DDA 2010, lecture 4: Applications of Ramsey's theorem

- Using Ramsey's theorem, we can show that these problems can't be solved in $O(1)$ rounds:
  - finding large independent sets in cycles
  - graph colourings and maximal matchings in cycles
  - better than 2-approximation of vertex cover
  - and many more…

DDA 2010, lecture 4a: Introduction and background

- Hardness of graph colouring and other symmetry-breaking problems

Graph colouring

- Graph colouring is a central symmetry-breaking primitive in distributed algorithms
  - Colouring can be used to **schedule** the actions of the nodes: e.g., neighbours don't transmit simultaneously
  - Given a graph colouring, we can **solve other problems**: maximal independent set, maximal matching, etc.
  - We can use colours to **simulate greedy algorithms**: finding small dominating sets, etc.

Graph colouring

- Graph colouring is a central symmetry-breaking primitive in distributed algorithms
- Many problems are as difficult as graph colouring
  - Given an algorithm that finds a maximal independent set, we can use it to find a graph colouring, and vice versa
- To understand the capabilities of distributed algorithms, it is important to know how fast we can find a graph colouring

Hardness of graph colouring

- Cole–Vishkin algorithm can be used to colour cycles in *almost* constant running time: $O(\log^* n)$
  - assuming we have unique identifiers
- Could we get *exactly* constant running time?
  - it seems very difficult to come up with an O(1)-time algorithm for graph colouring…
  - but how could one possibly prove that no such algorithm exists?
  - there are infinitely many algorithms!

Hardness of graph colouring

- Cole–Vishkin algorithm can be used to colour cycles in *almost* constant running time: $O(\log^* n)$
  - assuming we have unique identifiers
- Could we get *exactly* constant running time?
- This was resolved by Nathan Linial in 1992:
  - 3-colouring an n-cycle requires $\Omega(\log^* n)$ rounds
  - Cole–Vishkin technique is within constant factor of the best possible algorithm!

Hardness of other problems

- Linial's result shows that it is not possible to solve these problems in cycles in $O(1)$ time:
  - vertex colouring, edge colouring, maximal independent set, maximal matching, …
- Naor and Stockmeyer (1995): generalisations
  - using Ramsey's theorem
- What about other problems?

Hardness of other problems

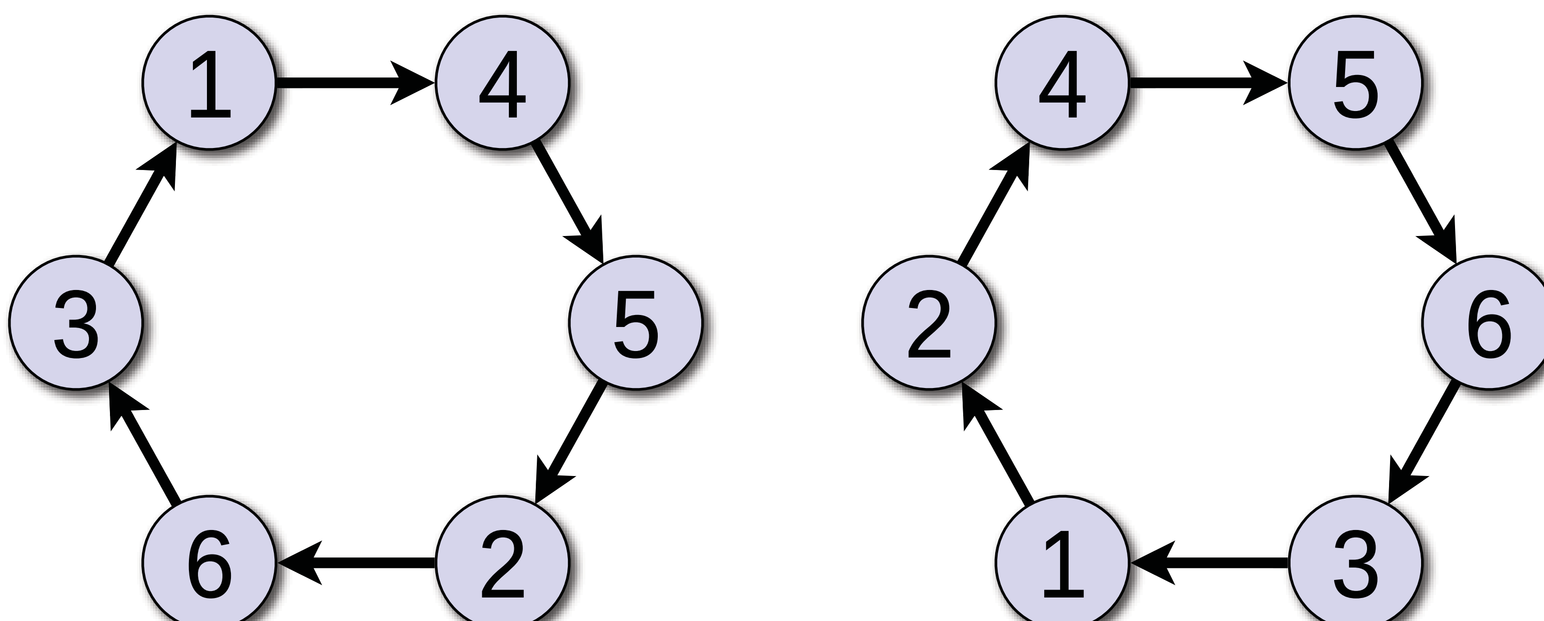
- Linial: we can't find **maximal** independent sets in constant time
- However, could we perhaps find a "**fairly large**" independent set in constant time?
  - e.g., an independent set with at least $n/10$ nodes?
- We will see that this is not possible, either
  - strong negative result
  - proof uses Ramsey's theorem

DDA 2010, lecture 4b: Finding a non-trivial independent set

- Czygrinow et al. (2008)
  - constant-time algorithms can’t find large independent sets in cycles

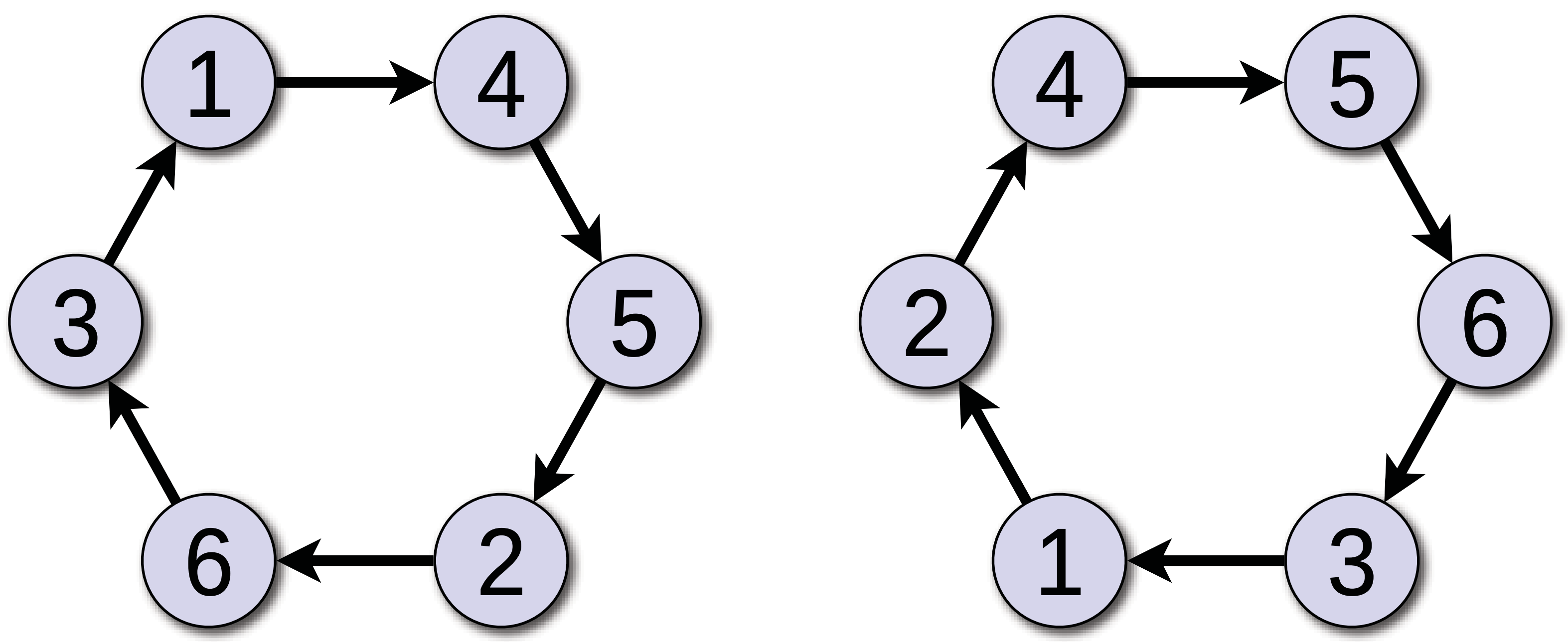
Lower-bound result for finding large independent sets

- Numbered directed *n*-cycle:
  - directed *n*-cycle, each node has outdegree = indegree = 1
  - node identifiers are a permutation of {1, 2, …, *n*}



Lower-bound result for finding large independent sets

- We will show that the problem is difficult even if we have a numbered directed cycle
  - general case of cycles with unique IDs at least as hard



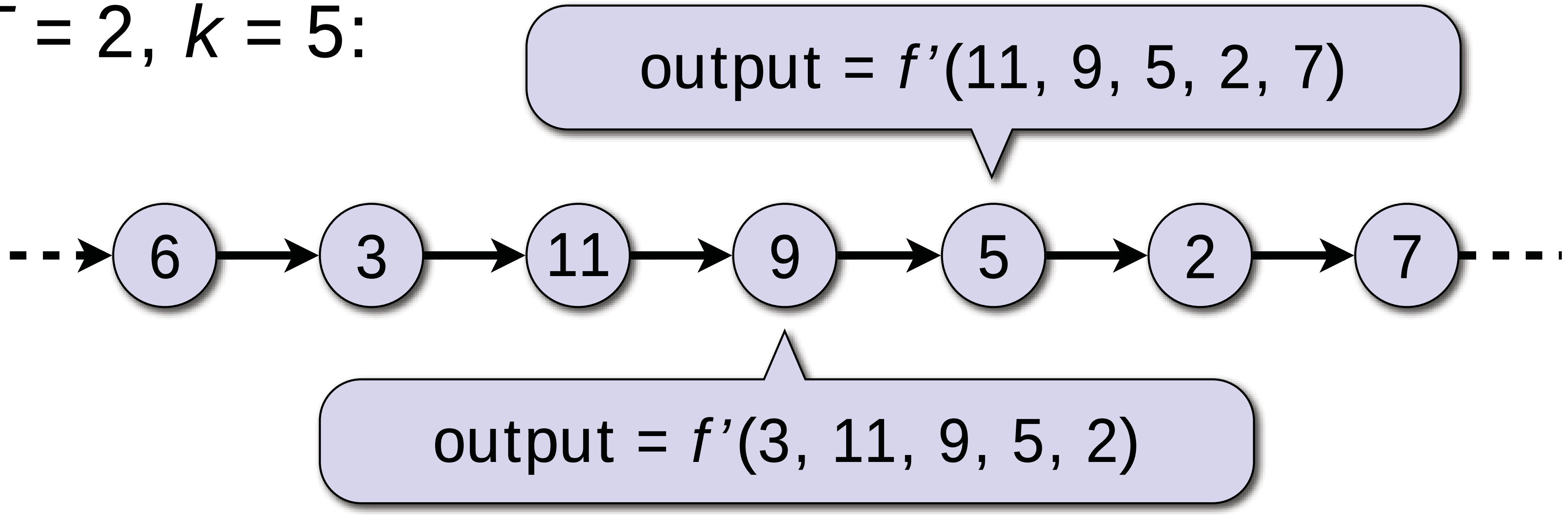
Lower-bound result for finding large independent sets

- Fix any $\varepsilon > 0$ and running time T (constants)
- Algorithm A finds a feasible independent set in any numbered directed cycle in time T
- **Theorem:** For a sufficiently large n there is a numbered directed n-cycle C in which A outputs an independent set with $\leq \varepsilon n$ nodes
  - can't find an independent set with > $0.001n$ nodes
  - not even if the running time is 1000000 rounds

Lower-bound result for finding large independent sets

- Let T be the running time of A, let $k = 2T + 1$
- The output of a node is a function f' of a sequence of k integers (unique IDs)

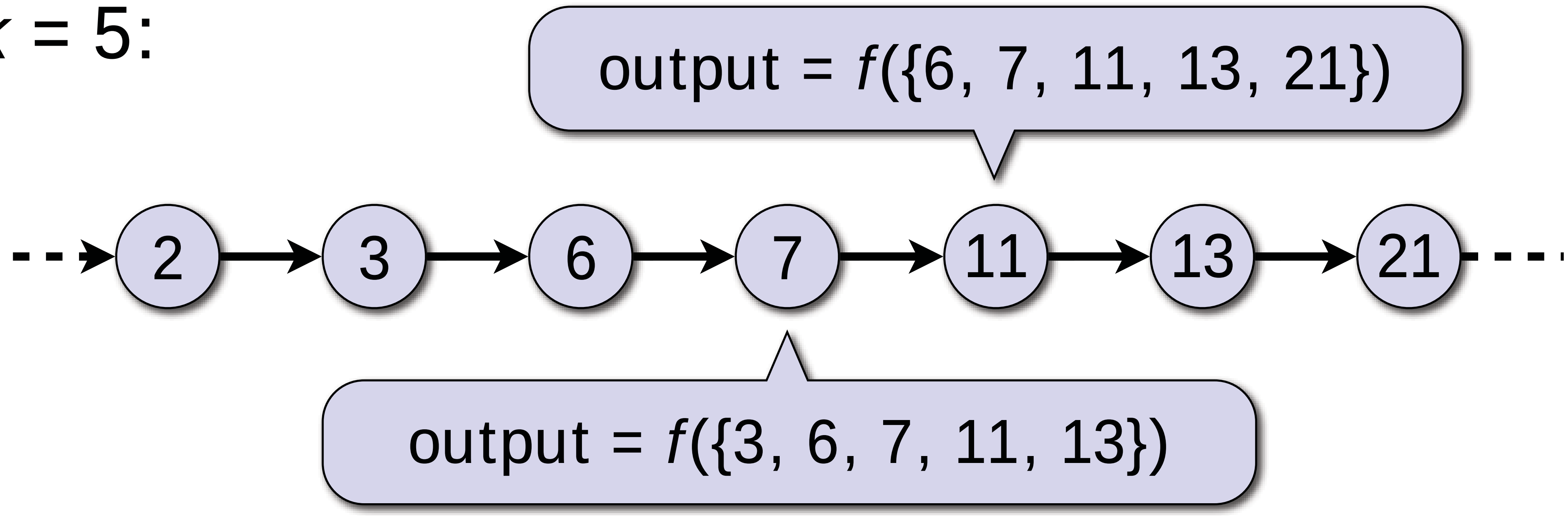
$T = 2$, $k = 5$:



Lower-bound result for finding large independent sets

- Lets focus on **increasing** sequences of IDs
- Then the output of a node is a function *f* of a **set** of *k* integers

$k = 5$:



Lower-bound result for finding large independent sets

- Hence we have assigned a colour $f(X) \in \{0, 1\}$ to each k-subset $X \subset \{1, 2, \ldots, n\}$

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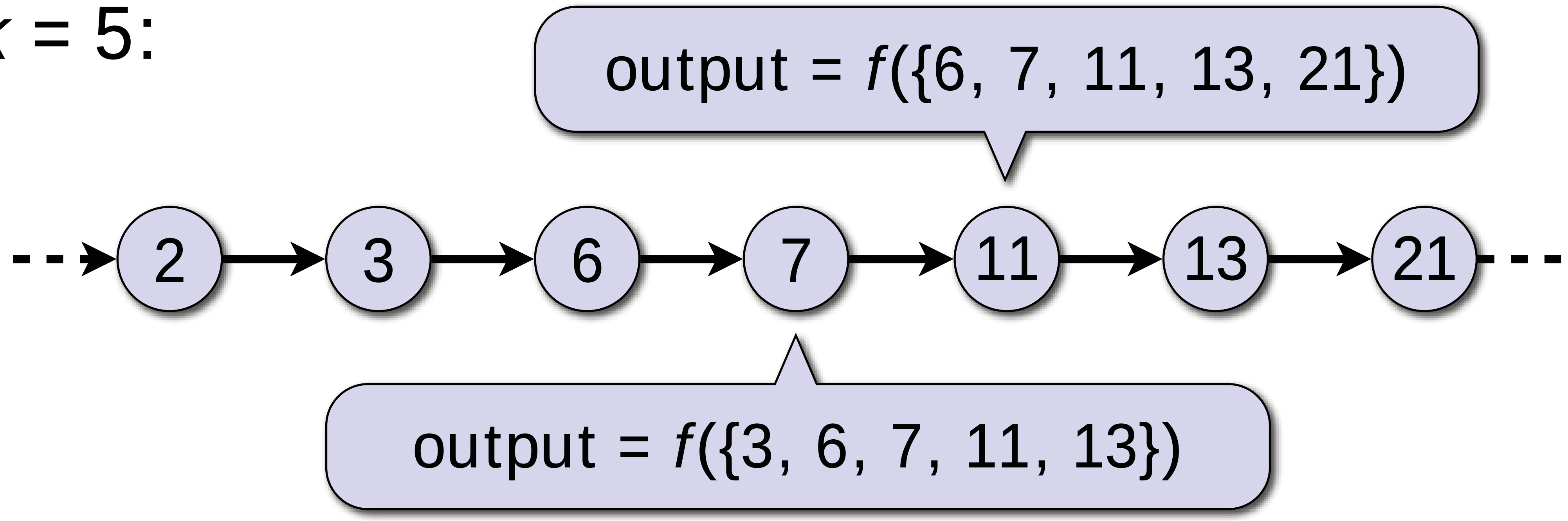
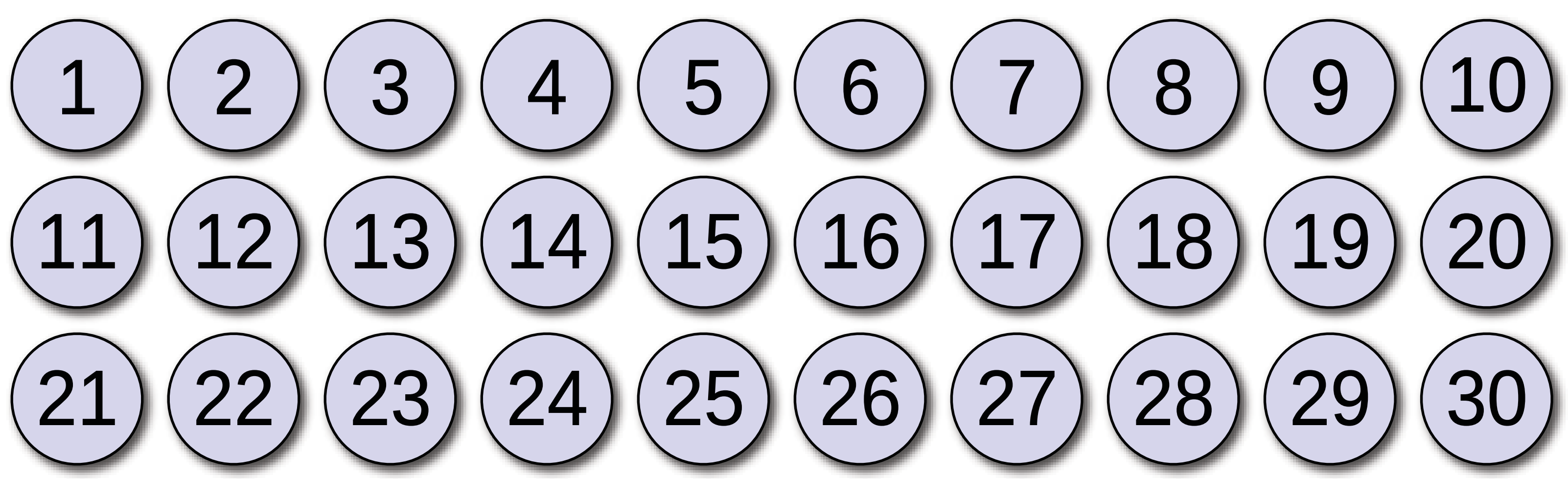


Lower-bound result for finding large independent sets

- Hence we have assigned a colour $f(X) \in \{0, 1\}$ to each k-subset $X \subset \{1, 2, \ldots, n\}$
- Fix a large m (depends on k and ε)
- Ramsey: If n is sufficiently large, we can find an m-subset $A \subset \{1, 2, \ldots, n\}$ s.t. all k-subset $X \subset A$ have the same colour

Lower-bound result for finding large independent sets

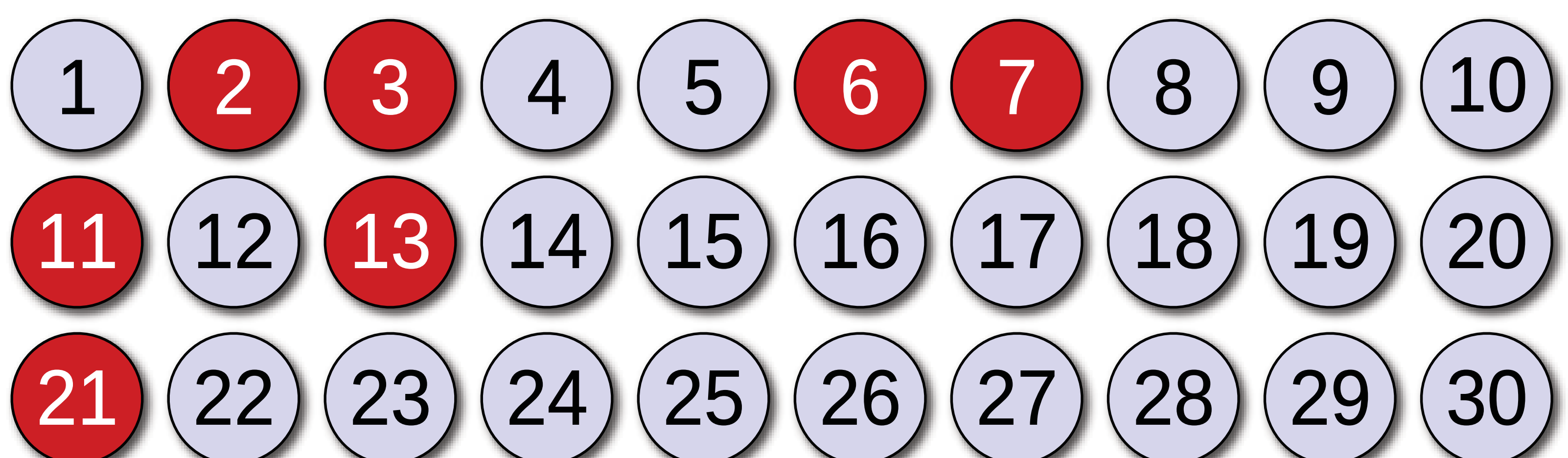
- That is, if the ID space is sufficiently large…



Lower-bound result for finding large independent sets

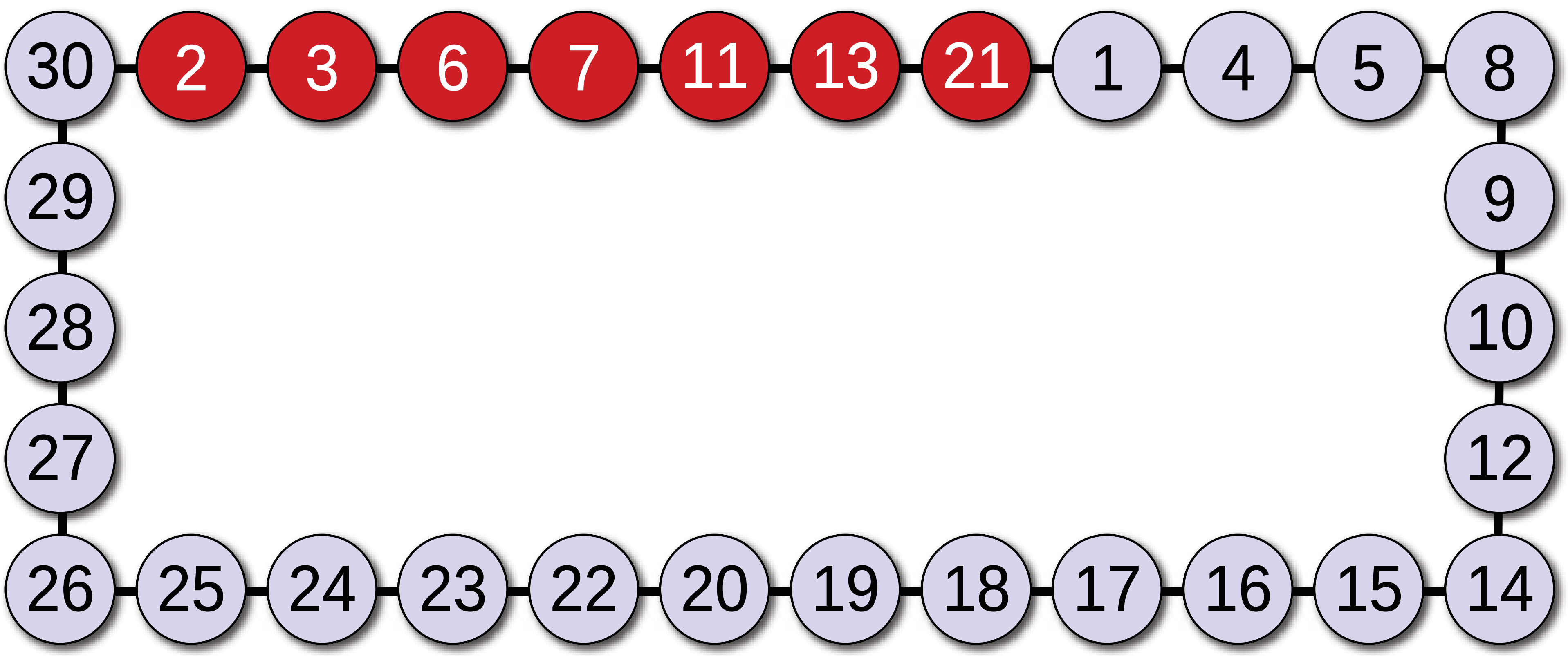
- That is, if the ID space is sufficiently large, we can find a **monochromatic** subset of *m* IDs…

$f(\{2, 3, 6, 7, 11\}) = f(\{2, 3, 6, 7, 13\}) =$
$f(\{2, 3, 6, 7, 21\}) = f(\{2, 3, 6, 11, 13\}) =$
$\ldots = f(\{6, 7, 11, 13, 21\})$



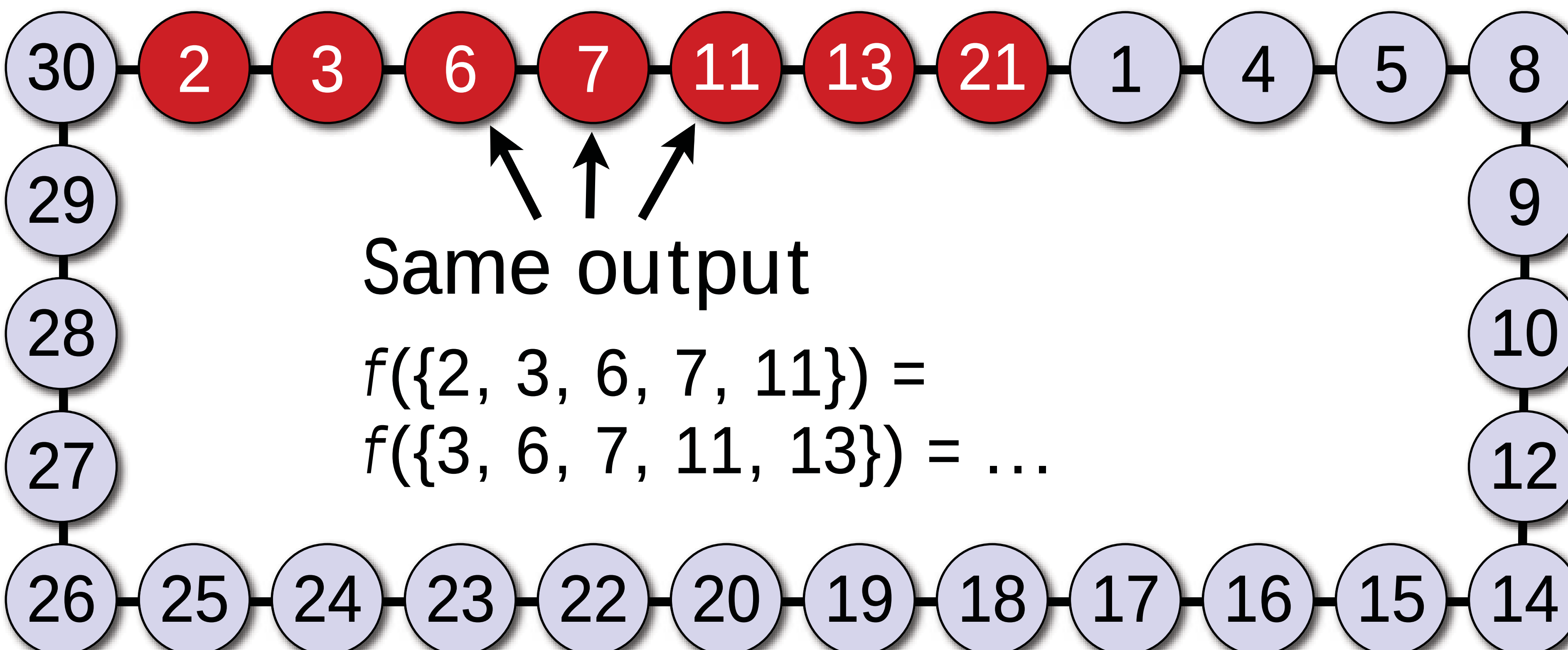
Lower-bound result for finding large independent sets

- Construct a numbered directed cycle: monochromatic subset as consecutive nodes



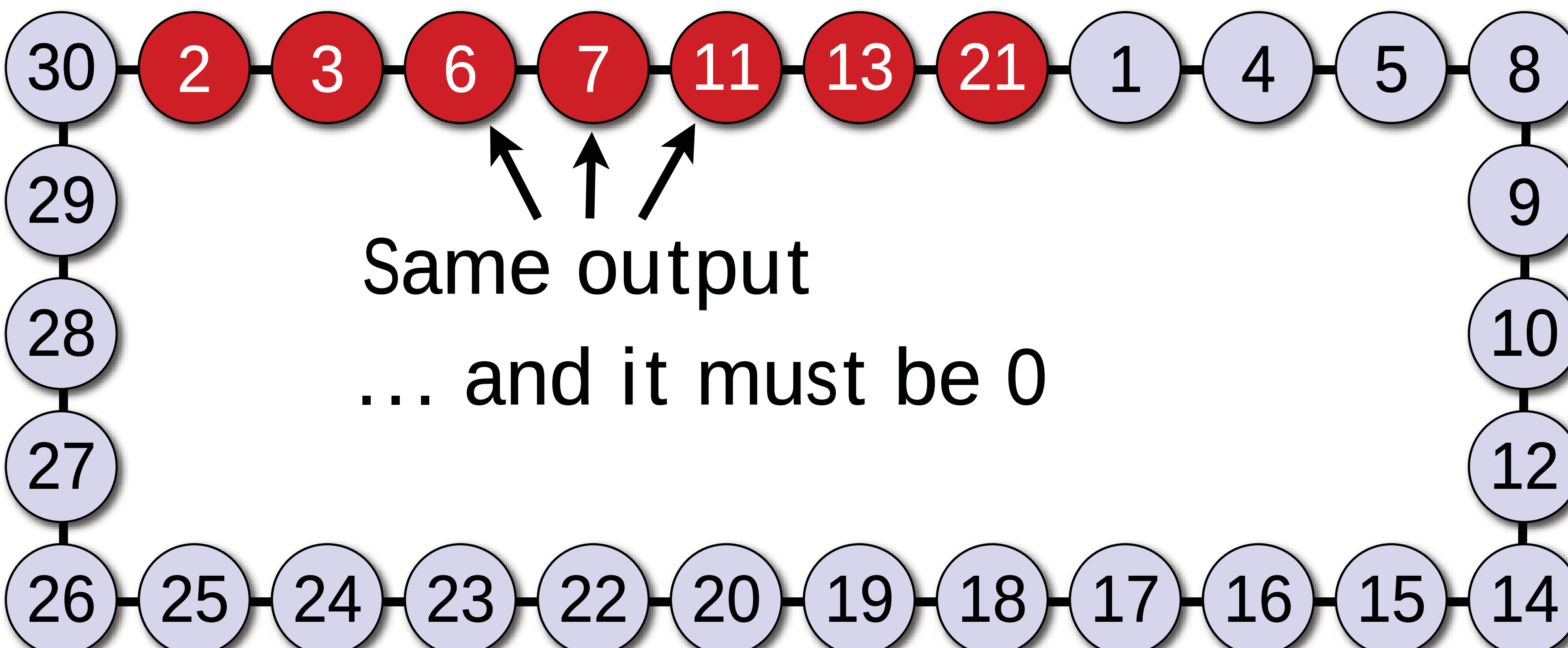
Lower-bound result for finding large independent sets

- Construct a numbered directed cycle: monochromatic subset as consecutive nodes



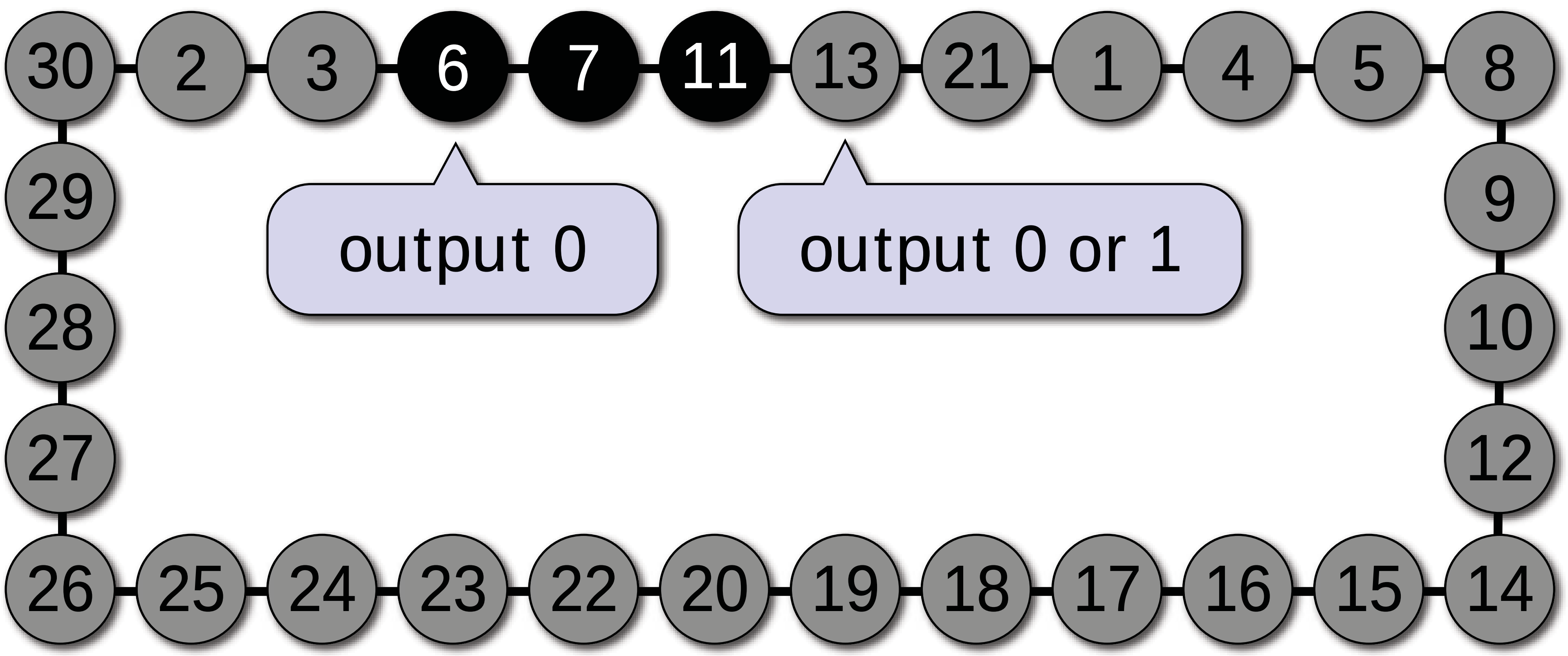
Lower-bound result for finding large independent sets

- Construct a numbered directed cycle: monochromatic subset as consecutive nodes



Lower-bound result for finding large independent sets

- Hence there is an n-cycle with a chain of $m - 2T$ nodes that output 0

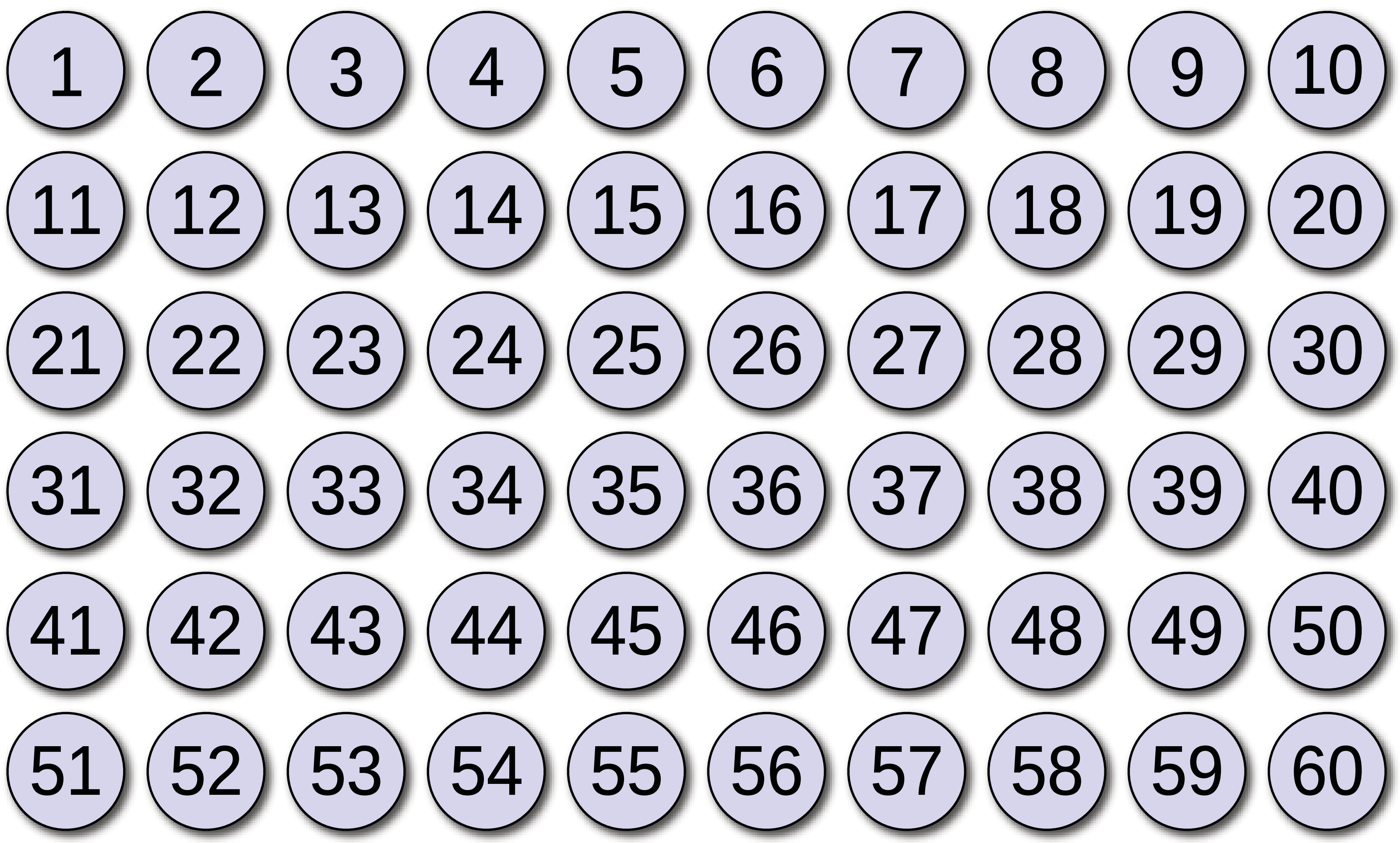


Lower-bound result for finding large independent sets

- Hence there is an n-cycle with a chain of $m - 2T$ nodes that output 0
- We can choose as large m as we want
  - Good, more “black” nodes that output 0
- However, n increases rapidly if we increase m
  - Bad, more “grey” nodes that might output 1
- Trick: choose “unnecessarily large” n so that we can apply Ramsey’s theorem repeatedly

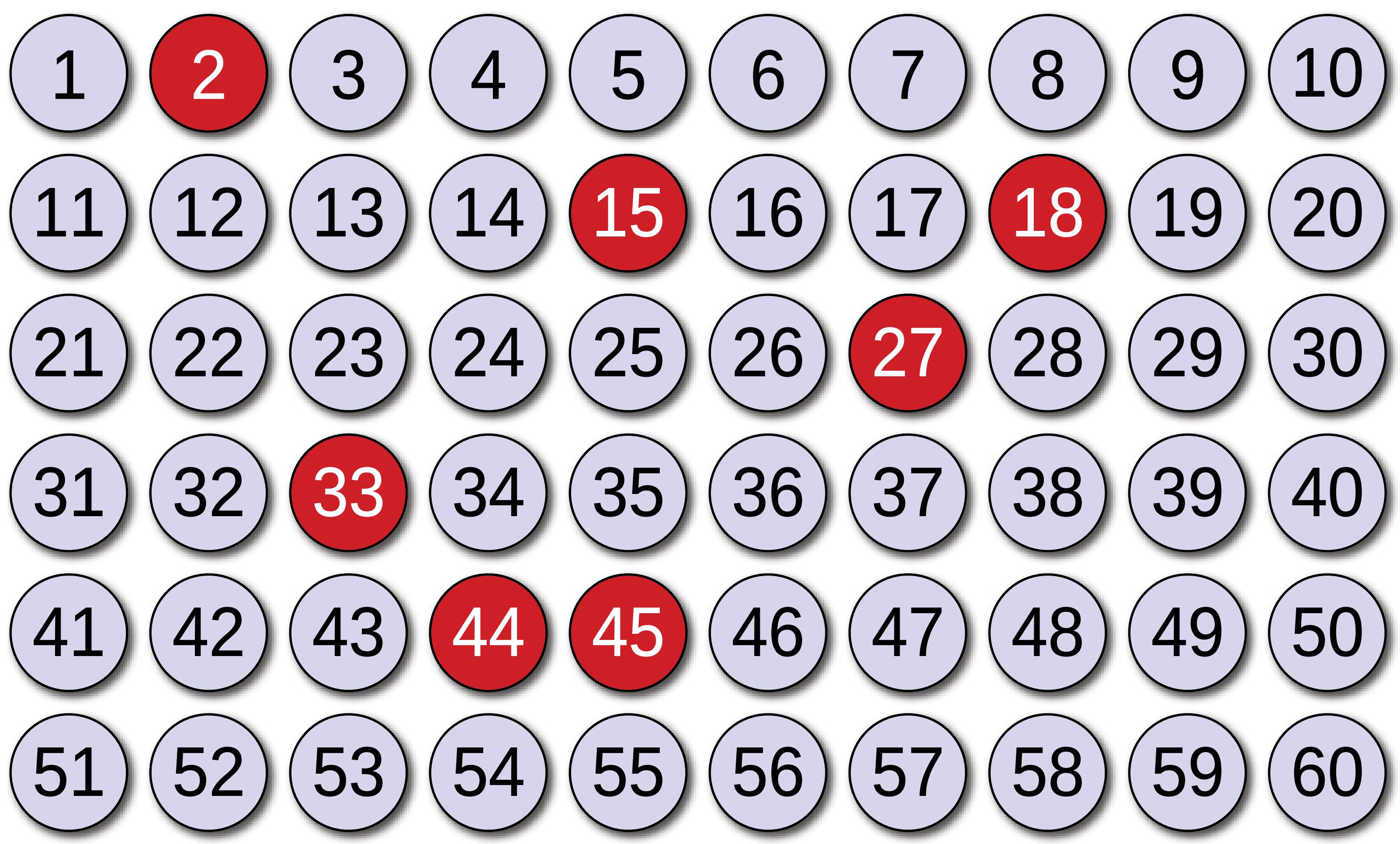
Lower-bound result for finding large independent sets

- Huge ID space…



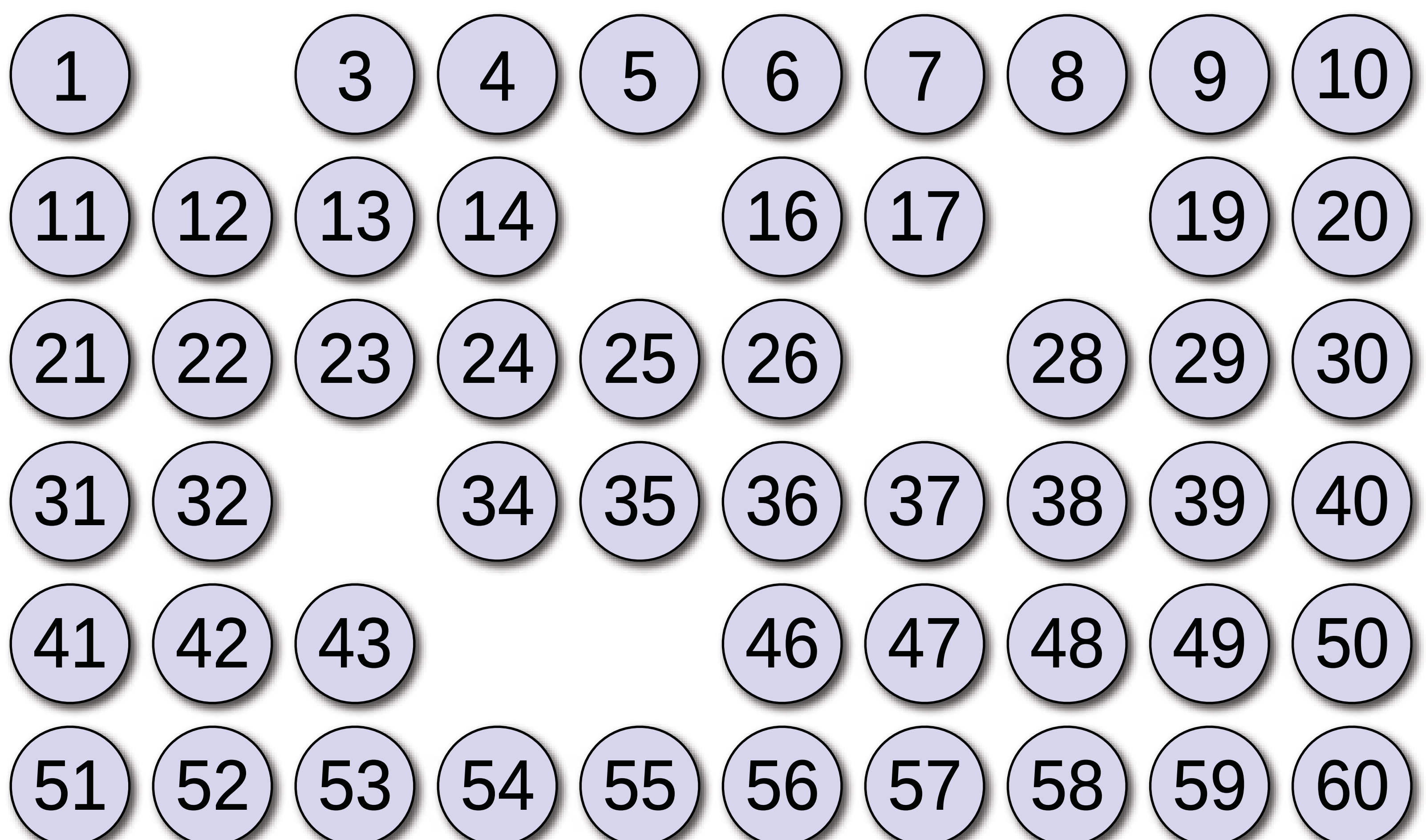
Lower-bound result for finding large independent sets

- Find a monochromatic subset of size *m*…



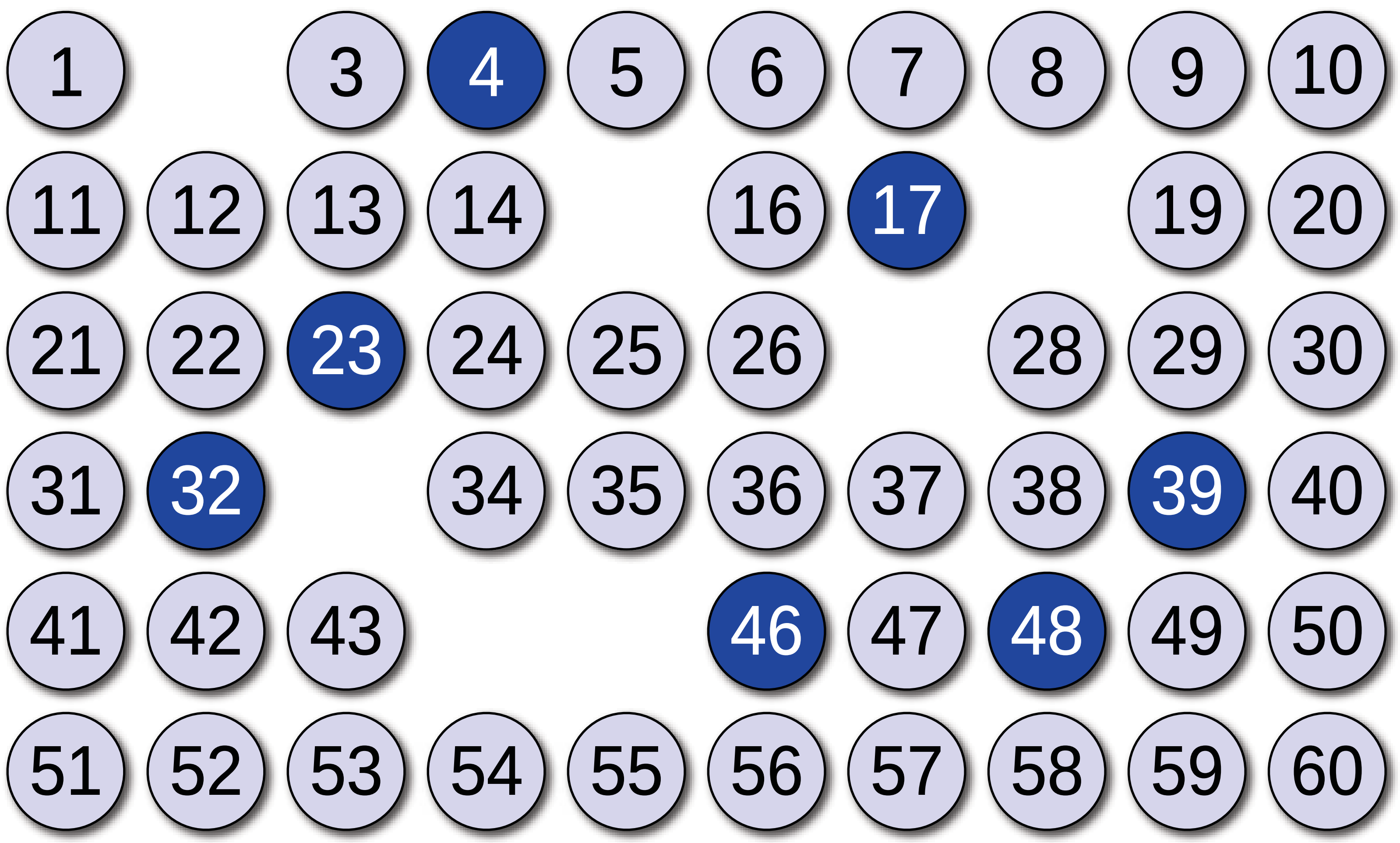
Lower-bound result for finding large independent sets

- Delete these IDs…



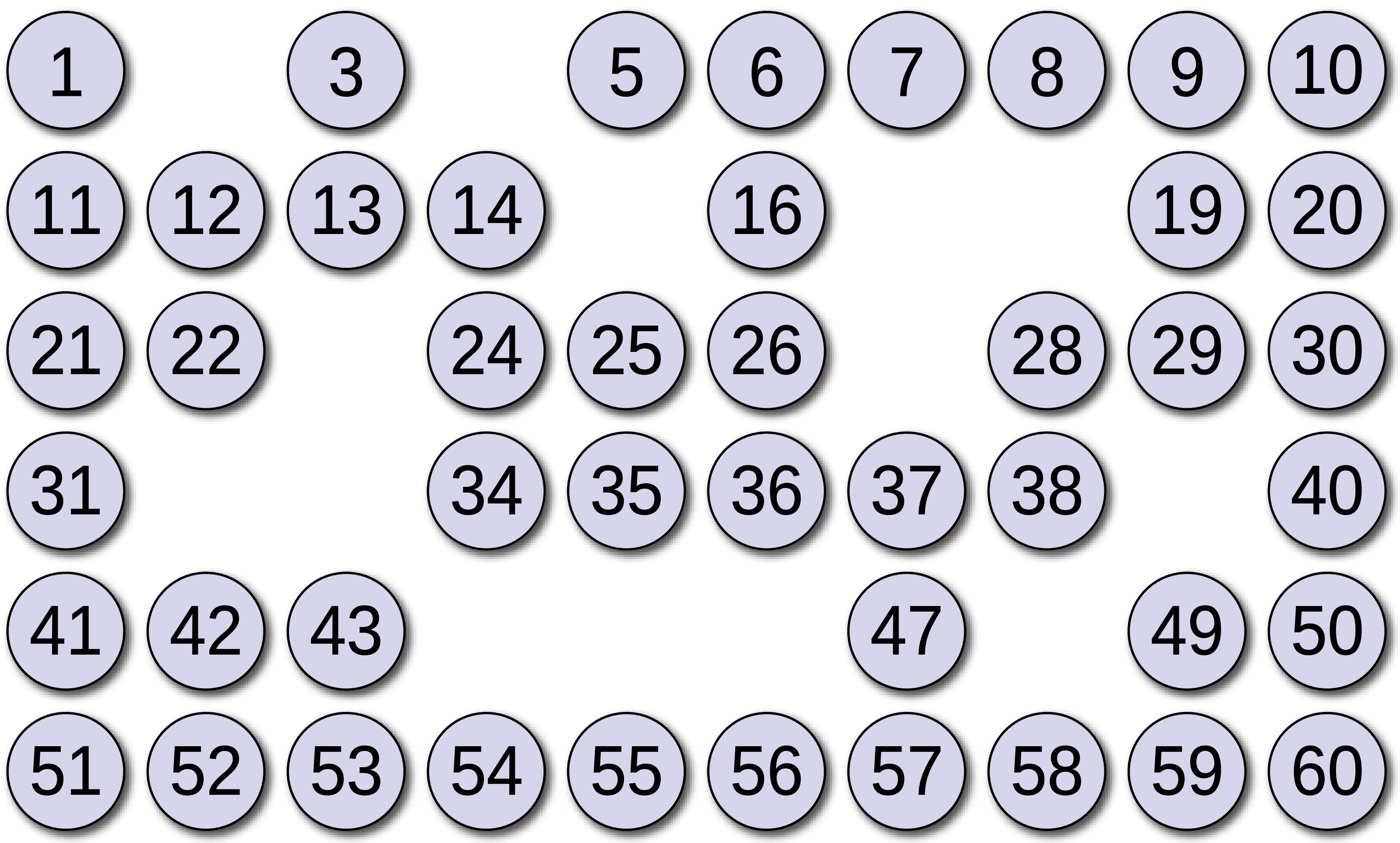
Lower-bound result for finding large independent sets

- Still sufficiently many IDs to apply Ramsey…



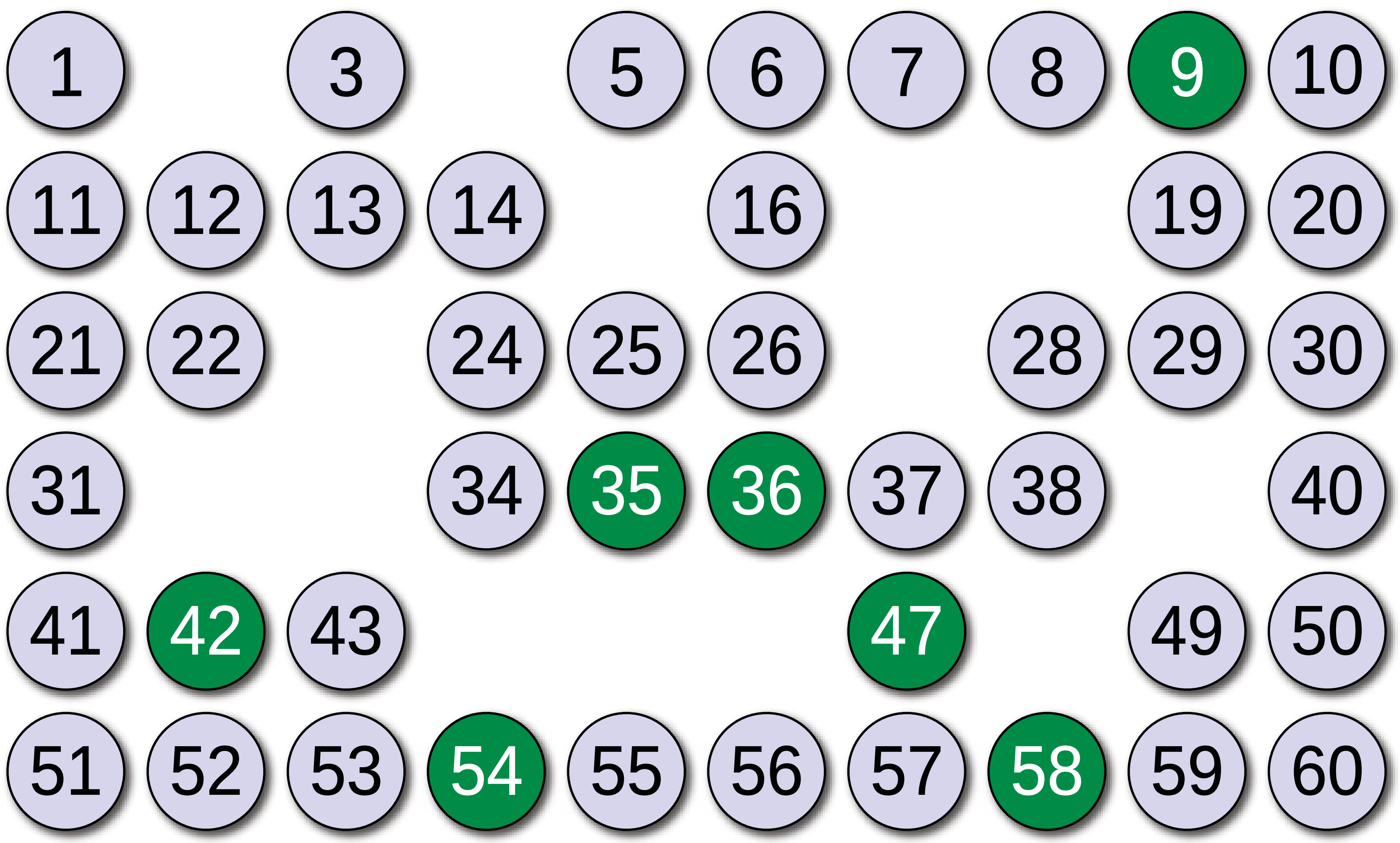
Lower-bound result for finding large independent sets

- Repeat…



Lower-bound result for finding large independent sets

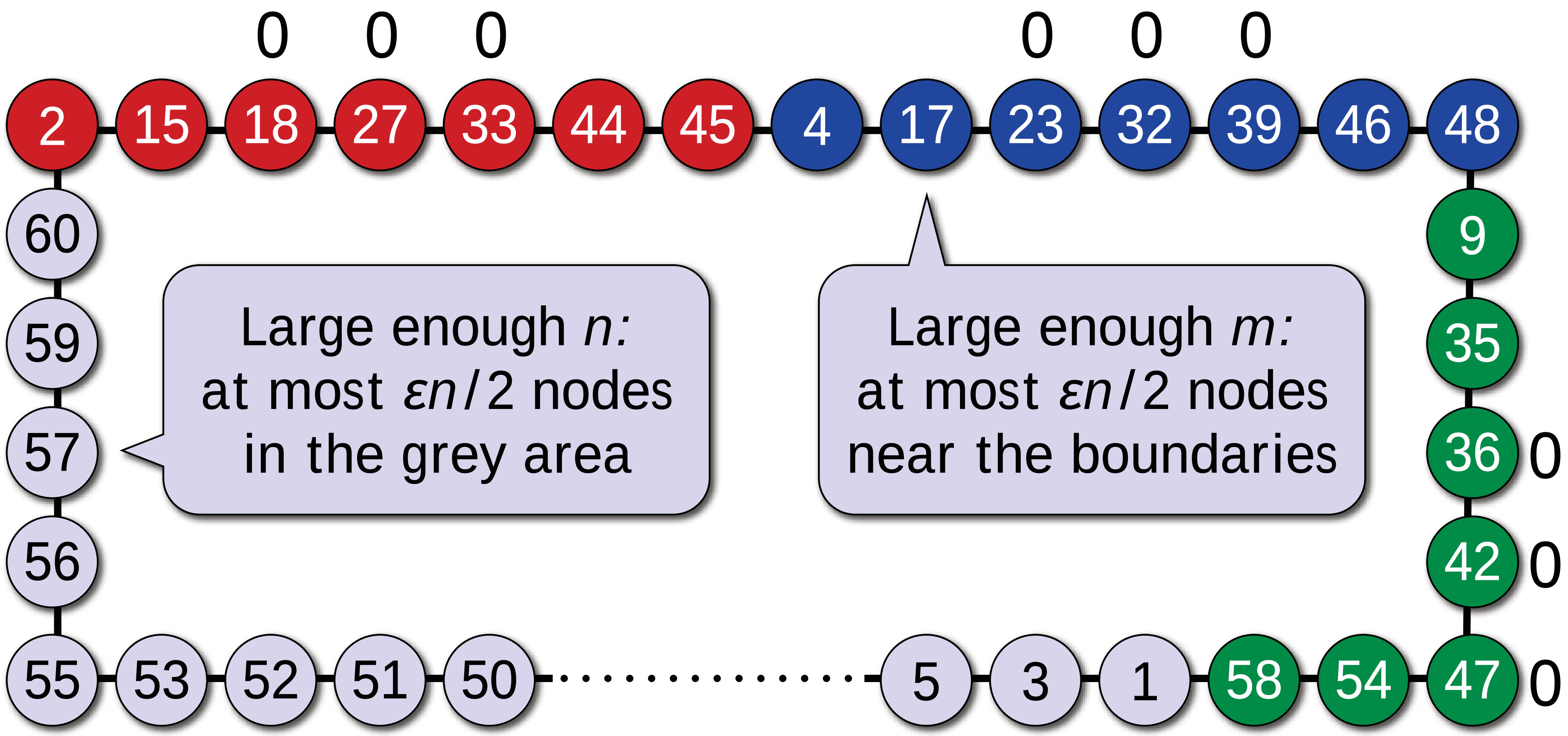
- Repeat until stuck



Lower-bound result for finding large independent sets

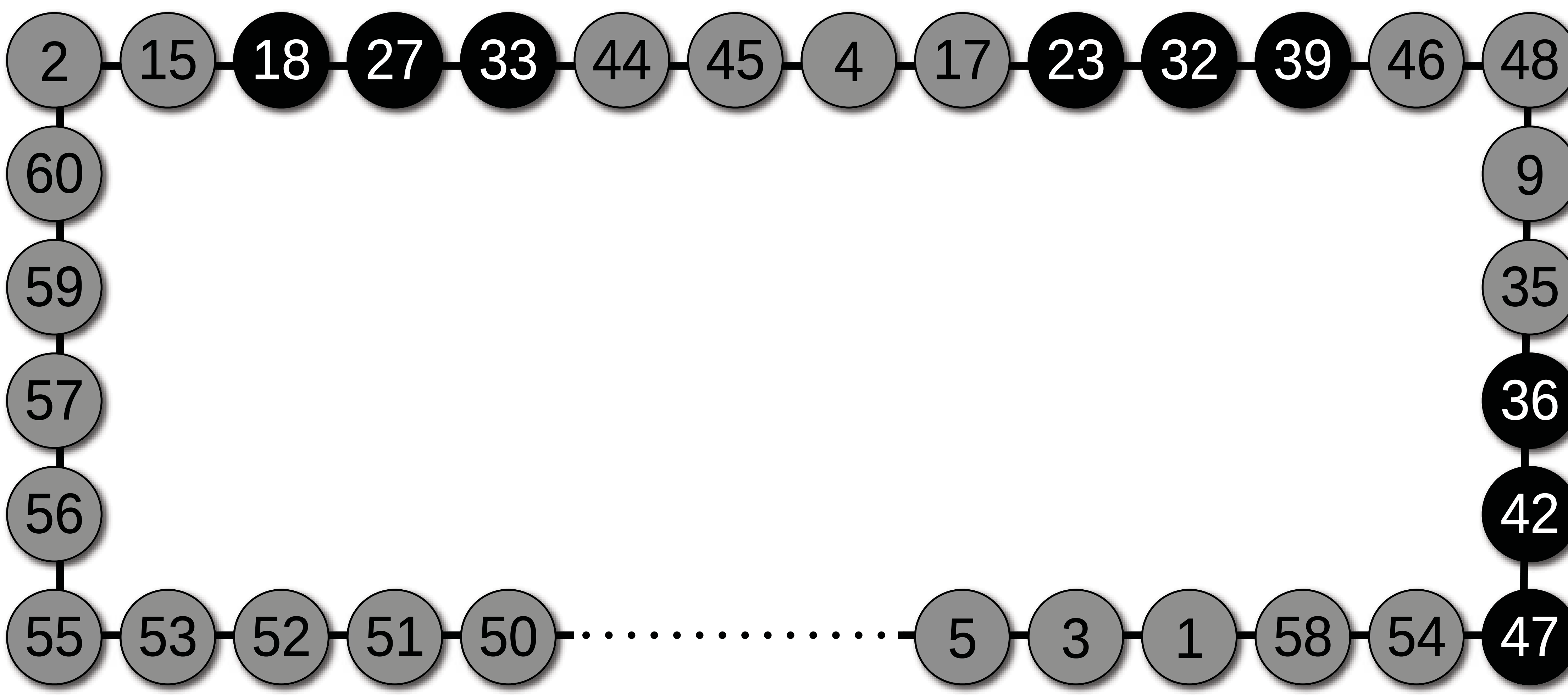
- Several monochromatic subsets + some leftovers

Lower-bound result for finding large independent sets



Lower-bound result for finding large independent sets

- Thus *A* outputs an independent set with ≤ *εn* nodes



DDA 2010, lecture 4c: Corollaries

- Finding “anything” non-trivial in cycles is not possible in constant time

A strong negative result

- We have used Ramsey's theorem to show that constant-time algorithms can't find large independent sets in cycles
  - moreover, we can get a $\Omega(\log^* n)$ lower bound on the running time of any algorithm that finds a large independent set
  - trick: use a power tower upper bound for $R_2(n; k)$
- What implications do we have?

A strong negative result

- If we could find a graph colouring…
  - we could find a maximal independent set…
  - which is an independent set with at least $n/3$ nodes
  - contradiction
- Corollary: graph colouring can’t be solved in constant time in cycles
  - we got Linial’s result as a simple corollary…

A strong negative result

- If we could find a $(2 - \varepsilon)$-approximation of vertex cover…
  - we would have a vertex cover with at most $n - \varepsilon n/2$ nodes in an n-cycle (even n)
  - its complement is an independent set with at least $\varepsilon n/2$ nodes
  - contradiction
- This is tight: it *is* possible to find a 2-approximation in time independent of n

A strong negative result

- Using Ramsey's theorem, we are able to show that these problems can't be solved in $O(1)$ time:
  - vertex colouring, edge colouring, …
  - maximal independent set, maximal matching, …
  - $(2 - \varepsilon)$-approximation of vertex cover
  - $(\Delta + 1 - \varepsilon)$-approximation of dominating set…
- Next lecture: something *positive* with $O(1)$ running time…