Instructions. There are three questions, each of them worth 20 points. For the minimum passing grade of 1/5, you will need approximately 30 points, and for the highest grade of 5/5, you will need approximately 50 points. You can answer in English, Finnish, or Swedish.

Definitions. All questions are related to *deterministic distributed algorithms in the port-numbering model*. We will use the following definitions in this exam.

- $\mathbf{\Pi_{MIS}}$ is the distributed graph problem of finding a **maximal independent set**. That is, if $N = (V, P, p)$ is a simple port-numbered network, and G is the underlying graph of N, then $f \in \Pi_{\mathrm{MIS}}(N)$ if f encodes a maximal independent set of G. Note that f has to be a function $f : V \to \{0, 1\}$.

- $\mathbf{\Pi_6}$ is the distributed graph problem of finding a **6-colouring**. That is, if $N = (V, P, p)$ is a simple port-numbered network, and G is the underlying graph of N, then $f \in \Pi_6(N)$ if f encodes a proper vertex colouring of G with 6 colours. Note that f has to be a function $f : V \to \{1, 2, \dots, 6\}$. Remember that a 6-colouring does not need to use all colours; a 2-colouring is also a 6-colouring.

- $\mathbf{\Pi_{256}}$ is the distributed graph problem of finding a **256-colouring**.

- $\mathscr{F}_{\mathrm{C}}$ is the family of **cycle graphs**. That is, $G \in \mathscr{F}_{\mathrm{C}}$ if G is a connected 2-regular simple undirected graph.

Question 1. Design a distributed algorithm A that solves problem Π_{MIS} on graph family $\mathscr{F}_{\mathrm{C}}$ given Π_6. That is, given a cycle graph with a 6-colouring, the algorithm has to find a maximal independent set. Present your algorithm in a formally precise manner, using the *state machine formalism*. You will have to define

· States_A, the set of states,
· Msg_A, the set of possible messages,
· $\mathrm{init}_{A,d}\colon \mathrm{Input}_A \to \mathrm{States}_A$, the function that initialises the state machine,
· $\mathrm{send}_{A,d}\colon \mathrm{States}_A \to \mathrm{Msg}_A^d$, the function that constructs outgoing messages, and
· $\mathrm{receive}_{A,d}\colon \mathrm{States}_A \times \mathrm{Msg}_A^d \to \mathrm{States}_A$, the function that processes incoming messages.

Note that the set of local inputs is $\mathrm{Input}_A = \{1, 2, \dots, 6\}$ and the set of local outputs (stopping states) is $\mathrm{Output}_A = \{0, 1\}$. Prove that your algorithm is correct. What can be said about its running time?

Question 2. Design a distributed algorithm A that solves problem Π_6 on graph family $\mathscr{F}_{\mathrm{C}}$ given Π_{256}. The running time should be less than 100 communication rounds. An informal description of the algorithm (e.g., using pseudo code) is sufficient. You can use algorithm DPBit from the course material as a subroutine. Remember that DPBit reduces the number of colours from 2^x to $2x$ in one communication round in *directed pseudoforests*, but in this question we are only given an *undirected* cycle with some port numbering and some 256-colouring, so we cannot directly apply DPBit.

Question 3. Prove that there is no distributed algorithm that solves problem Π_6 on graph family $\mathscr{F}_{\mathrm{C}}$ given Π_{MIS}. That is, given a cycle graph and a maximal independent set, it is not possible to find a 6-colouring. You can use the following result that is familiar from the course material.

> *Theorem.* Assume that A is a distributed algorithm, $X = \mathrm{Input}_A$ is a set of local inputs, $N = (V, P, p)$ and $N' = (V', P', p')$ are port-numbered networks, $f : V \to X$ and $f' : V' \to X$ are arbitrary functions, and $\phi : V \to V'$ is a covering map from (N, f) to (N', f'). Let $x_0, x_1, \dots$ be the execution of A on (N, f), and let $x'_0, x'_1, \dots$ be the execution of A on (N', f'). Then for each $t = 0, 1, \dots$ and each $v \in V$ we have $x_t(v) = x'_t(\phi(v))$.