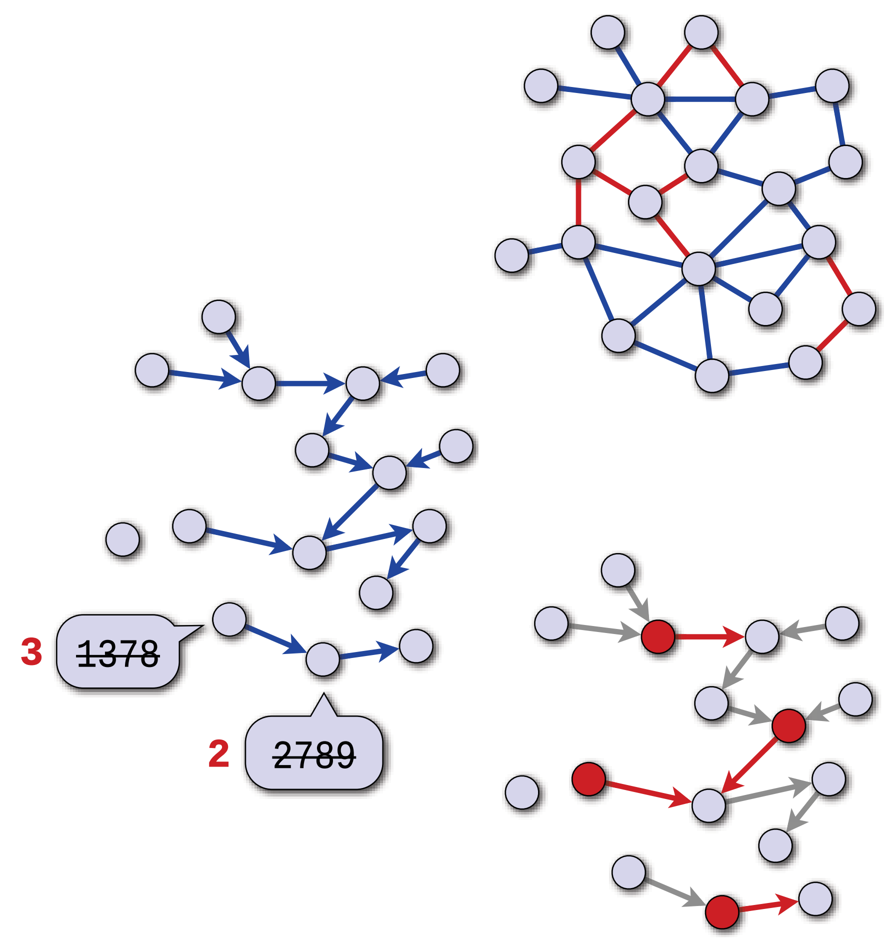
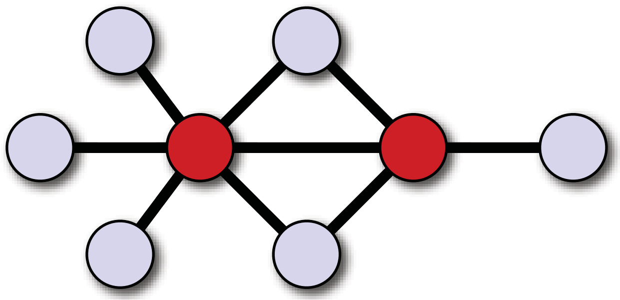
DDA 2010, lecture 7: Local news

- Some recent work in our research group
  - algorithm for vertex covers
  - application of Cole–Vishkin technique in port-numbering model

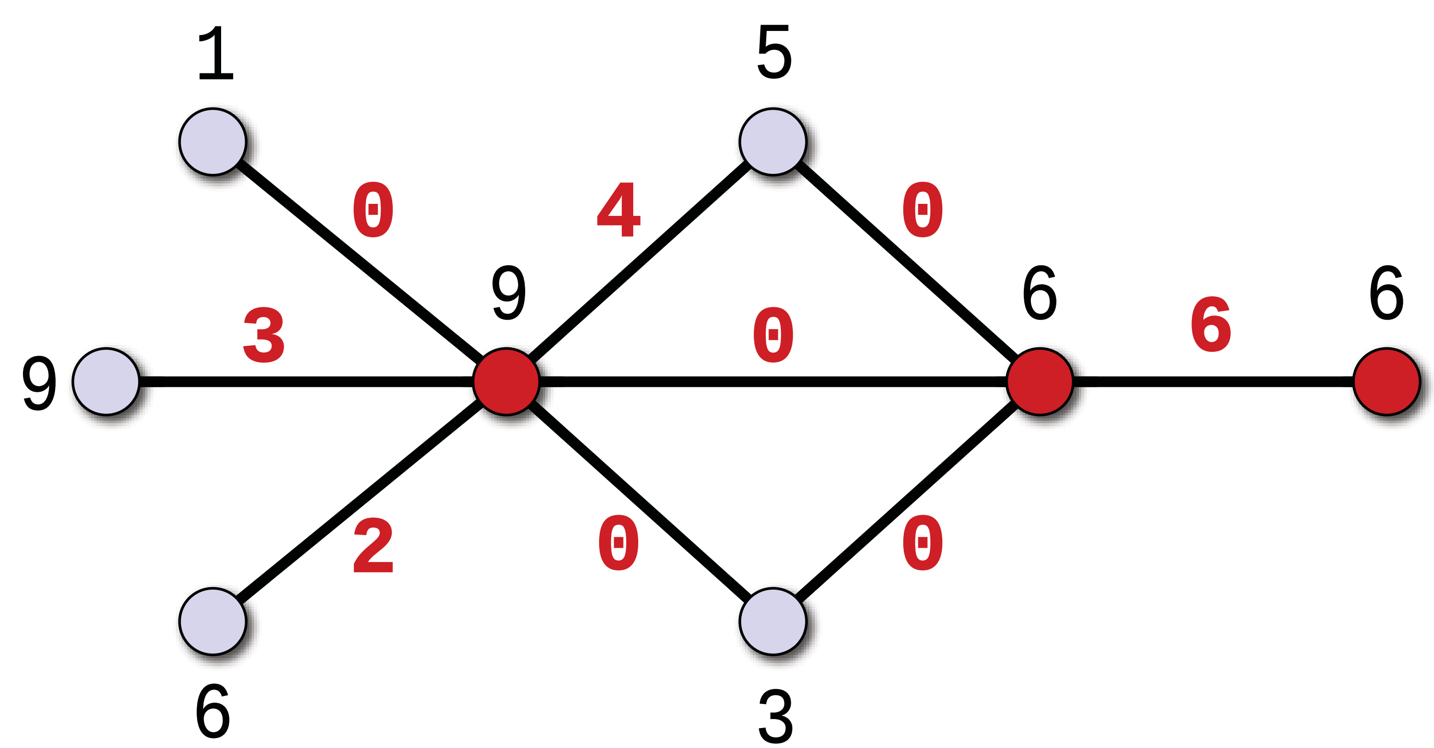


Research problem



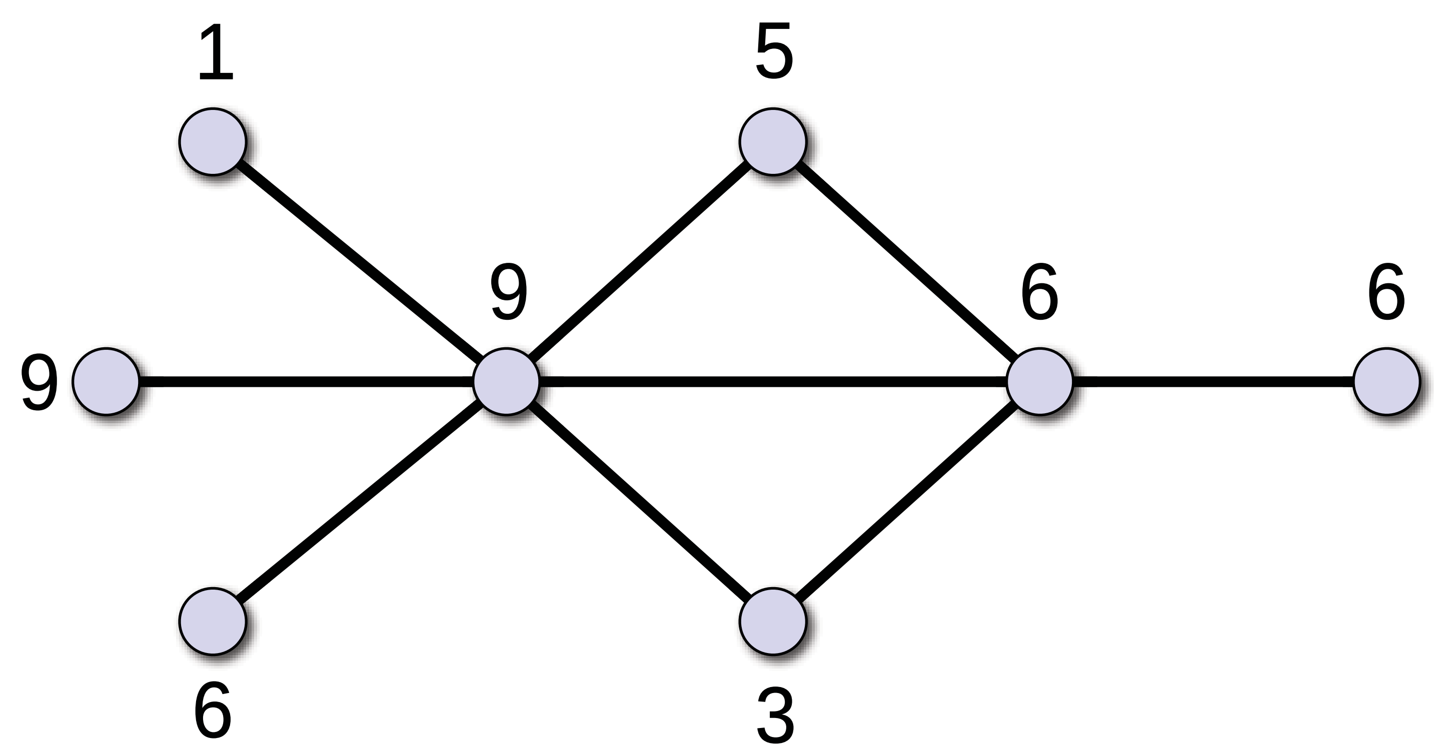
- Goal: finding a 2-approximation of **minimum vertex cover**
  - fast: time independent of *n*
  - port-numbering model
- From lecture 4:
  - even if we had unique identifiers, it's not possible to find $(2 - \varepsilon)$-approximation in constant time
  - hence approximation factor 2 is the best possible

DDA 2010, lecture 7a:
Vertex covers and edge packings



Vertex cover in the port-numbering model

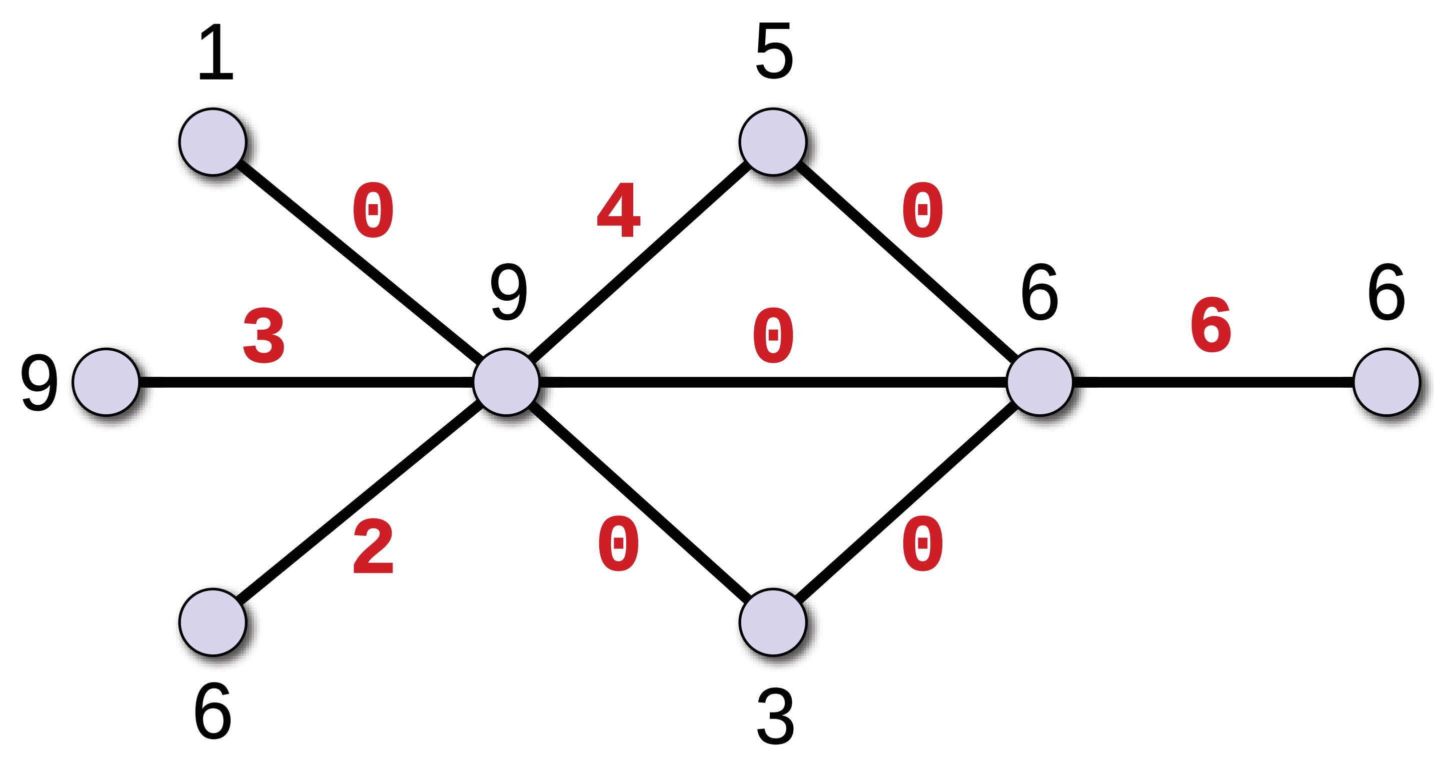
- Convenient to study a more general problem: minimum-weight vertex cover
  - **More general problems are sometimes easier to solve?**



Notation:
$w(v)$ = weight of v

Edge packings and vertex covers

- **Edge packing**: weight $y(e) \geq 0$ for each edge e
  - Packing constraint: $y[v] \leq w(v)$ for each node v, where $y[v]$ = total weight of edges incident to v



Edge packings and vertex covers

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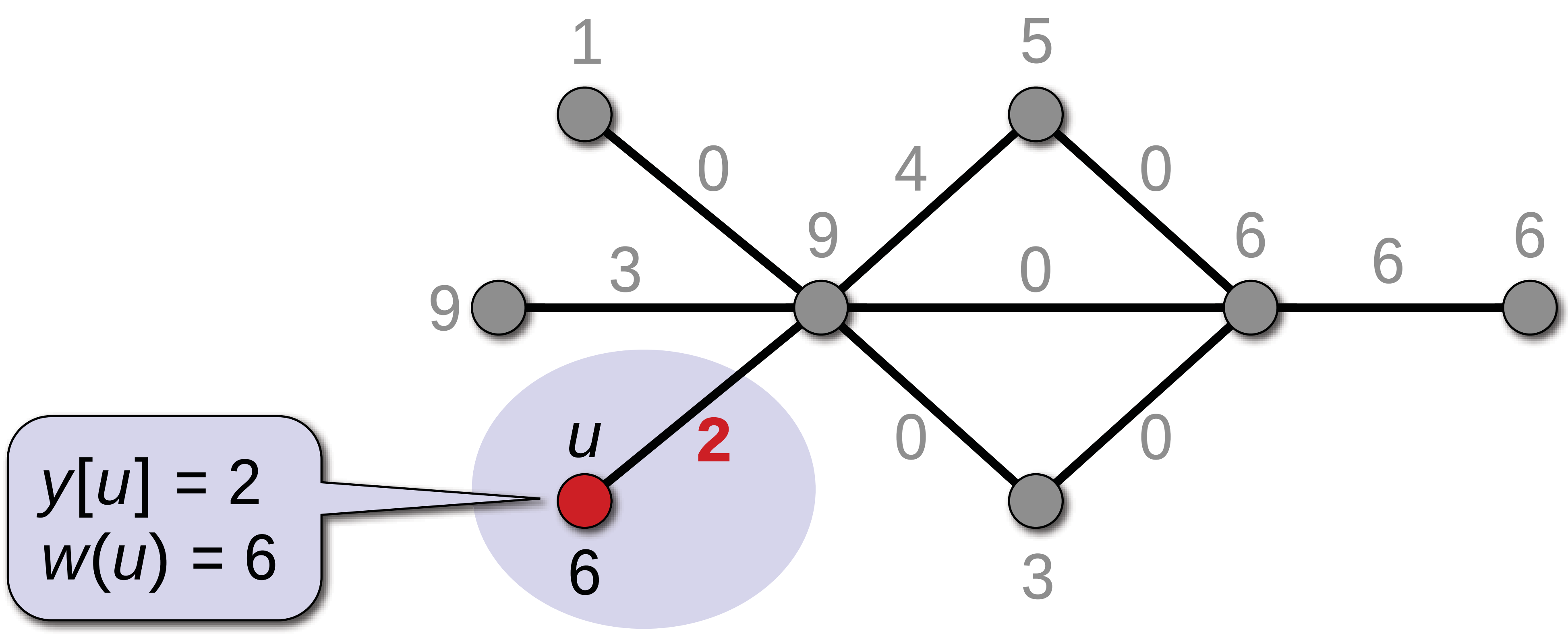

Edge packings and vertex covers

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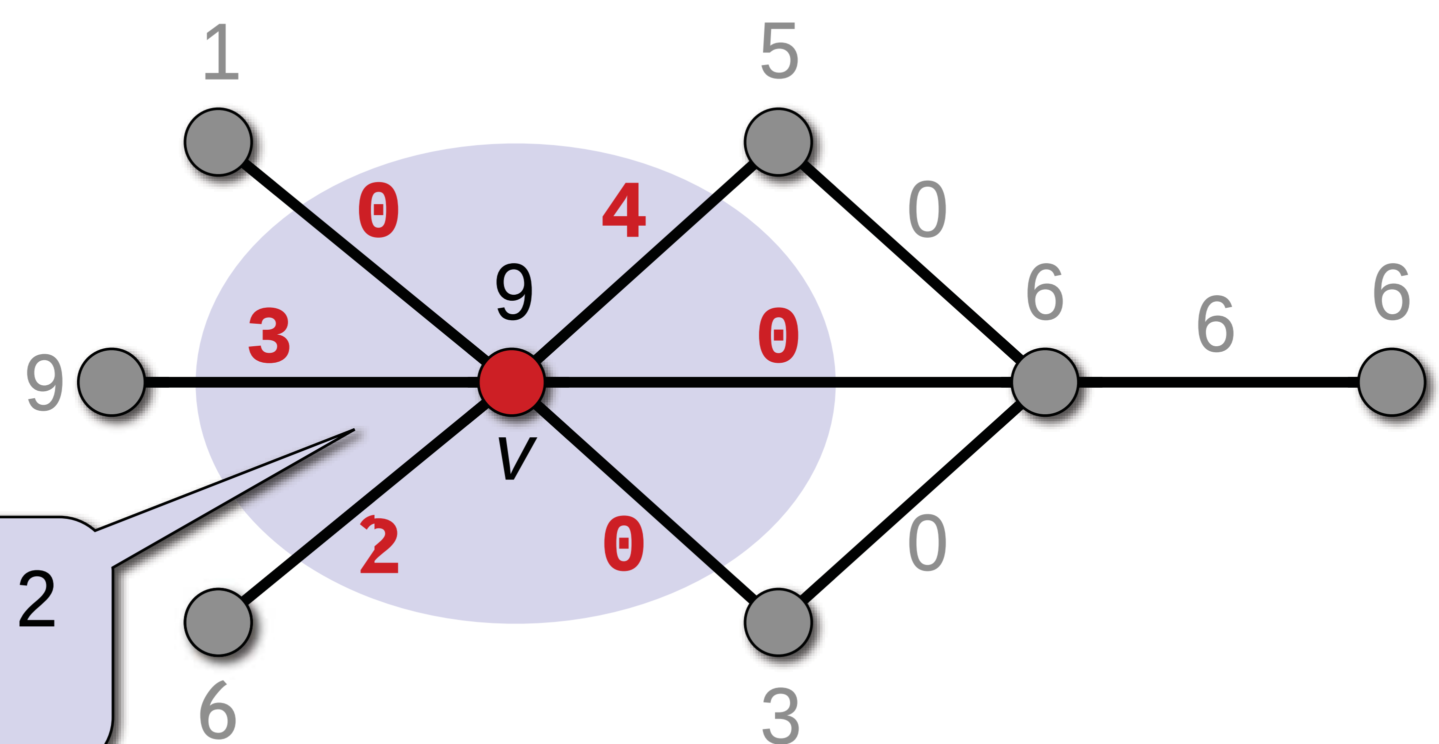
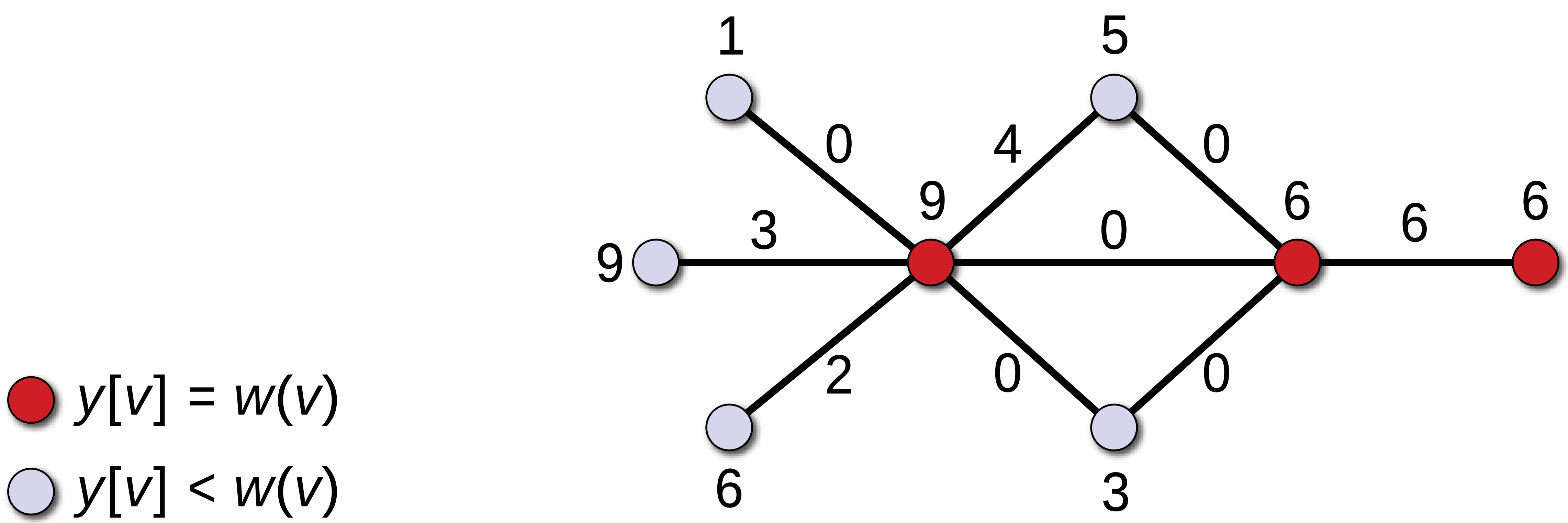


$y[v] = 3 + 0 + 4 + 0 + 0 + 2$
$w(v) = 9$

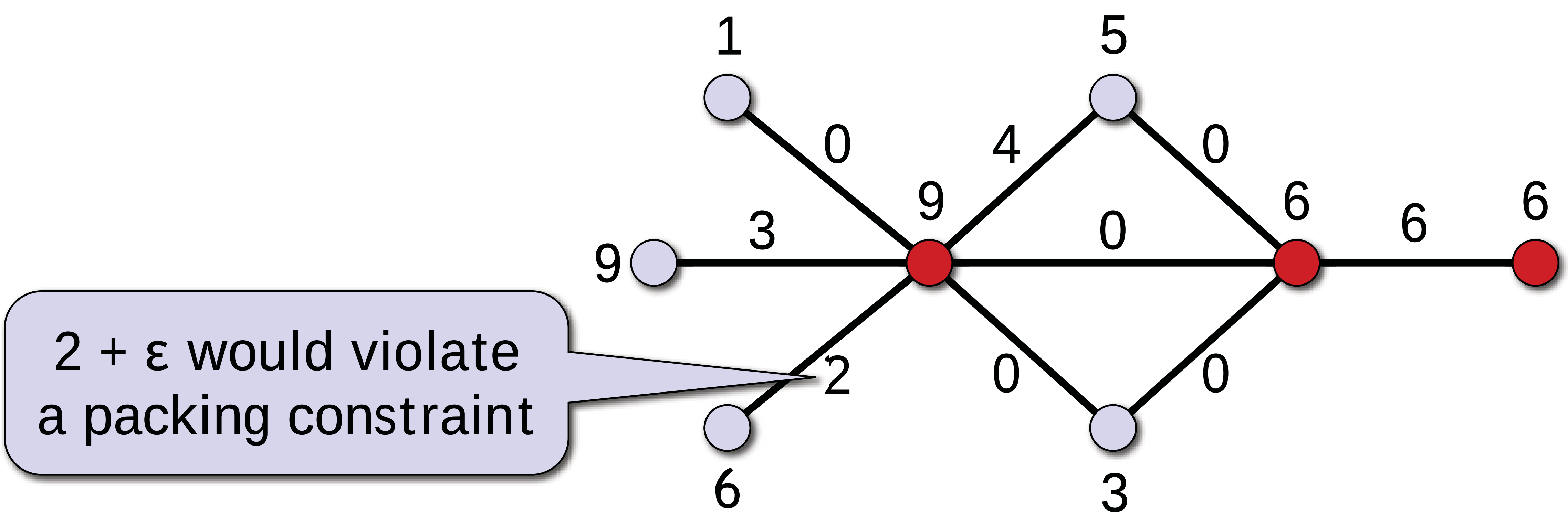
Edge packings and vertex covers

- Node v is **saturated** if $y[v] = w(v)$
  - Total weight of edges incident to v is *equal* to $w(v)$, i.e., the packing constraint holds with equality



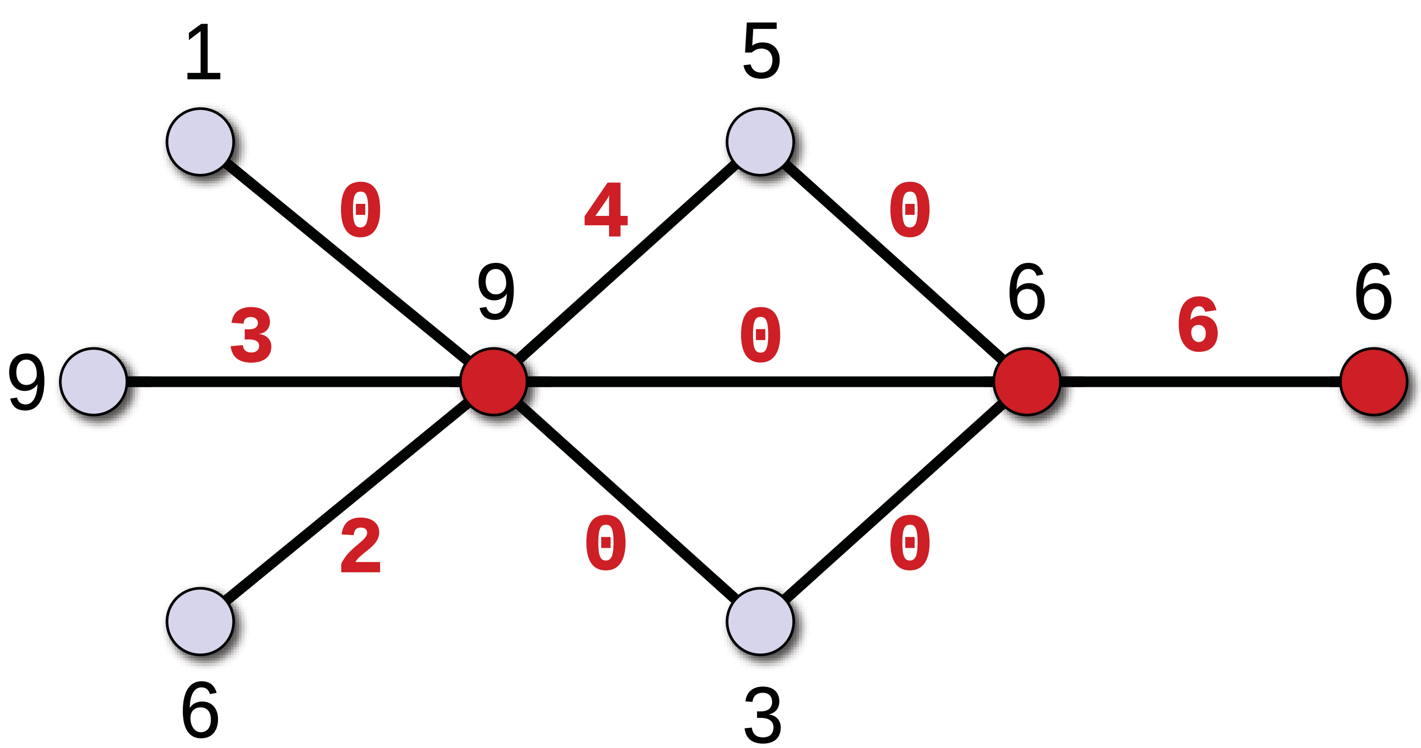
Edge packings and vertex covers

- Edge e is **saturated** if at least one endpoint of e is saturated
  - Equivalently: edge weight $y(e)$ can't be increased



Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  $\iff$ none of the edge weights $y(e)$ can be increased
  $\iff$ saturated nodes form a vertex cover!



Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find:
  - Centralised setting: NP-hard
  - Distributed setting: integer problem (choose 0 or 1), symmetry-breaking issues
- Maximal edge packing y easy to find:
  - Centralised setting: trivial greedy algorithm
  - Distributed setting: linear problem, no symmetry-breaking issues (?)

Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find
- Maximal edge packing y easy to find?
- Saturated nodes $C(y)$ in y: **2-approximation** of C^*
  - Textbook proof: LP-duality,
    relaxed complementary slackness
  - Minimum-weight fractional vertex cover and
    maximum-weight edge packing are *dual problems*
  - But there's a simple and more elementary proof…

Edge packings and vertex covers

$\sum_{v \in C(y)} w(v)$	Total weight of saturated nodes
$= \sum_{v \in C(y)} y[v]$	Saturated nodes have $y[v] = w(v)$
$= \sum_{e \in E} y(e) \, \lvert e \cap C(y) \rvert$	Interchange the order of summation
$\leq 2 \sum_{e \in E} y(e) \, \lvert e \cap C^* \rvert$	Each edge is covered at least *once* by C^* and at most *twice* by $C(y)$
$= 2 \sum_{v \in C^*} y[v]$	Interchange the order of summation
$\leq 2 \sum_{v \in C^*} w(v)$	All nodes have $y[v] \leq w(v)$

Edge packings and vertex covers

$\sum_{v \in C(y)} \sum_{e \in E:\, v \in e} y(e)$

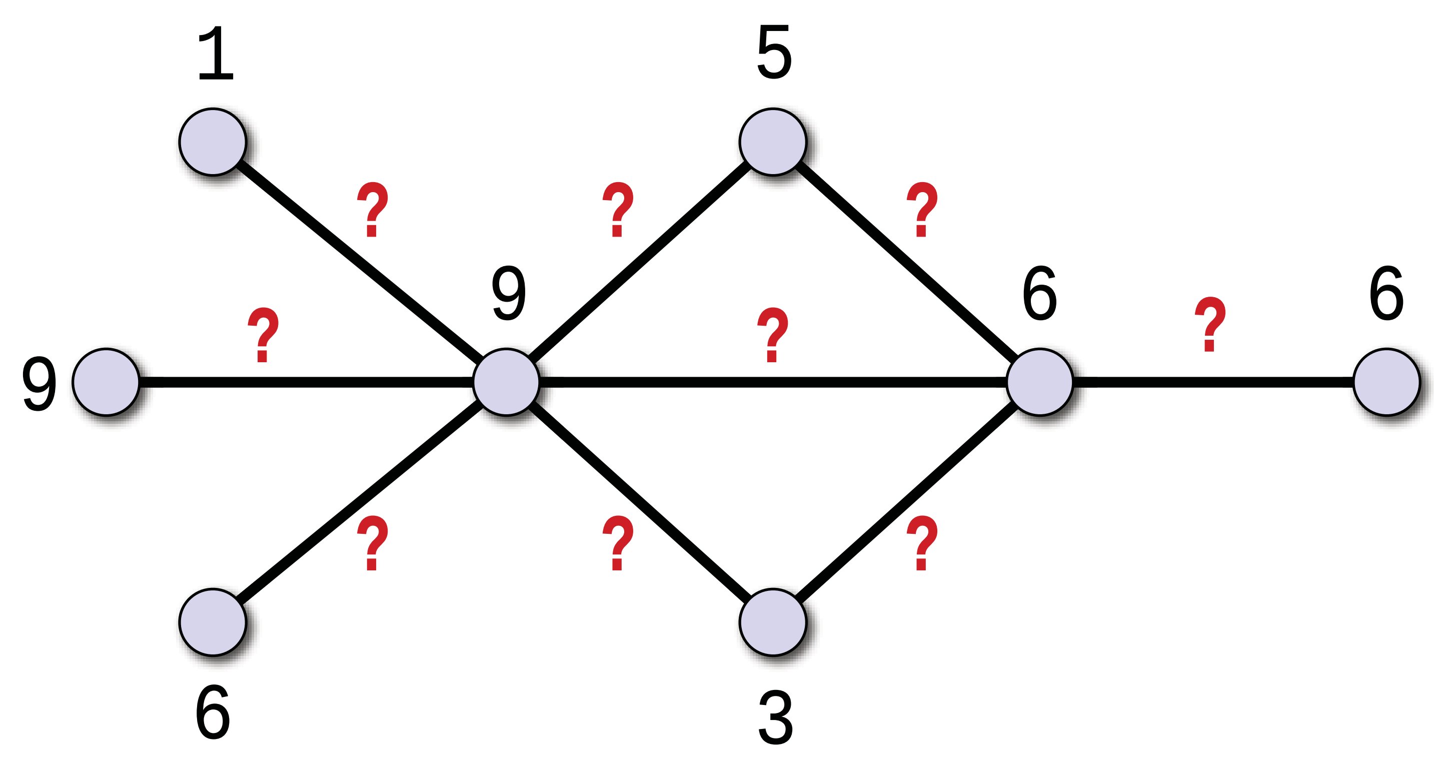
$\sum_{e \in E} \sum_{v \in C(y):\, v \in e} y(e)$

$\sum_{v \in C(y)} w(v)$	ted nodes
$= \sum_{v \in C(y)} y[v]$	$y[v] = w(v)$
$= \sum_{e \in E} y(e)\, \lvert e \cap C(y) \rvert$	Interchange the order of summation
$\leq 2 \sum_{e \in E} y(e)\, \lvert e \cap C^* \rvert$	Each edge is covered at least *once* by C^* and at most *twice* by $C(y)$
$= 2 \sum_{v \in C^*} y[v]$	Interchange the order of summation
$\leq 2 \sum_{v \in C^*} w(v)$	All nodes have $y[v] \leq w(v)$

Summary

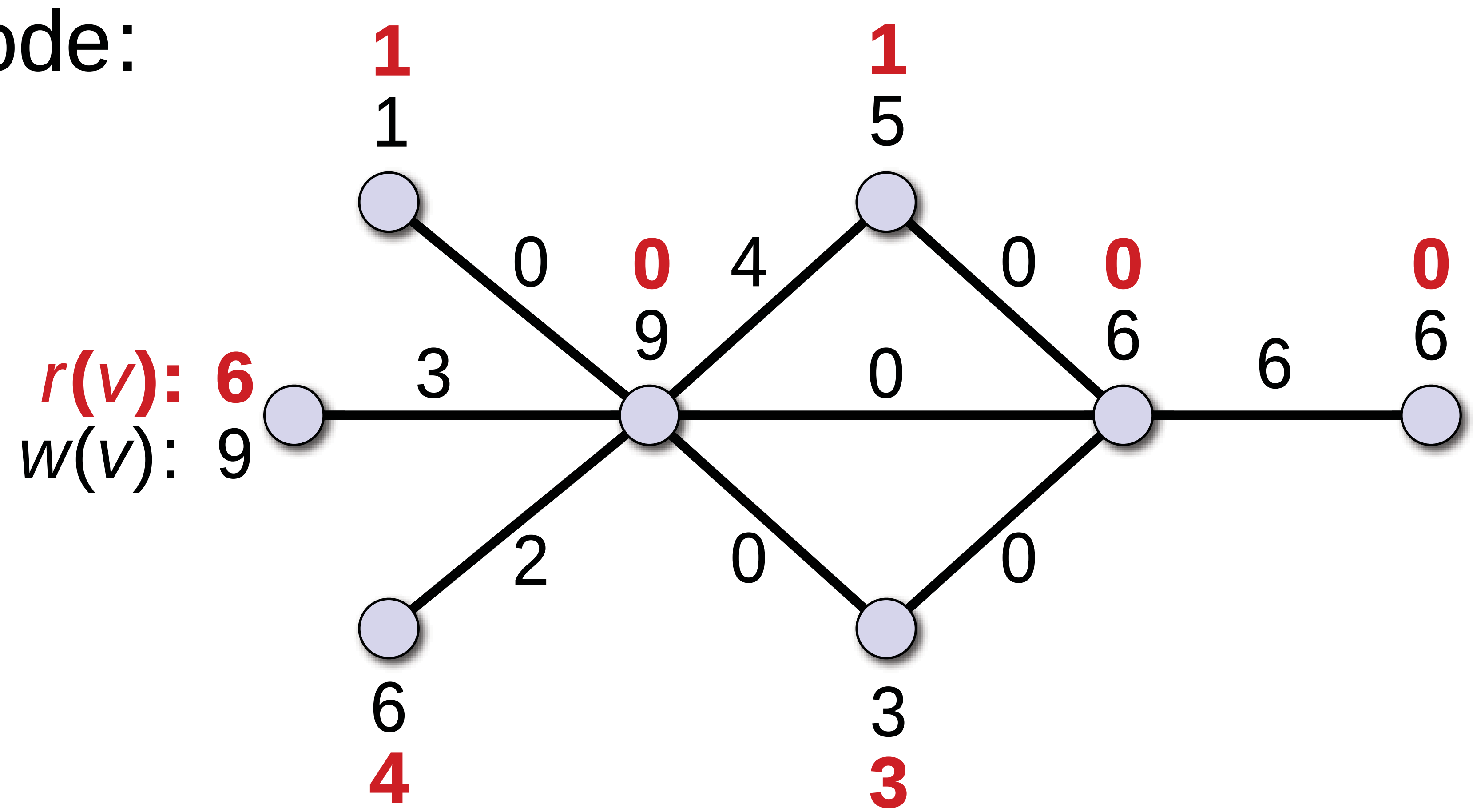
- Goal:
  - Find a 2-approximation of minimum-weight **vertex cover**
  - Deterministic algorithm in the **port-numbering** model
- Idea:
  - Find a **maximal edge packing**, take saturated nodes
- Coming up next:
  - Begin with a “greedy but safe” algorithm
  - We will see later how the Cole–Vishkin technique helps

DDA 2010, lecture 7b: Finding a maximal edge packing



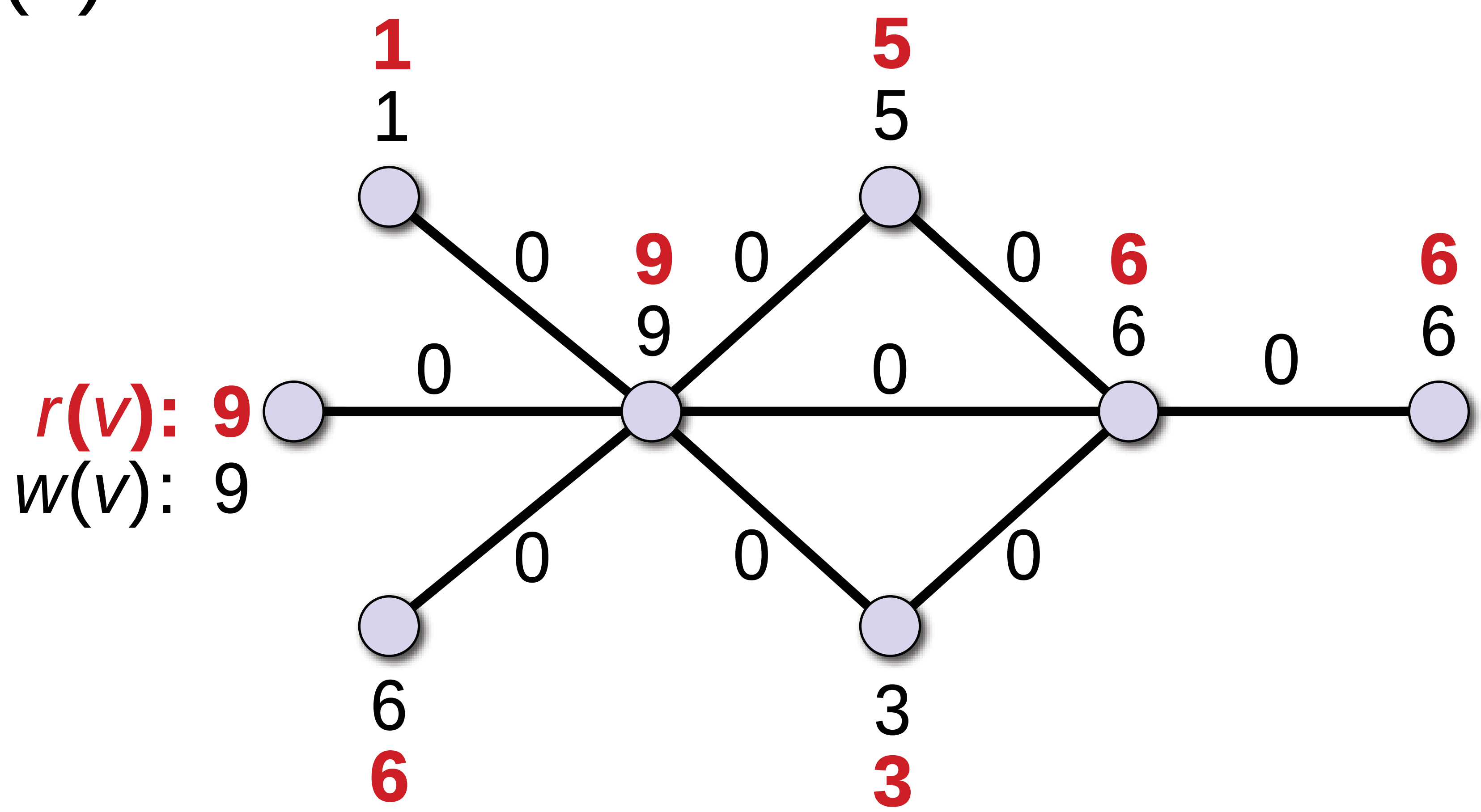
Finding a maximal edge packing: phase I

- $y[v]$ = total weight of edges incident to node v
- **Residual capacity** of node v: $r(v) = w(v) - y[v]$
- Saturated node: $r(v) = 0$



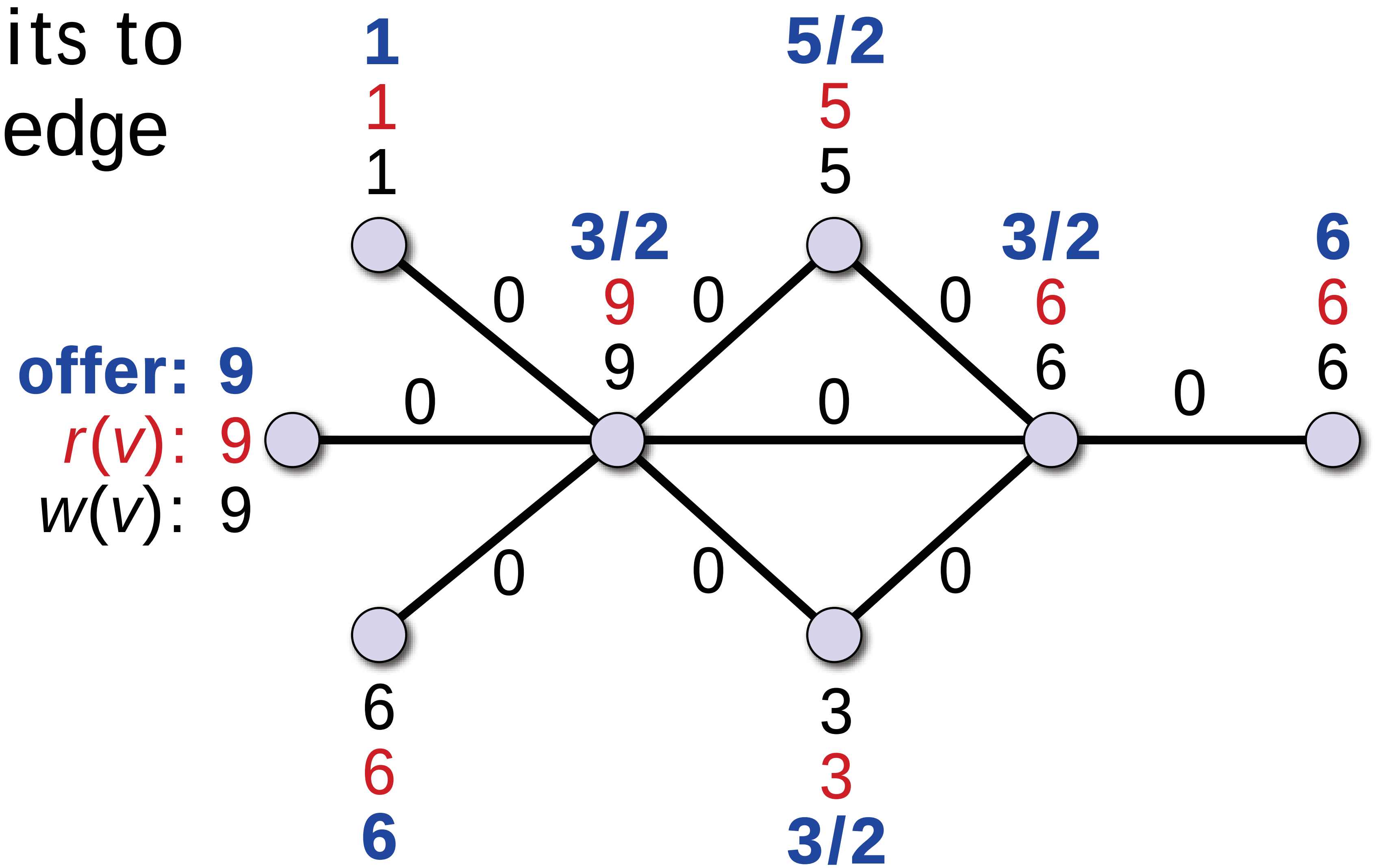
Finding a maximal edge packing: phase I

Start with a trivial edge packing $y(e) = 0$



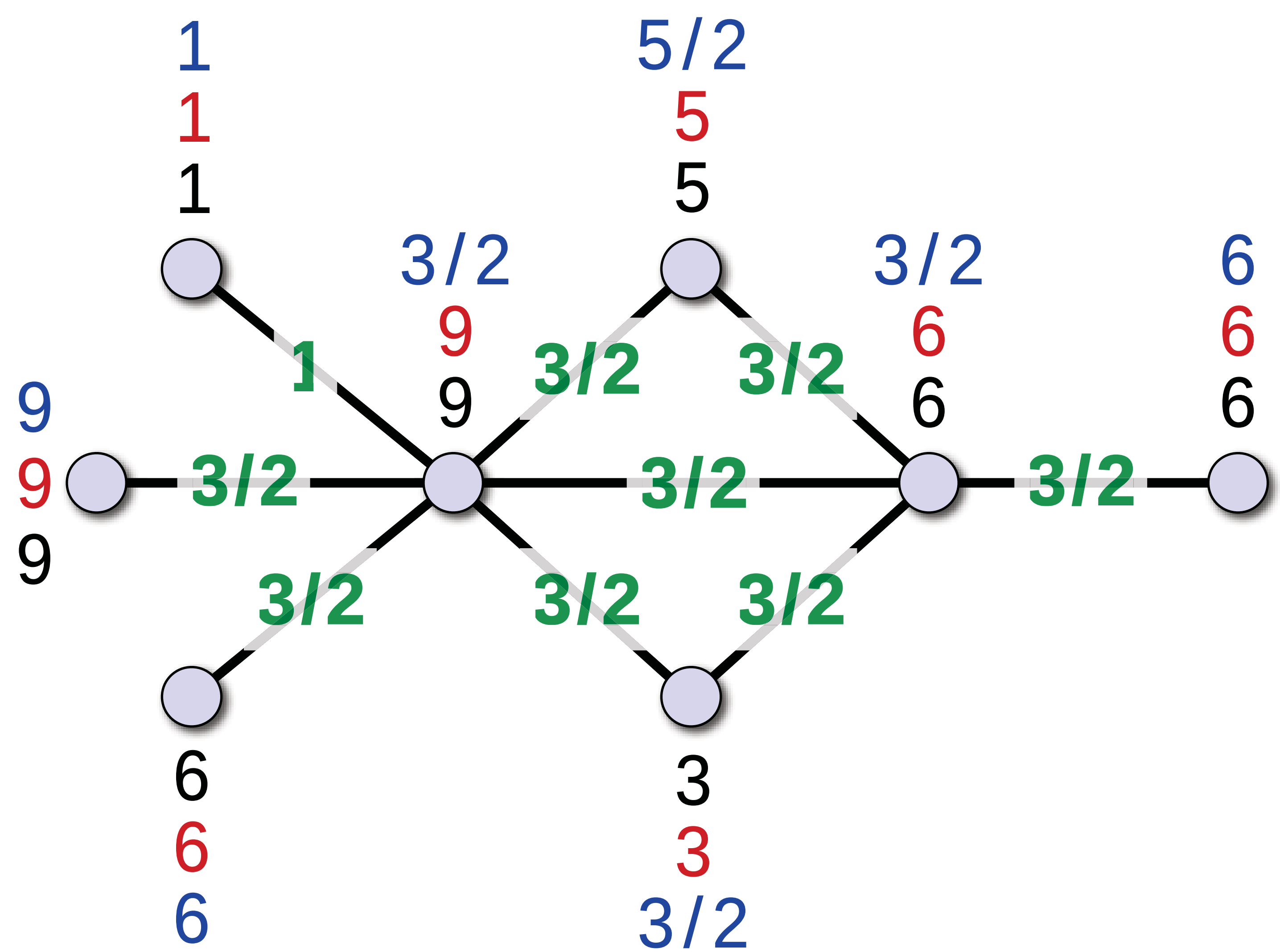
Finding a maximal edge packing: phase I

Each node v **offers** $r(v)\,/\,\deg(v)$ units to each incident edge



Finding a maximal edge packing: basic idea

Each edge **accepts** the smallest of the 2 offers it received

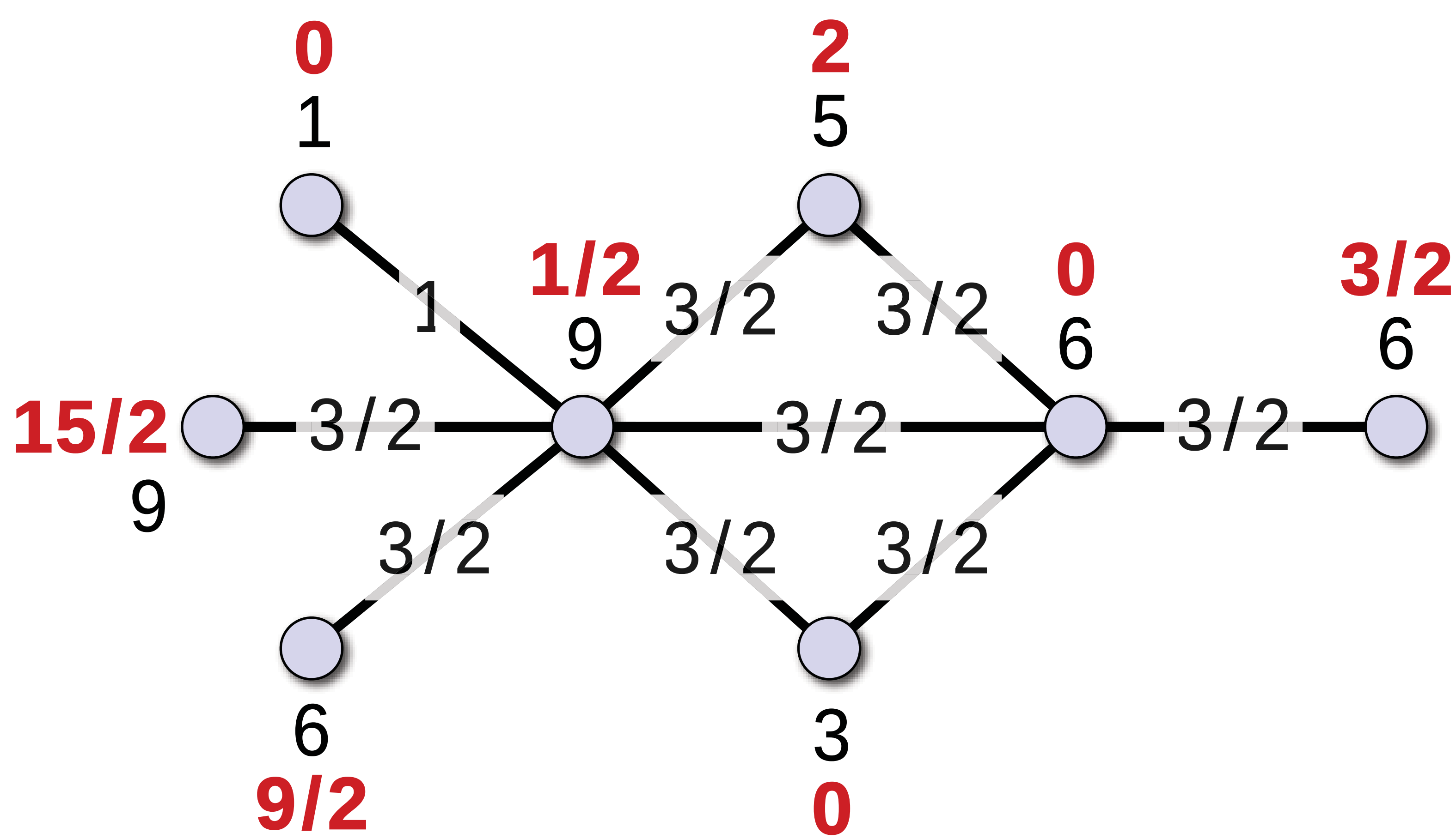


Increase $y(e)$ by this amount

- Safe, can’t violate packing constraints

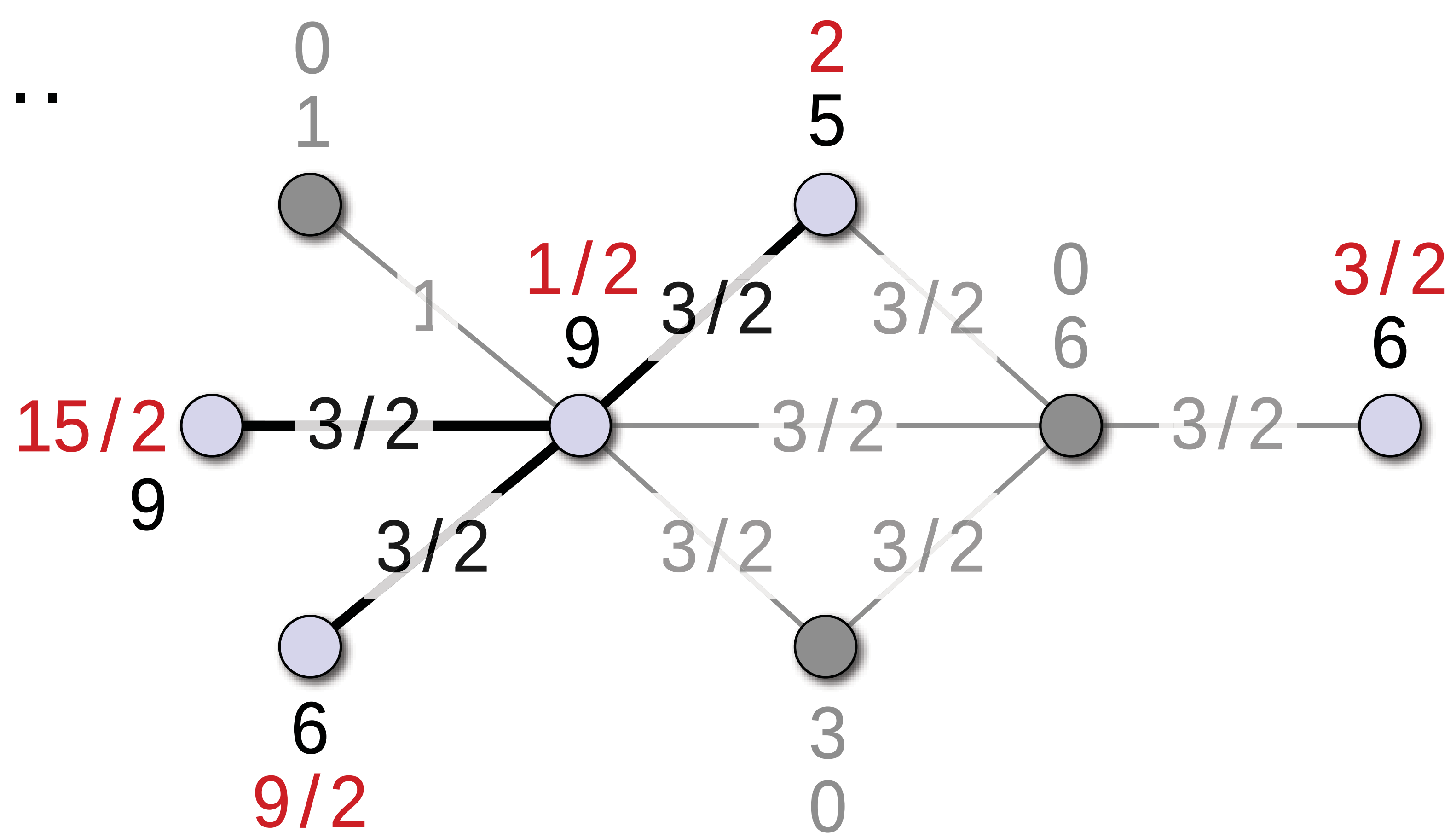
Finding a maximal edge packing: phase I

Update **residuals**…



Finding a maximal edge packing: phase I

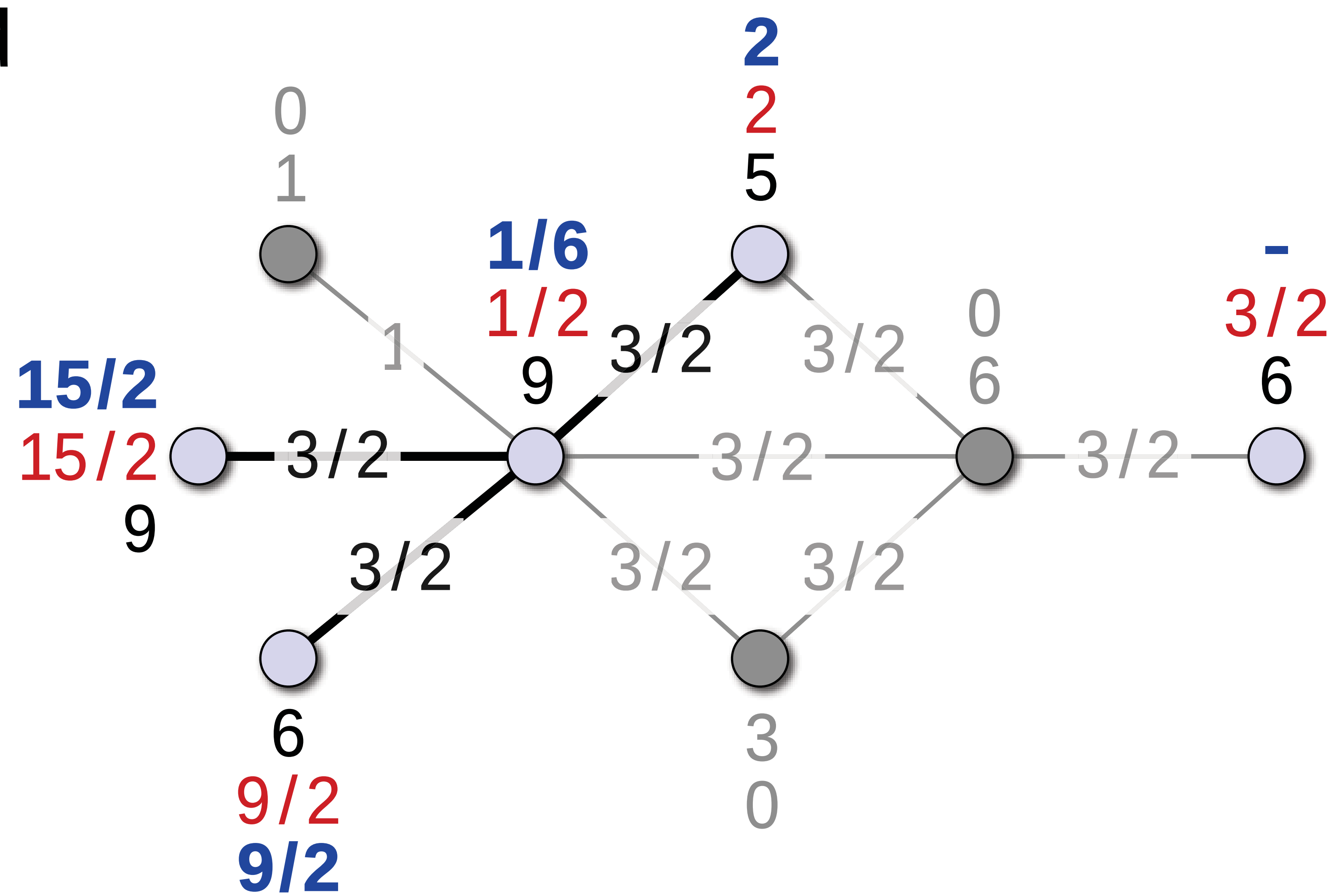
Update residuals, discard saturated nodes and edges…



Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat…

Offers…

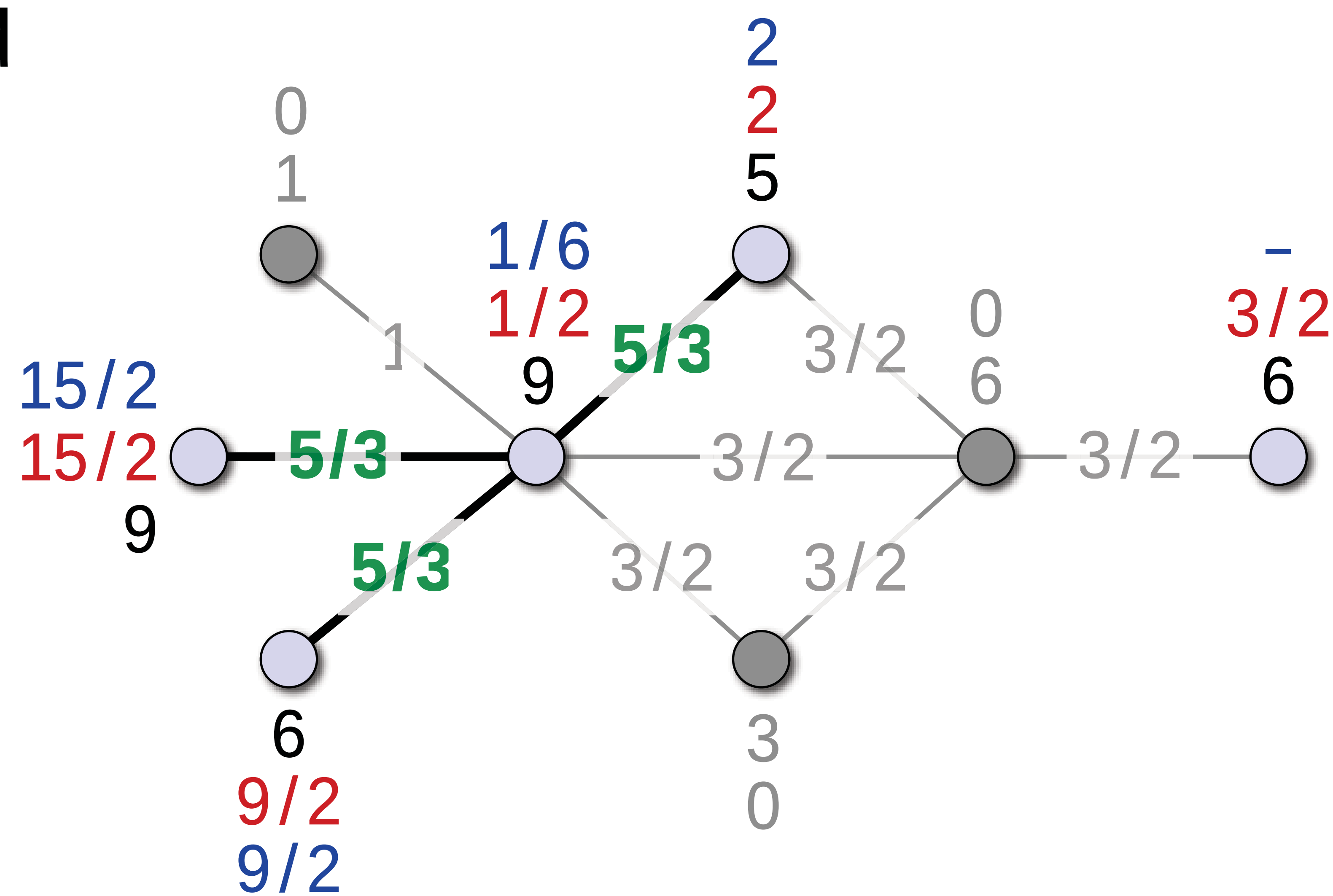


Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat…

Offers…

Increase weights…



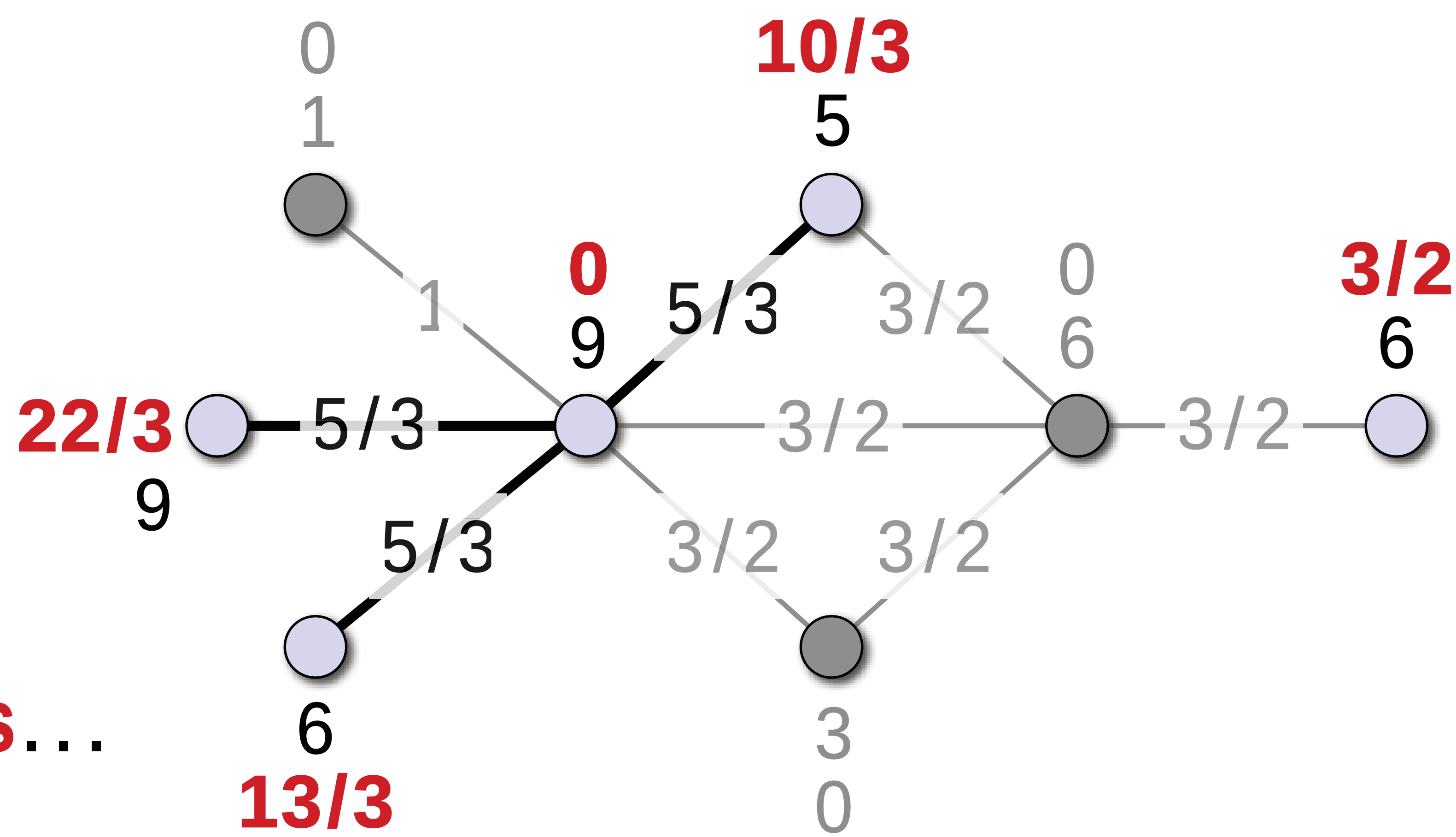
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat…

Offers…

Increase weights…

Update residuals…



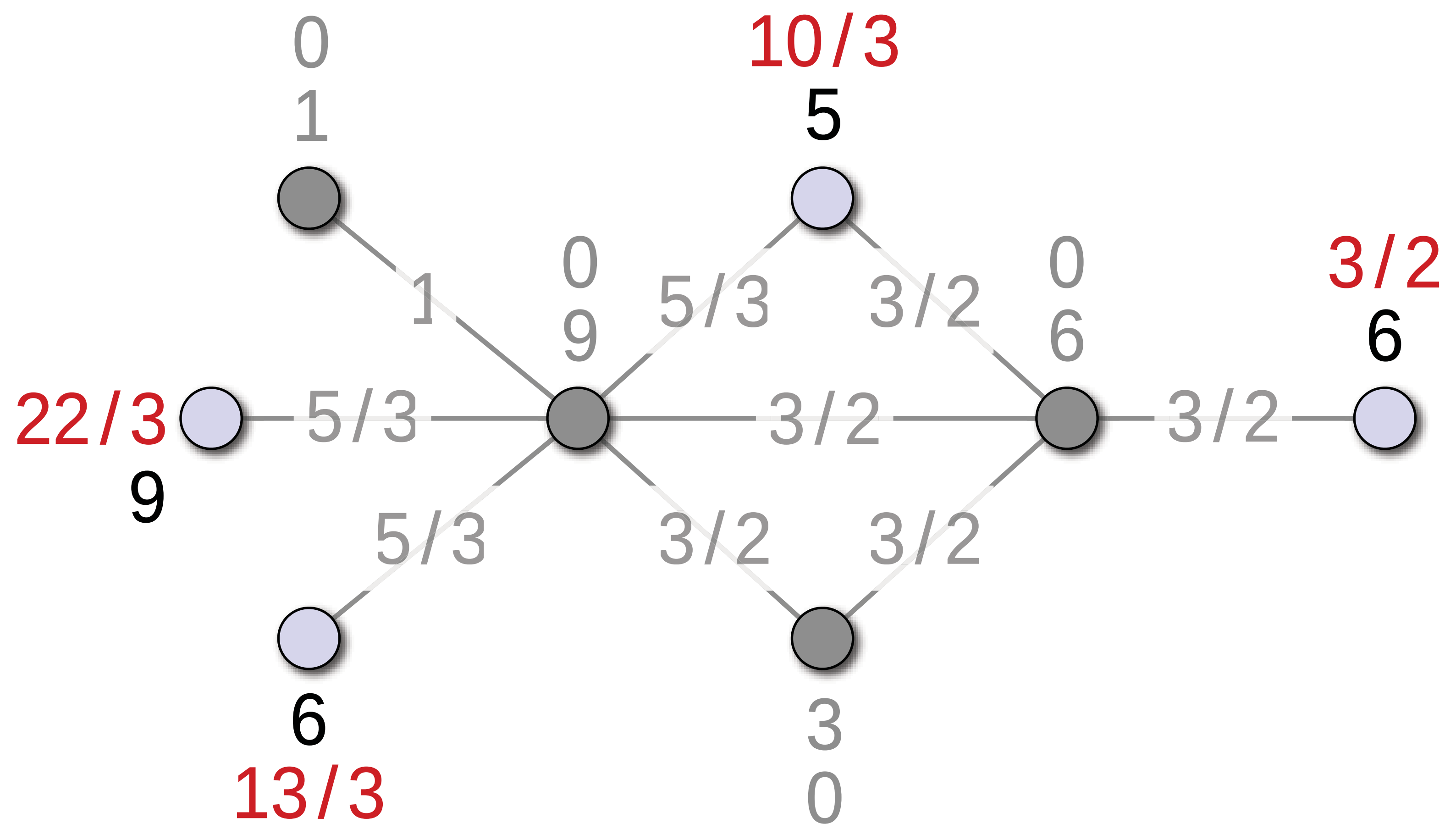
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat…

Offers…

Increase weights…

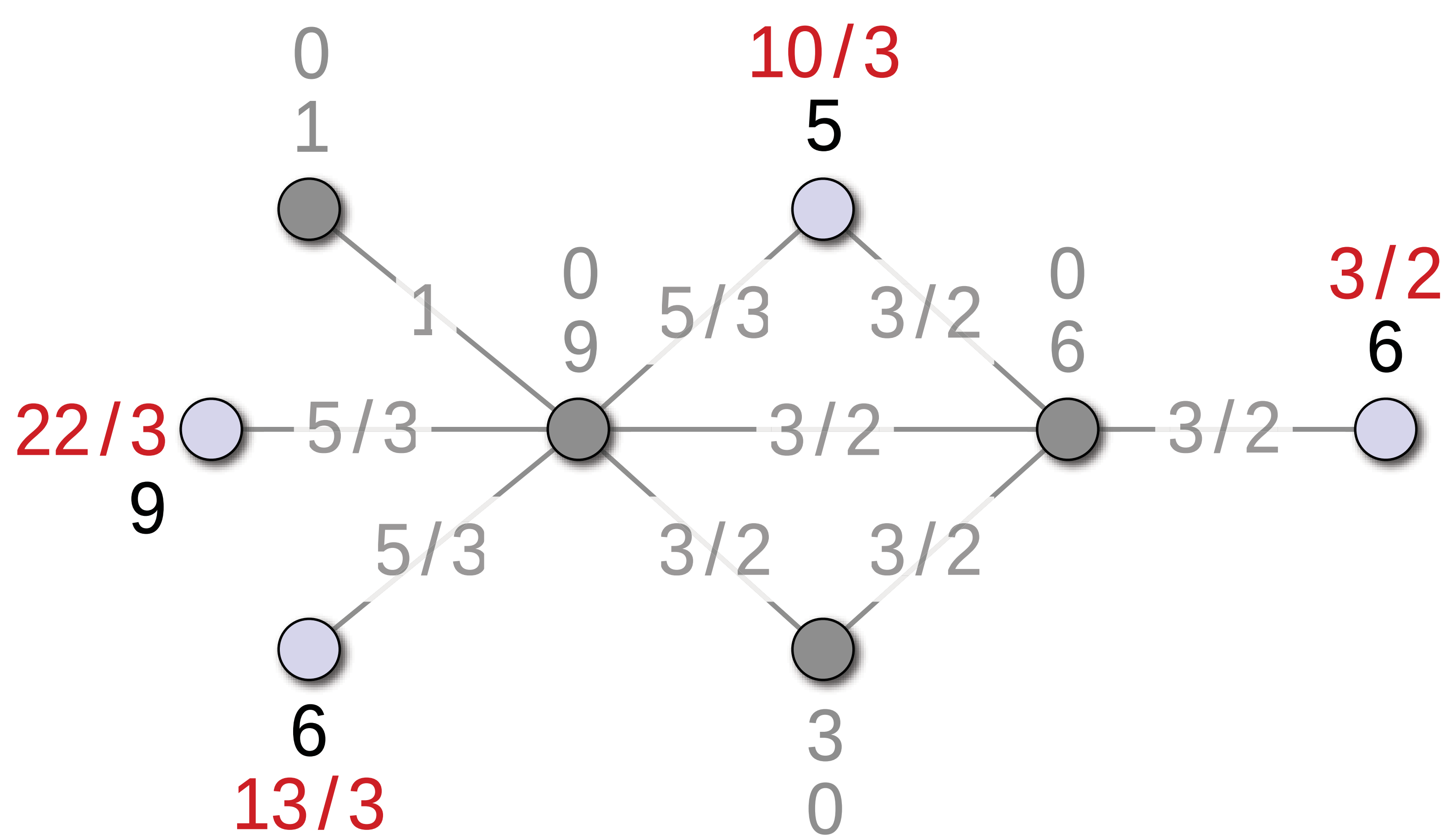
Update residuals and graph, etc.



Finding a maximal edge packing: phase I

This is a simple deterministic distributed algorithm

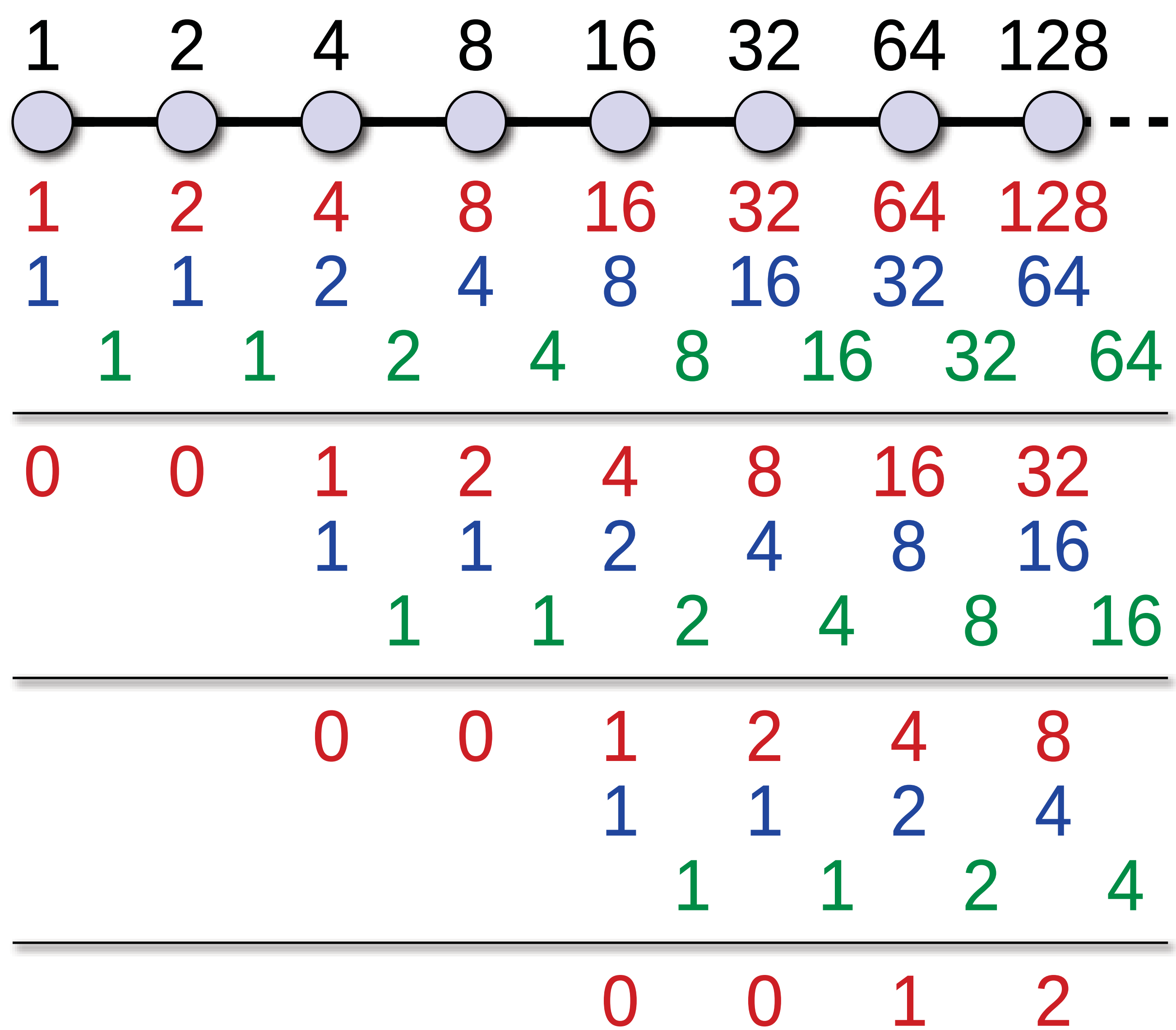
We are making some progress towards finding a maximal edge packing — but…



Finding a maximal edge packing: phase I

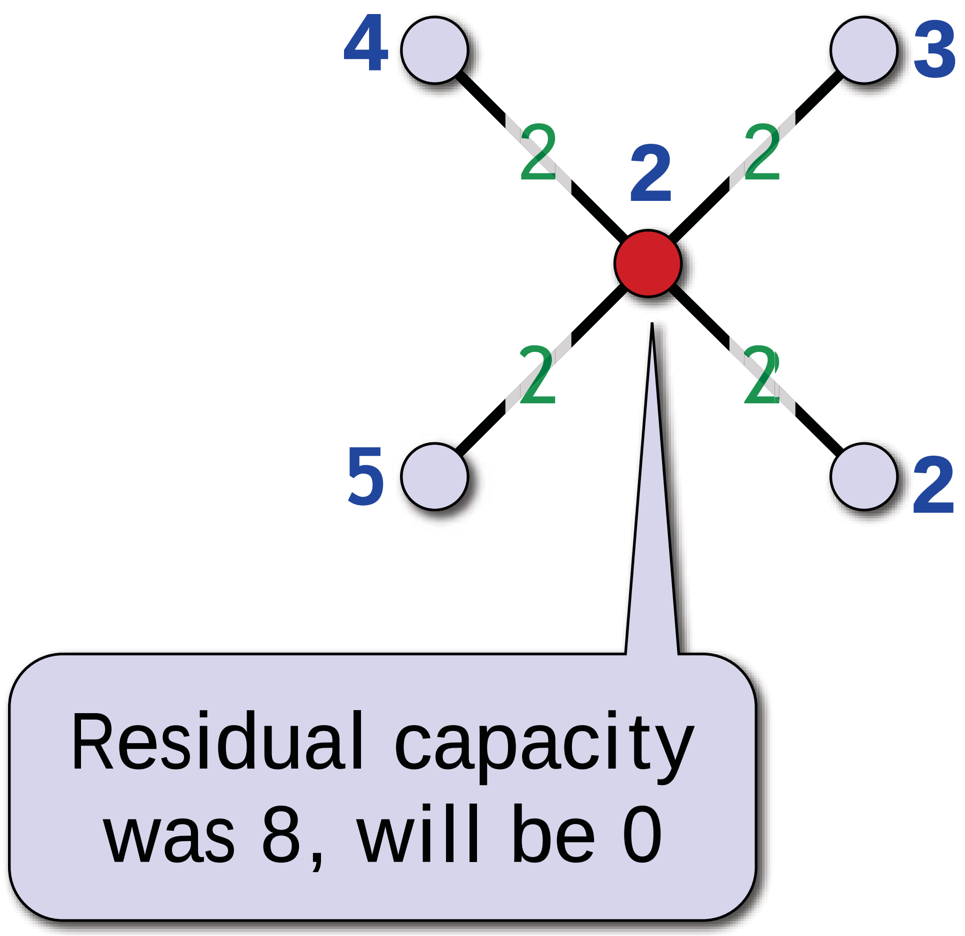
This is a simple deterministic distributed algorithm

We are making some progress towards finding a maximal edge packing — but this is **too slow**!



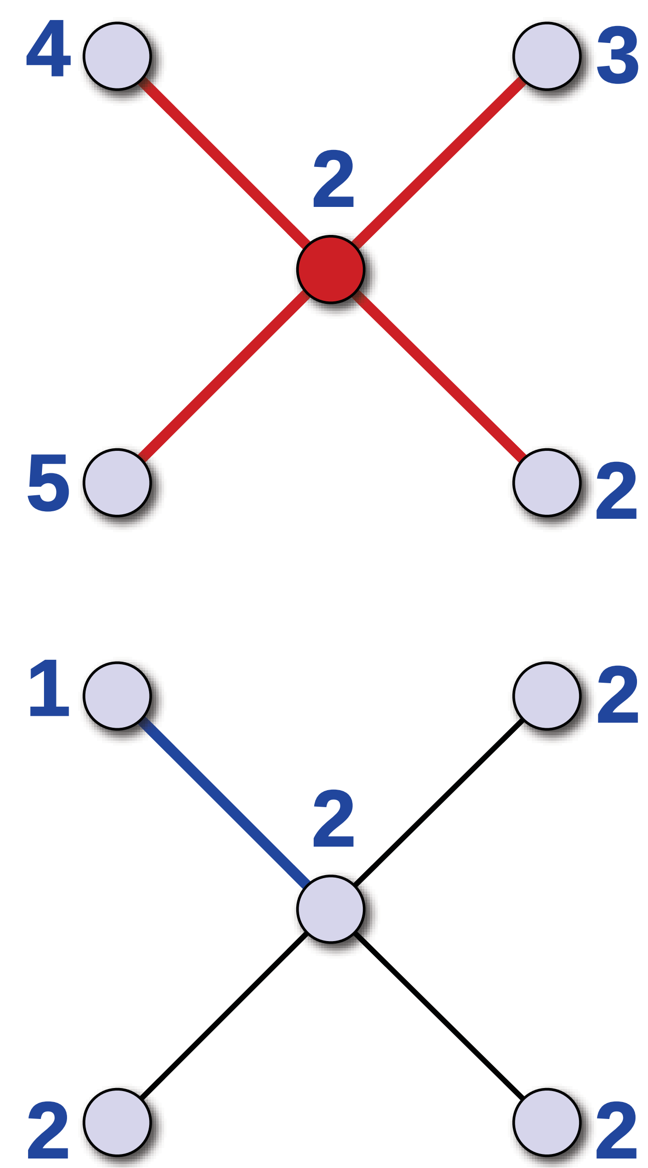
Finding a maximal edge packing: colouring trick

- Offer is a local minimum:
  - Node will be **saturated**
  - And all edges incident to it will be saturated as well



Finding a maximal edge packing: colouring trick

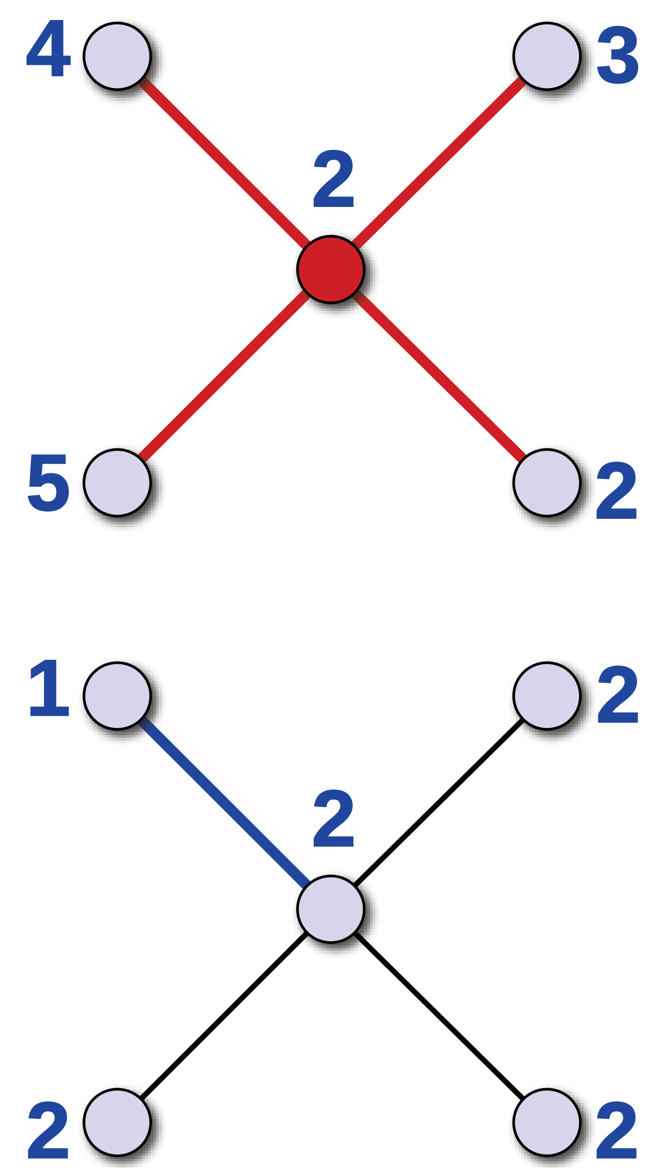
- Offer is a local minimum:
  - Node will be **saturated**
- Otherwise there is a neighbour with a different offer:
  - Interpret the offer sequences as “colours”
  - Nodes *u* and *v* have different colours: {*u*, *v*} is **multicoloured**



Finding a maximal edge packing: colouring trick

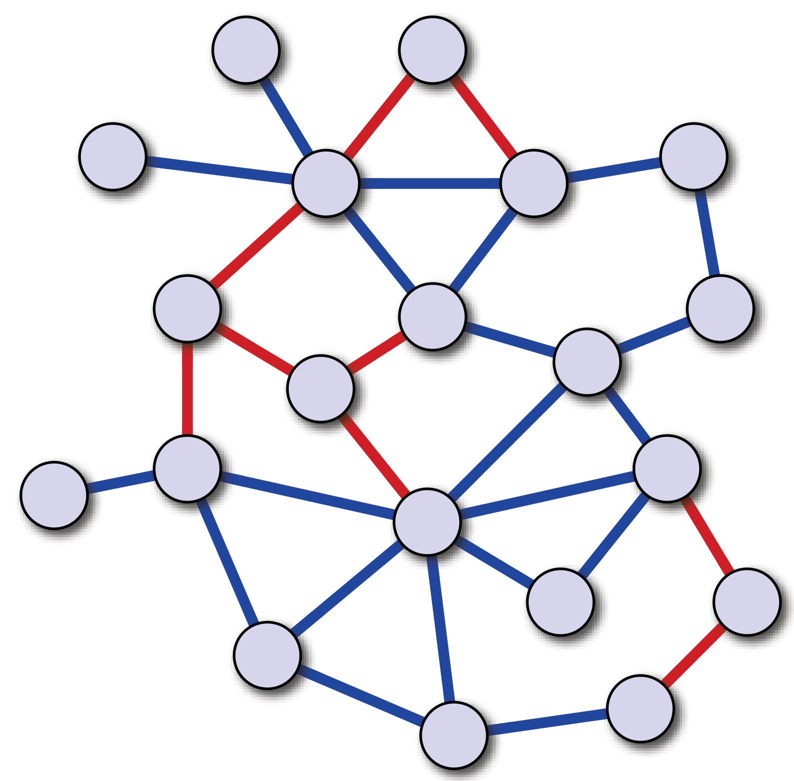
- Some progress guaranteed:
  - On each iteration, for each node, at least one incident edge becomes **saturated** or **multicoloured**
  - Such edges are be discarded in phase I: node degrees decrease by at least one on each iteration
  - Hence in Δ iterations all edges are saturated or multicoloured

Δ = maximum degree



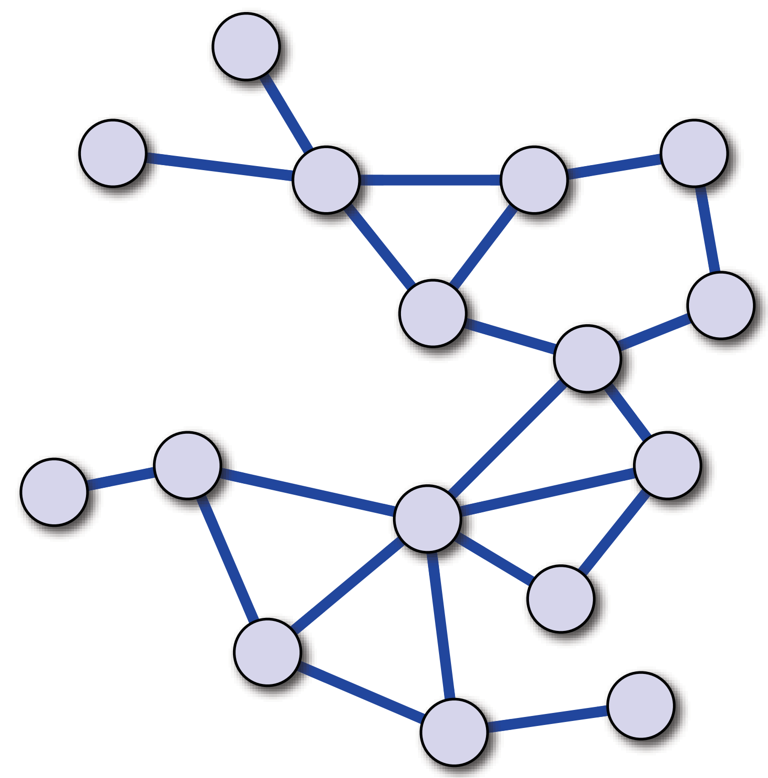
Finding a maximal edge packing: colouring trick

- Phase I: in Δ rounds all edges are **saturated** or **multicoloured**
  - Saturated edges are good — we’re trying to construct a maximal edge packing
  - Why are the multicoloured edges useful?



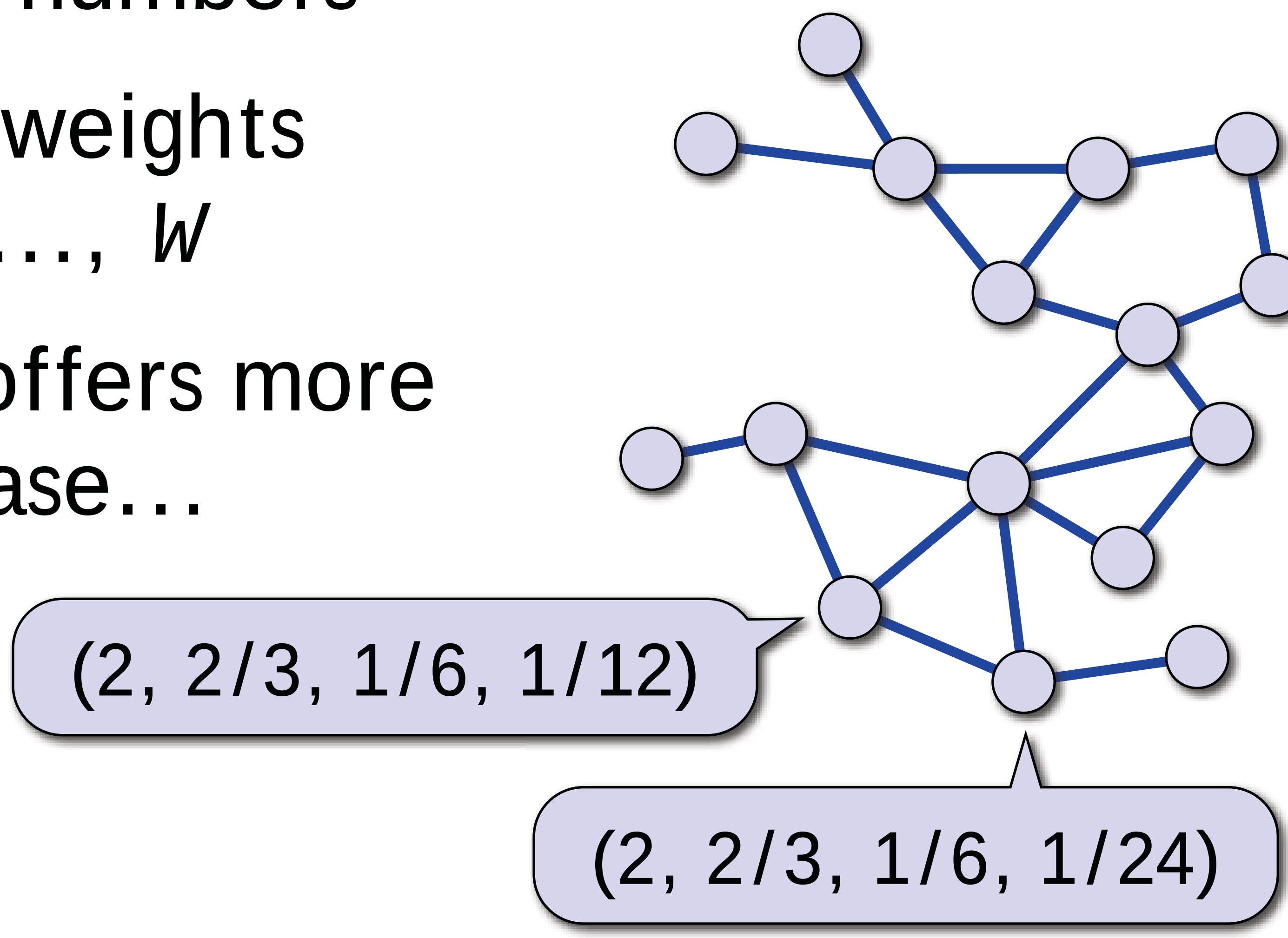
Finding a maximal edge packing: colouring trick

- Phase I: in Δ rounds all edges are **saturated** or **multicoloured**
  - Saturated edges are good — we're trying to construct a maximal edge packing
  - Why are the multicoloured edges useful?
  - Let's focus on unsaturated nodes and edges



Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ offers, which are rational numbers
- Assume that node weights are integers 1, 2, …, W
- Let’s analyse the offers more carefully in that case…



Finding a maximal edge packing: colouring trick

- Offers are rationals of the form $q/(\Delta!)^{\Delta}$
  - Proof idea: multiply weights by $(\Delta!)^{\Delta}$
  - Then $r(v)$ is a multiple of $(\Delta!)^{\Delta}$ before iteration 1
  - Offer $r(v)/\deg(v)$ is a multiple of $(\Delta!)^{\Delta-1}$ on iteration 1
  - $r(v)$ is a multiple of $(\Delta!)^{\Delta-1}$ after iteration 1

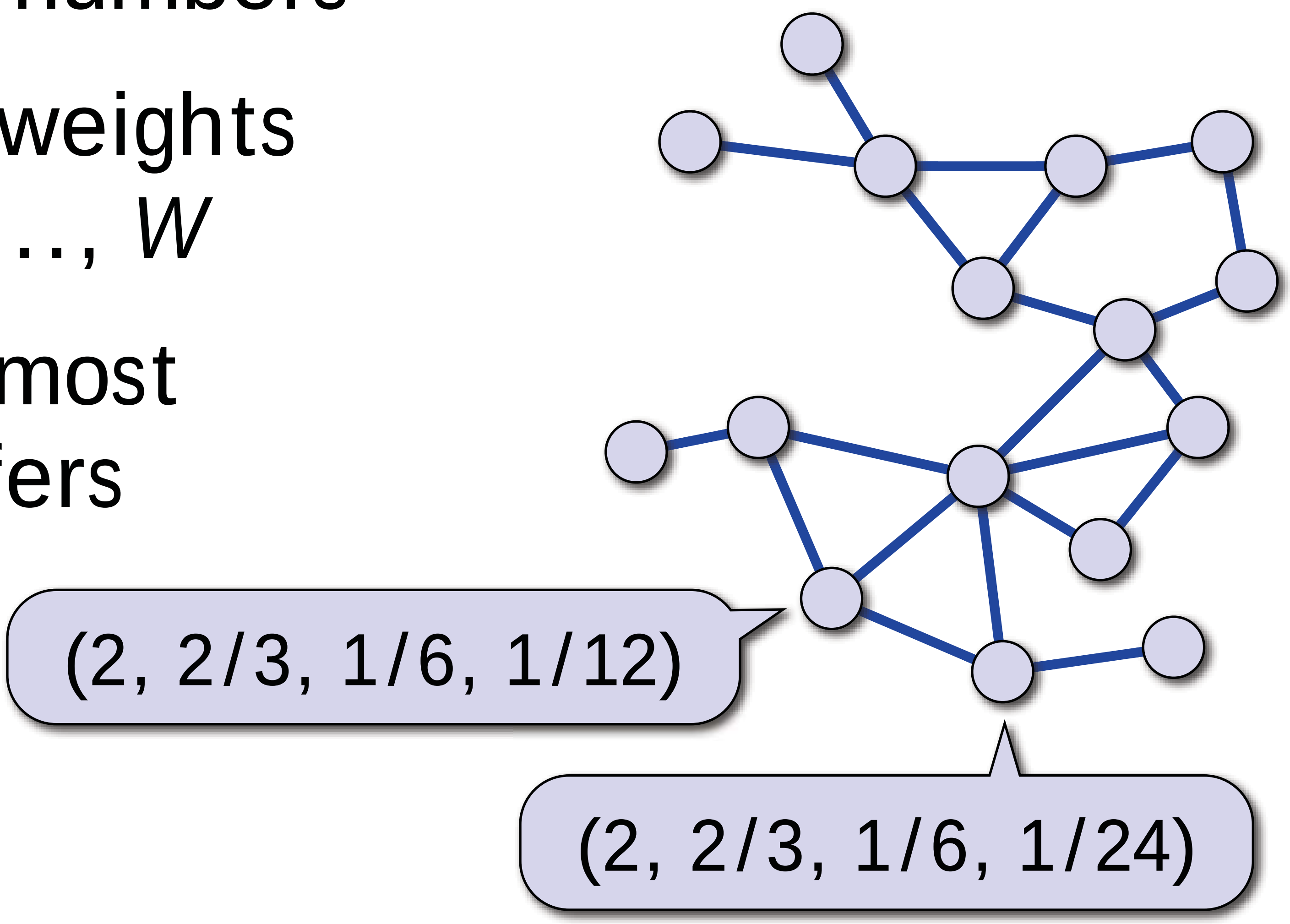
  ... (more formally: proof by induction)
  - $r(v)$ is a multiple of $\Delta!$ before iteration Δ
  - Offers are integers on iteration Δ

Finding a maximal edge packing: colouring trick

- Offers are rationals of the form $q/(\Delta!)^{\Delta}$
  - Proof idea: if we multiplied weights by $(\Delta!)^{\Delta}$, then the offers would integers throughout the algorithm
  - Without scaling, we get in the worst case $q/(\Delta!)^{\Delta}$
- If node weights are integers 1, 2, …, W, then offers are rationals between 0 and W
  - Offer of v is at most $r(v) \leq w(v) \leq W$
- There are at most $W(\Delta!)^{\Delta}$ possible offers!

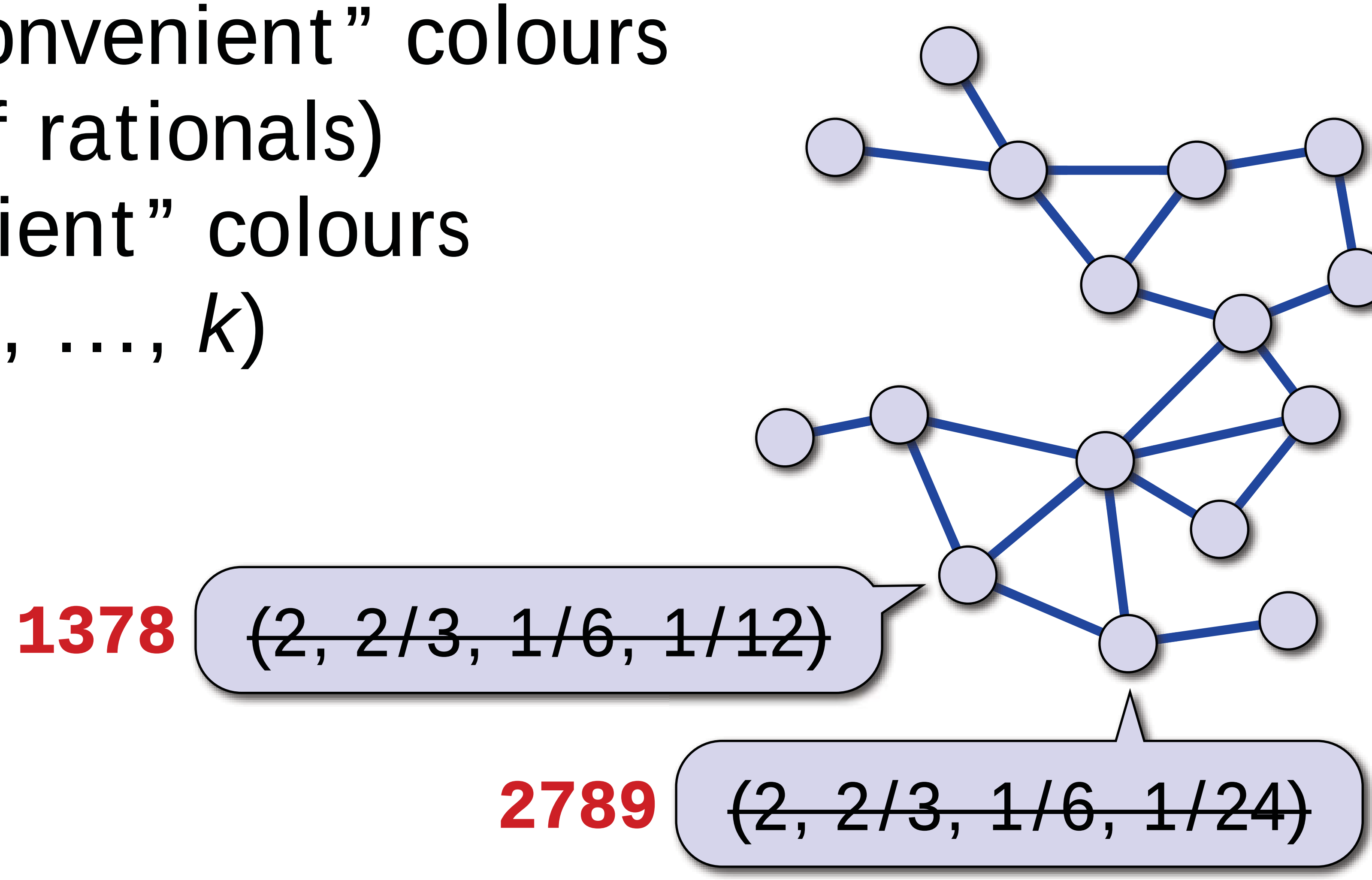
Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ offers, which are rational numbers
- Assume that node weights are integers 1, 2, …, W
- Then there are at most $W(\Delta!)^{\Delta}$ possible offers
- And hence only $k = (W(\Delta!)^{\Delta})^{\Delta}$ possible colours



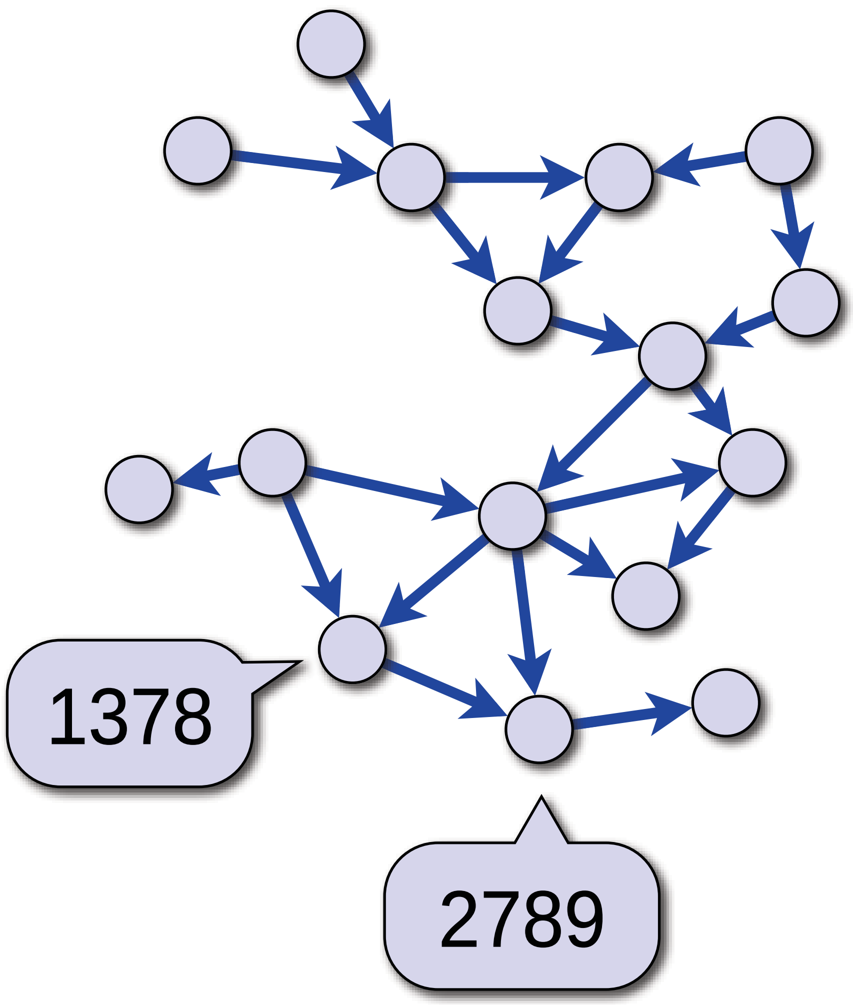
Finding a maximal edge packing: colouring trick

- Only $k = (W(\Delta!)^{\Delta})^{\Delta}$ possible colours
- Replace "inconvenient" colours (sequences of rationals) with "convenient" colours (integers 1, 2, ..., k)



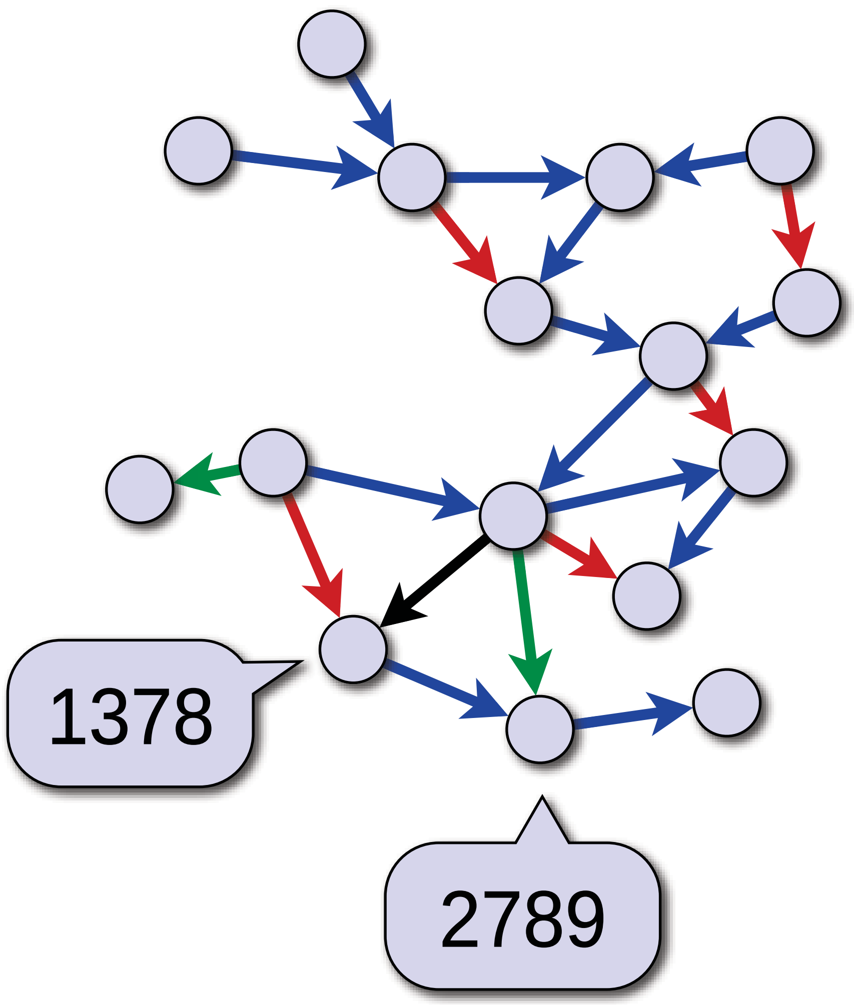
Finding a maximal edge packing: phase II

- We have a proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour (acyclic directed graph)



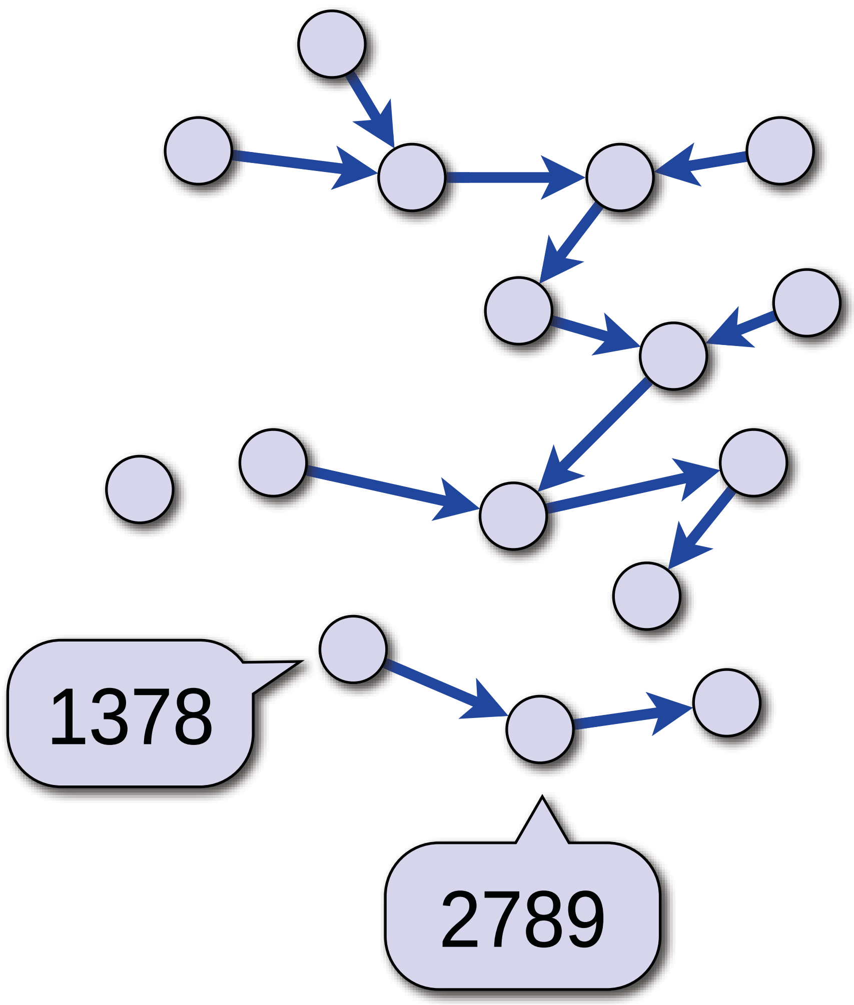
Finding a maximal edge packing: phase II

- We have a proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour (acyclic directed graph)
- Partition in Δ forests
    - Each node assigns its outgoing edges to different forests



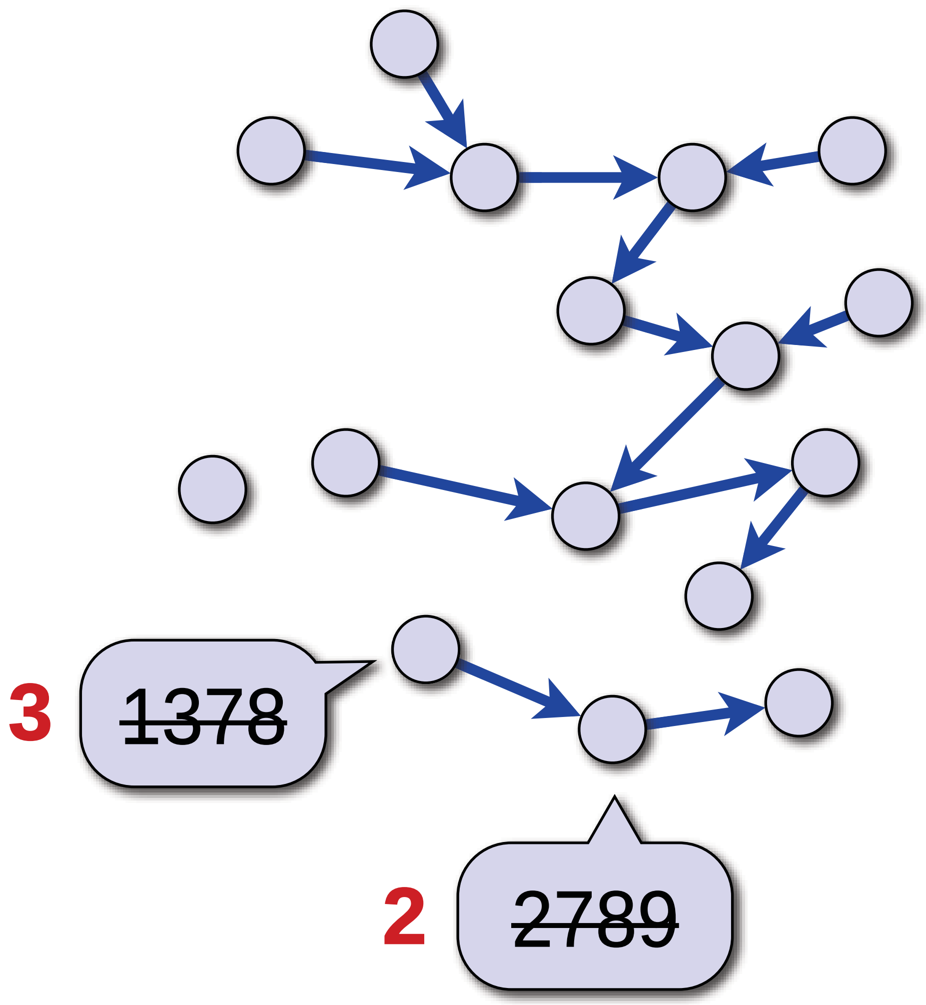
Finding a maximal edge packing: phase II

- For each forest in parallel…



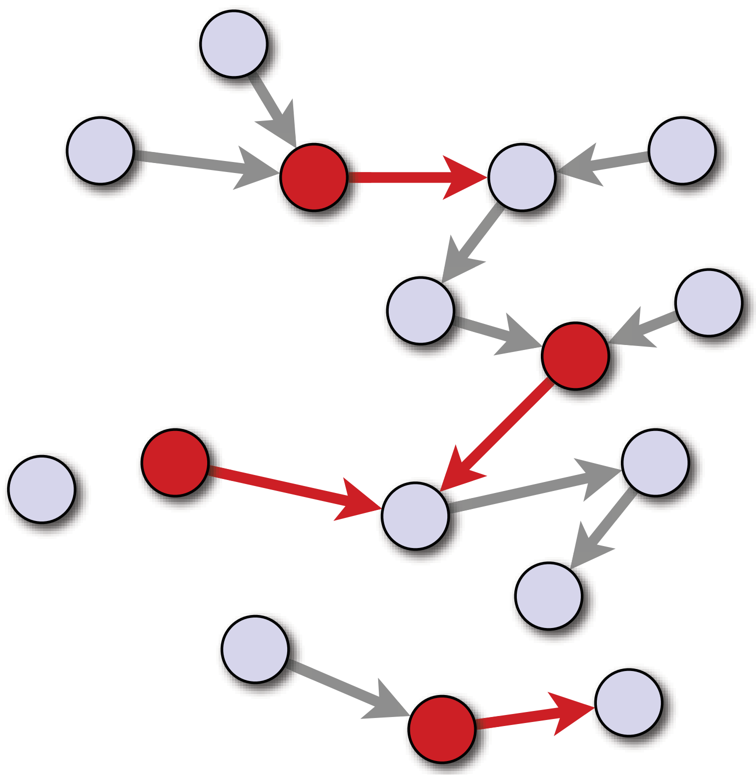
Finding a maximal edge packing: phase II

- For each forest in parallel:
  - Use Cole–Vishkin style colour reduction algorithm
  - Given a *k*-colouring, finds a **3-colouring** in time O(log* *k*)



Finding a maximal edge packing: phase II

- For each forest and each colour j = 1, 2, 3 in sequence:
  - Consider all outgoing edges of colour-j nodes



Finding a maximal edge packing: phase II

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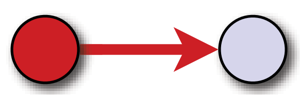
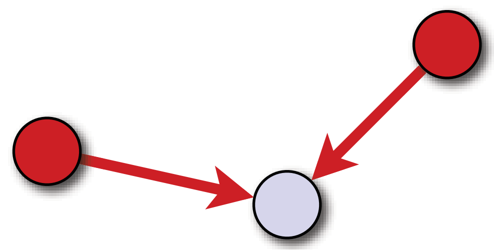
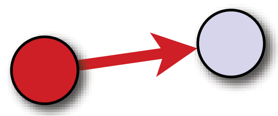


  - Node-disjoint stars: easy to saturate all such edges in parallel



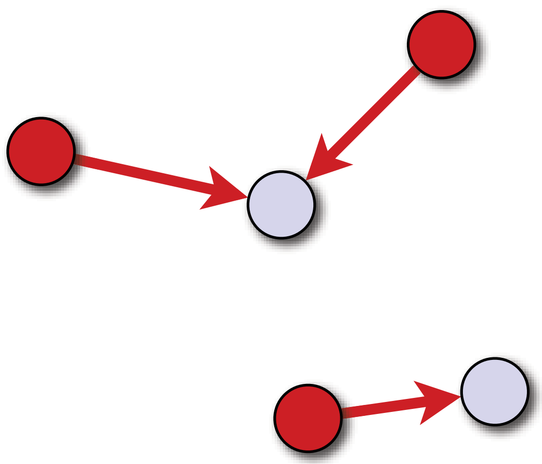
  - Two simple cases:
    - saturate centre



    - saturate all leaves

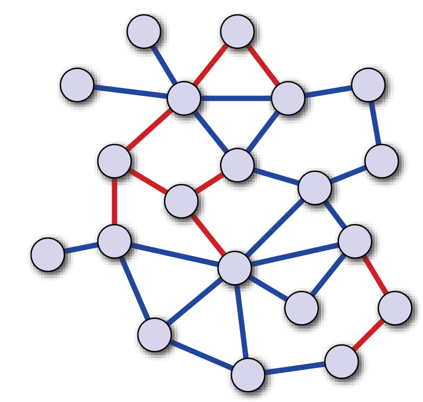
Finding a maximal edge packing: phase II

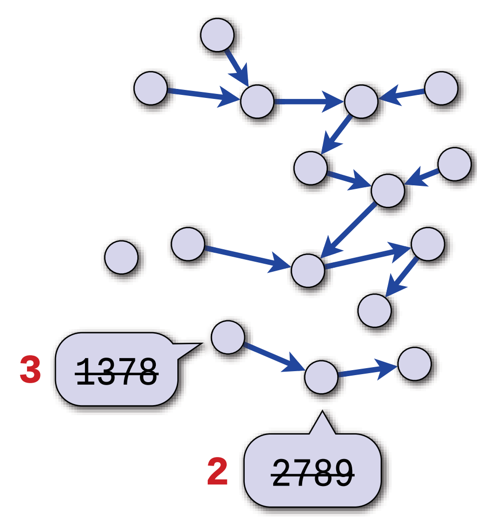
- This way we can saturate all multicoloured edges:
  - Each edge belongs to one forest, and its tail has colour 1, 2, or 3
  - We simply go through all forests and all colours and therefore saturate everything

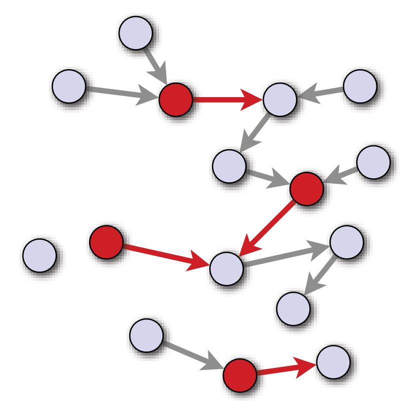


Finding a maximal edge packing: algorithm overview

- Phase I:
    - All edges become saturated or multicoloured
- Phase II:
    - Multicoloured edges are partitioned in Δ forests
    - Forests are 3-coloured
    - 3-coloured forests are saturated

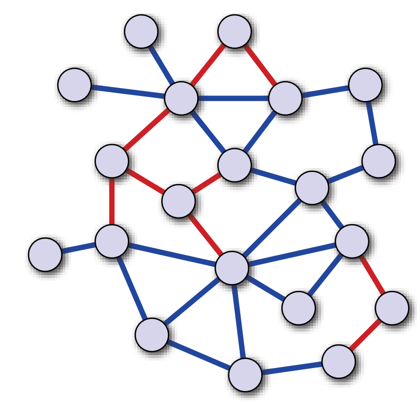


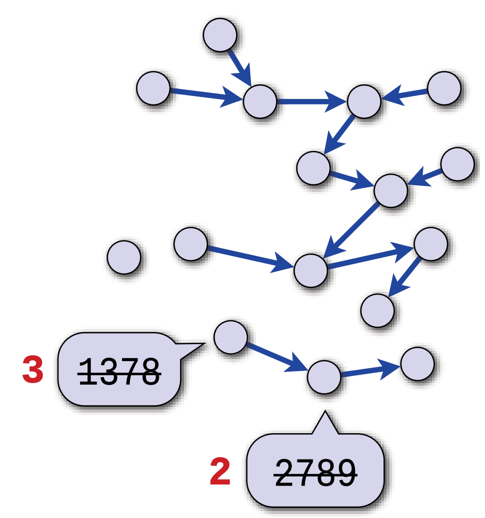


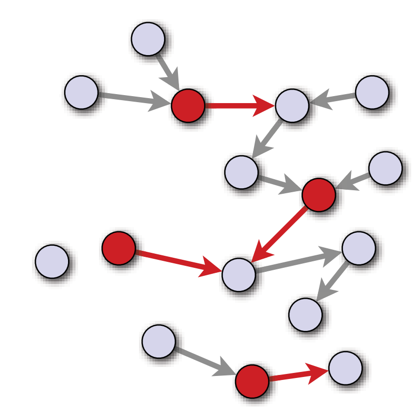


Finding a maximal edge packing: running time analysis

- Total running time:
    - All edges become saturated or multicoloured: $O(\Delta)$
    - Multicoloured forests are 3-coloured: $O(\log^* k)$
    - 3-coloured forests are saturated: $O(\Delta)$
- $O(\Delta + \log^* k) = O(\Delta + \log^* W)$
    - k is huge, but log* grows slowly

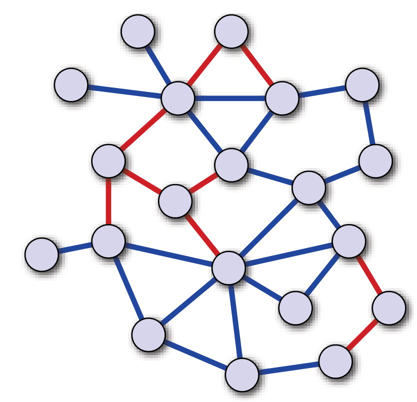


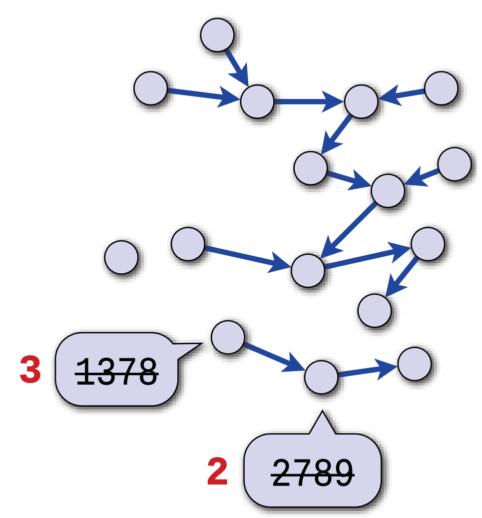


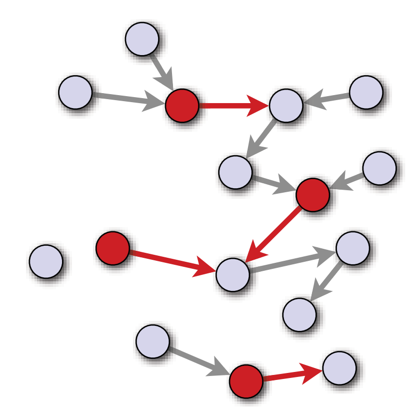


Finding a maximal edge packing: summary

- Maximal edge packing and 2-approximation of vertex cover in time $O(\Delta + \log^* W)$
  - W = maximum node weight
- Unweighted graphs: running time simply $O(\Delta)$, independent of n
- Everything can be implemented in the **port-numbering model**



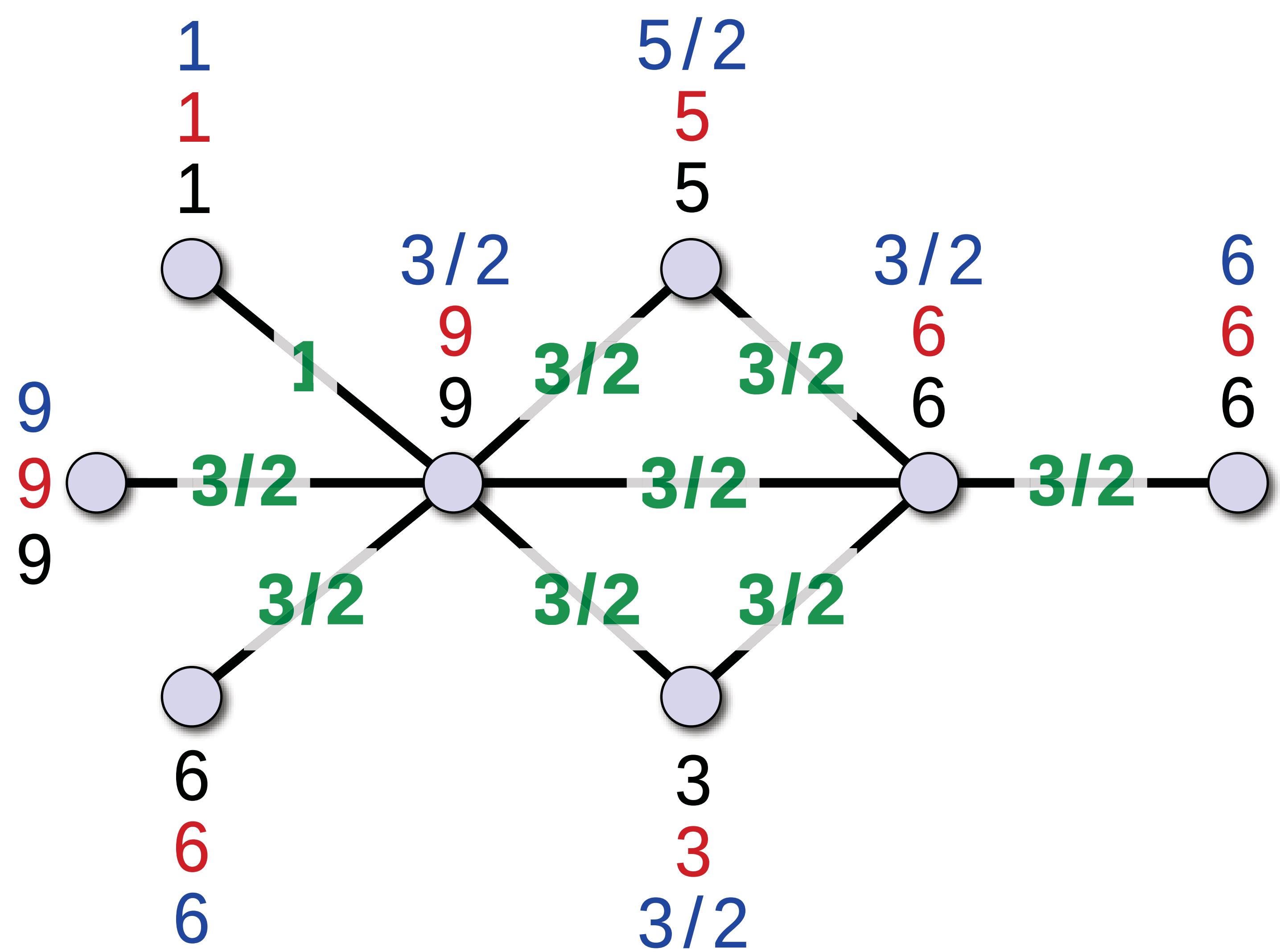




Finding a maximal edge packing: recap

Phase I:

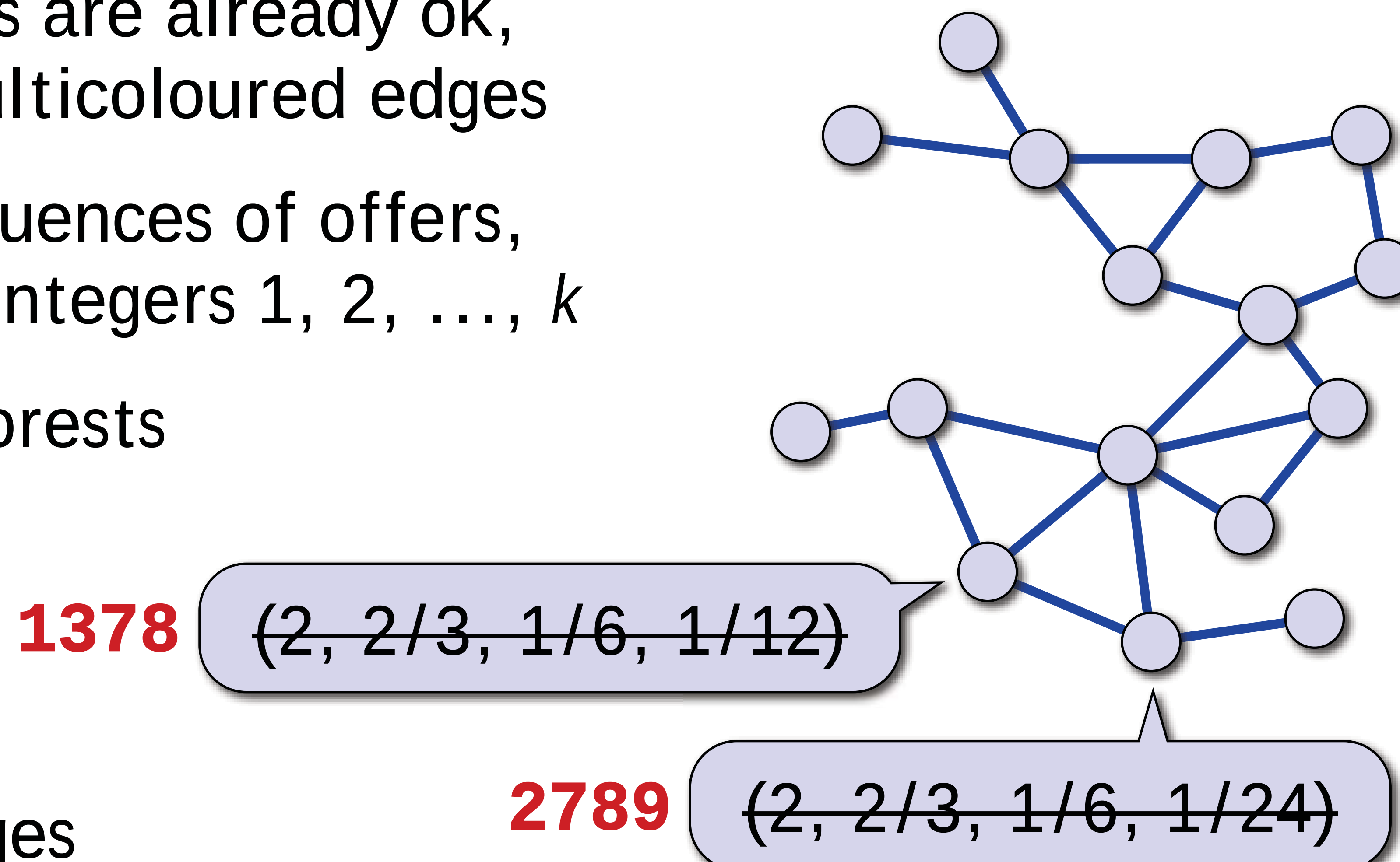
- **Residuals**
  $r(v) = w(v) - y[v]$
- **Offer** $r(v)\,/\,\deg(v)$
- **Accept minimum**, increase weights
- Progress: edges become *saturated* or *multicoloured* (different offers)



Finding a maximal edge packing: recap

Phase II:

- Saturated edges are already ok, we focus on multicoloured edges
- Colours are sequences of offers, re-colour with integers 1, 2, . . . , k
- Partition in Δ forests
- Cole–Vishkin: 3-colouring
- Use colours to saturate all edges



Finding a maximal edge packing: some intuition

- Regular graph with uniform weights:
  - Symmetry-breaking (e.g., graph colouring) is not possible in the port-numbering model
  - But it is trivial to find a maximal edge packing directly
- “Irregular” graph:
  - We have symmetry-breaking information, which can be used to find a graph colouring, which can be used to find a maximal edge packing
- Handling these two cases turns out to be enough!

Take-home messages

- Non-trivial problems can be solved in very restrictive models of distributed computing
- Generalise!
  - More difficult problems may be easier to solve: vertex cover → weighted vertex cover → weighted set cover…
- Cole–Vishkin technique is a powerful tool
  - Wide range of applications far beyond the textbook examples of colouring cycles with numerical IDs
  - log* of almost everything is something reasonable