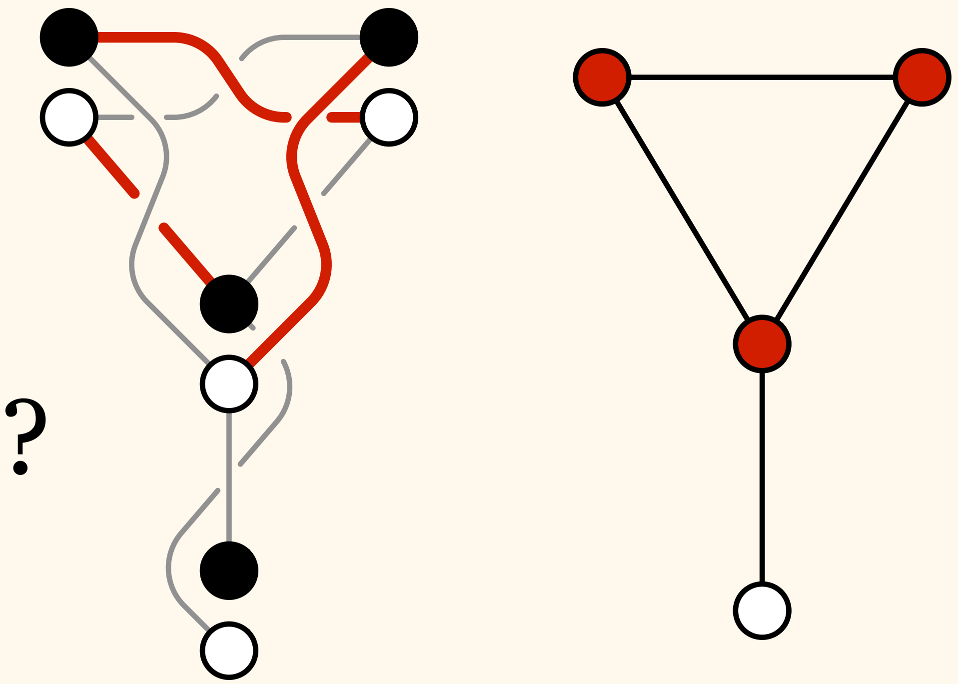
- C = nodes with at least one copy matched: 3-approximation of minimum vertex cover!
- Why vertex cover?
  - assume that there is an uncovered edge
  - conclude that M' is not maximal





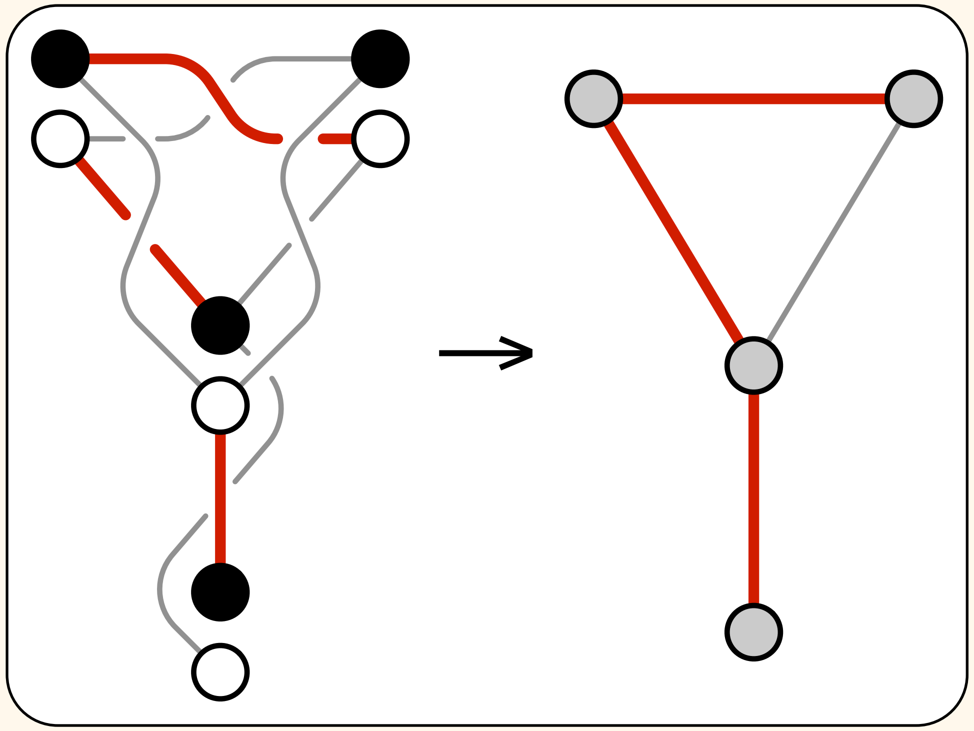
Vertex Cover

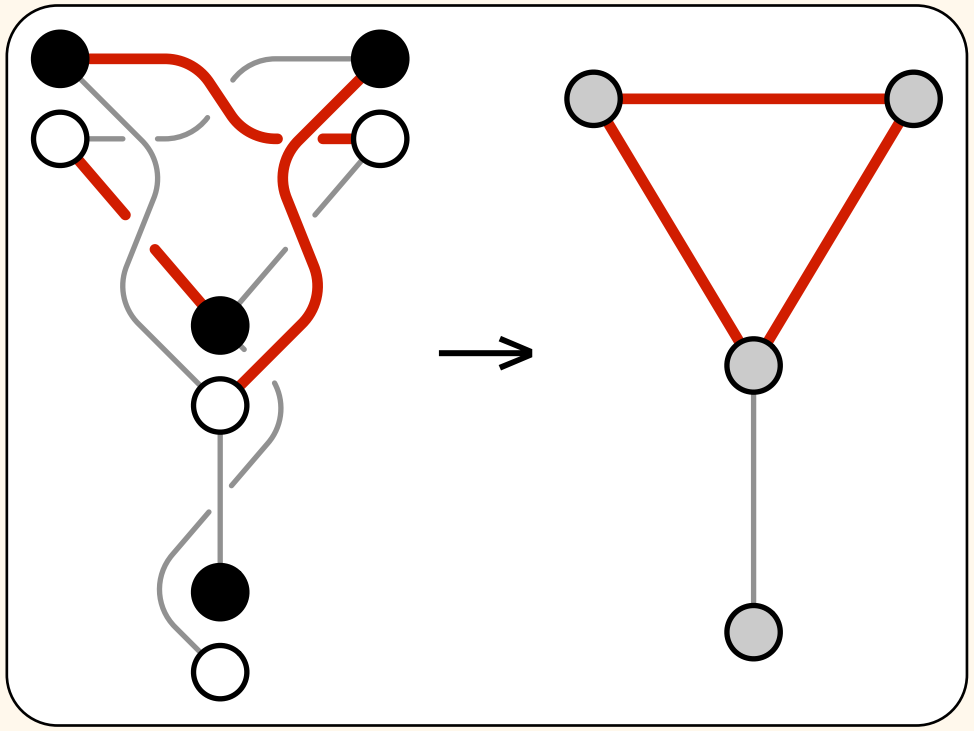
- C = nodes with at least one copy matched: 3-approximation of minimum vertex cover!
- Why vertex cover?
- Why 3-approximation?



Vertex Cover

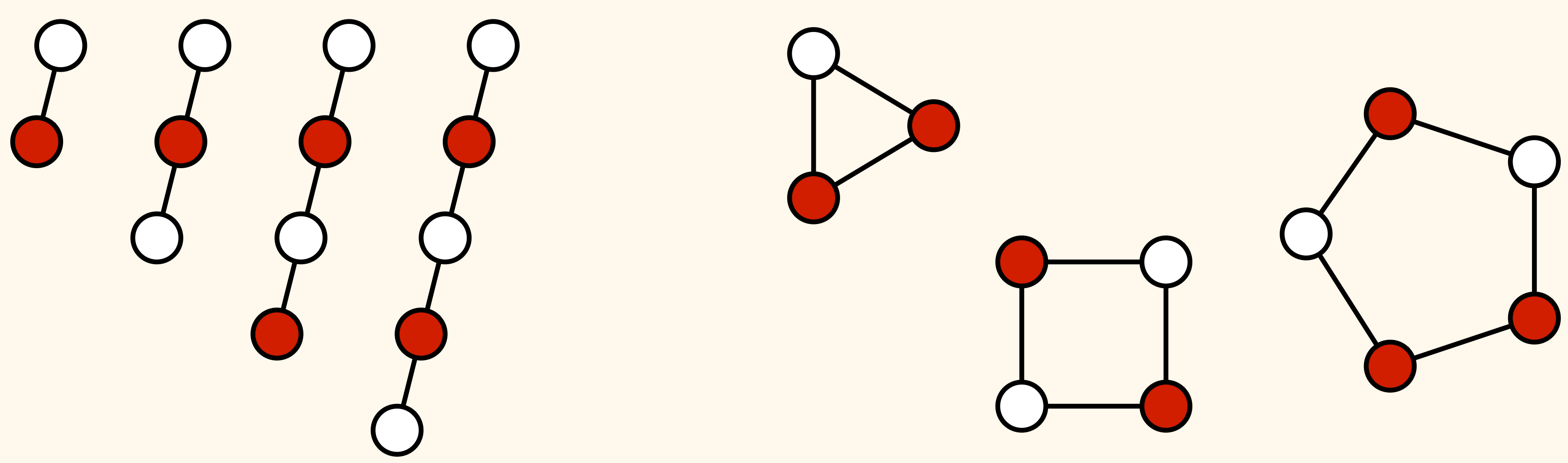
- Idea: matching in bipartite double cover
→ paths and/or cycles in original graph





Vertex Cover

- Any vertex cover contains at least 1/3 of nodes of any path or cycle
- 3-approximation if we take all of these



Summary

- We can solve non-trivial problems with distributed algorithms
  - e.g., 3-approximation of minimum vertex cover
- What next?
  - week 3: problems that cannot be solved at all
  - week 4: more positive results
  - weeks 5–6: what changes if the nodes have names?