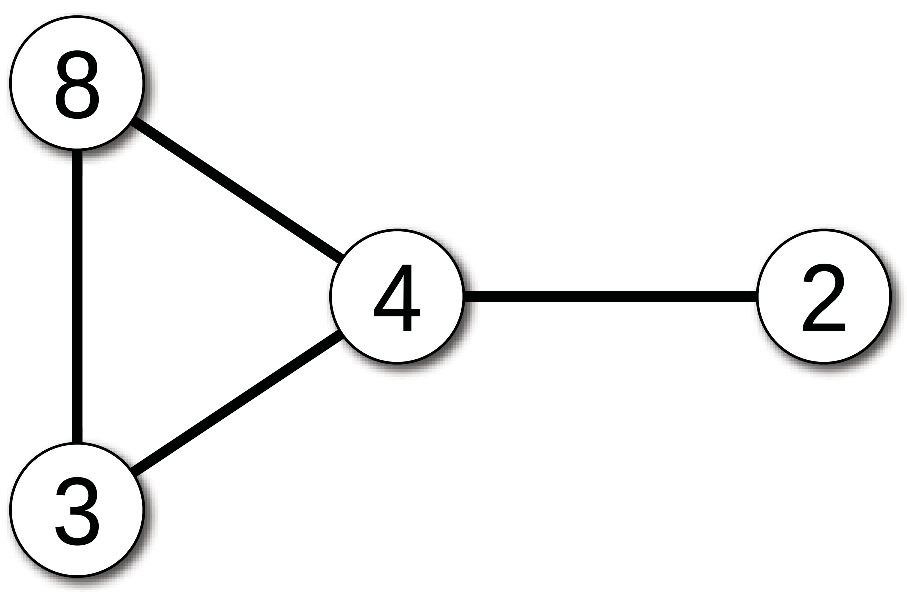
DDA 2010, lecture 2: Algorithms with running time $O(\log^* n)$

- Cole–Vishkin (1986):
  - colour reduction technique
  - colouring paths, cycles, trees
- Applications:
  - colouring arbitrary graphs

Unique identifiers



- Assumption:
  each node has
  a unique identifier
  in its local input
- Node identifiers
  are a subset of
  1, 2, …, poly(n)
- Chosen by adversary

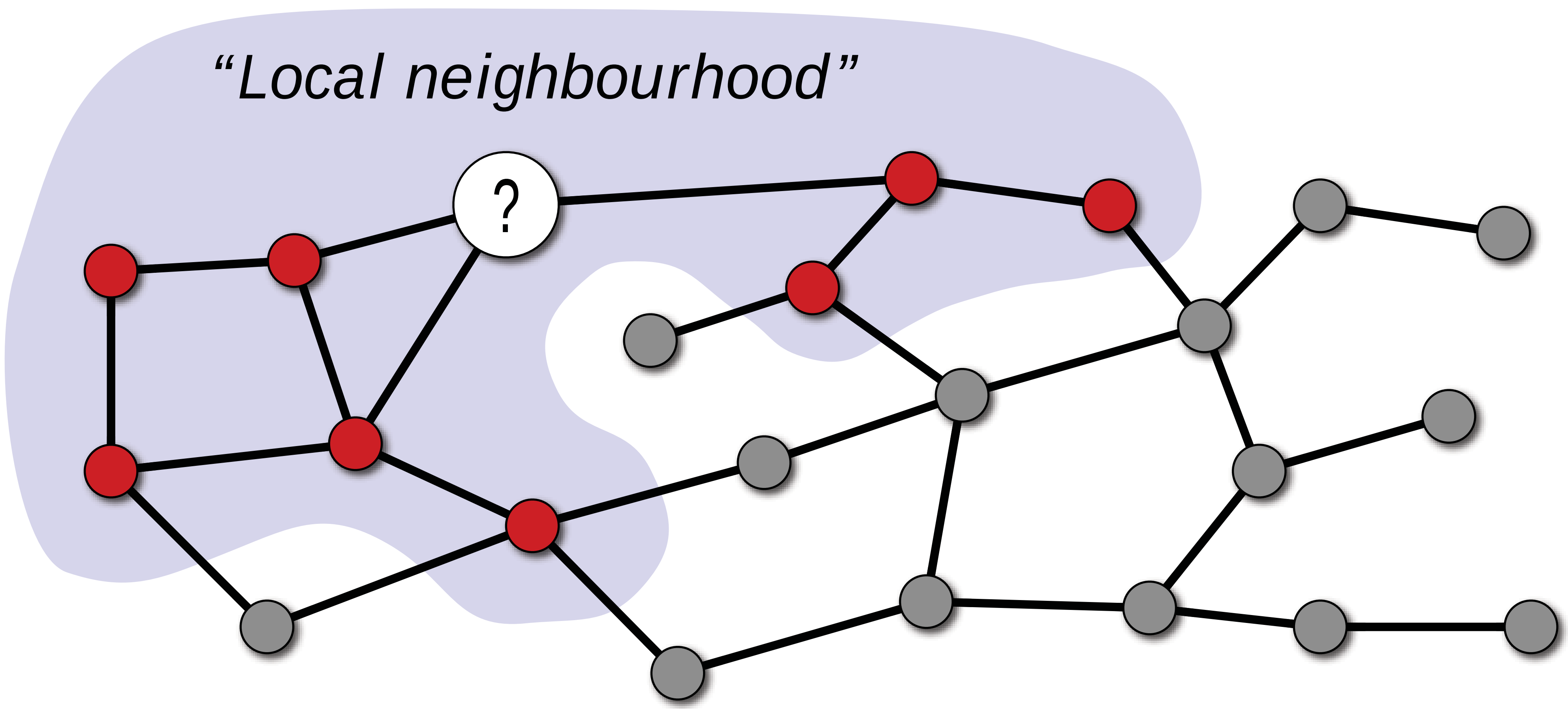
Algorithms for networks with unique identifiers

- With unique identifiers, "everything" can be solved in diameter(G) + 1 rounds
  - Algorithm: each node
    1. gathers full information about G (including all local inputs)
    2. solves the graph problem by brute force
    3. chooses its local output accordingly
- What can be solved **much faster**?

Algorithms for networks with unique identifiers

- Running time is $T \iff$
  output is a function of input within distance T

$T = 2$:



Algorithms for networks with unique identifiers

- We have seen a simple algorithm with running time $O(\Delta)$
- We will soon see other algorithms with running times such as $O(\Delta + \log^* n)$
  - these can be ***much*** smaller than diameter(G)
  - faster than just sending information across the network!
  - these algorithms use only "local information" to produce their local outputs
  - distributed algorithms in the strongest possible sense

DDA 2010, lecture 2a: Cole–Vishkin technique

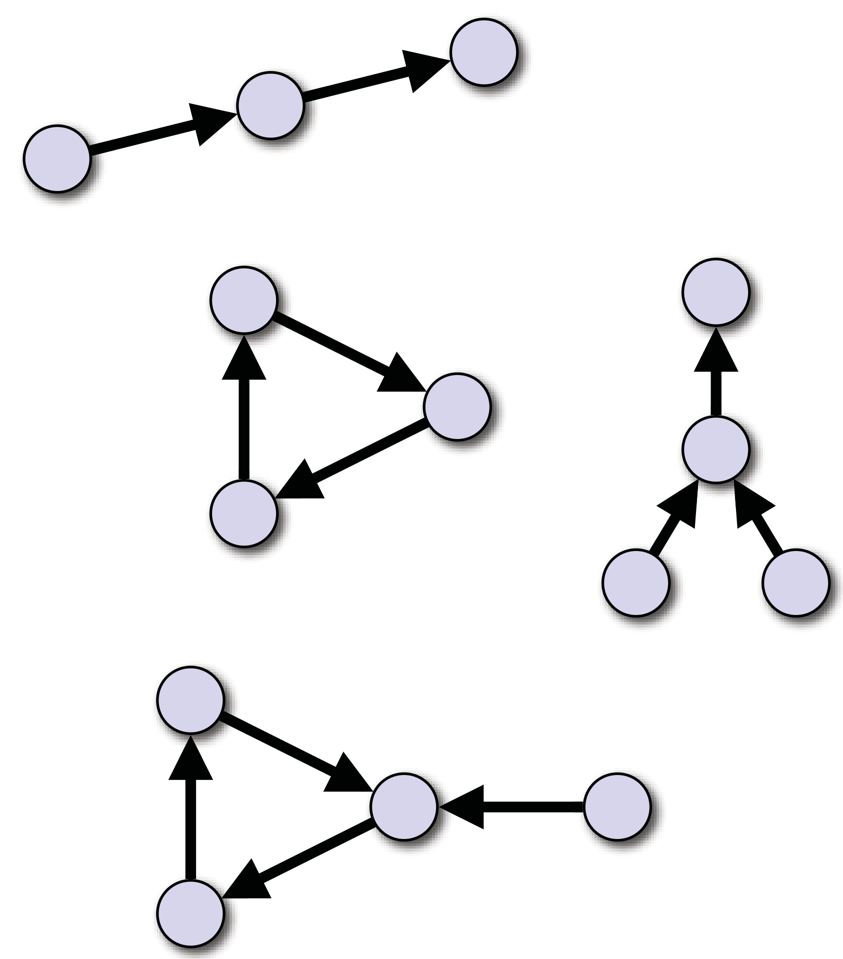
- Richard Cole and Uzi Vishkin (1986): "Deterministic coin tossing with applications to optimal parallel list ranking"
  - the original paper is about *parallel* algorithms and *linked list data structures*
  - however, the same technique can be used in *distributed* algorithms and *path graphs*

Colour reduction

- Cole–Vishkin algorithm is a **colour reduction** technique:
  - given a proper k_1-colouring of the graph, find a proper k_2-colouring
  - large k_1, small k_2
- **Note**: unique identifiers form a colouring!
  - hence we often have $k_1 = \text{poly}(n)$, $k_2 = O(1)$: given unique identifiers, find an $O(1)$-colouring
- **Convention**: colours are integers $0, 1, \ldots, k - 1$

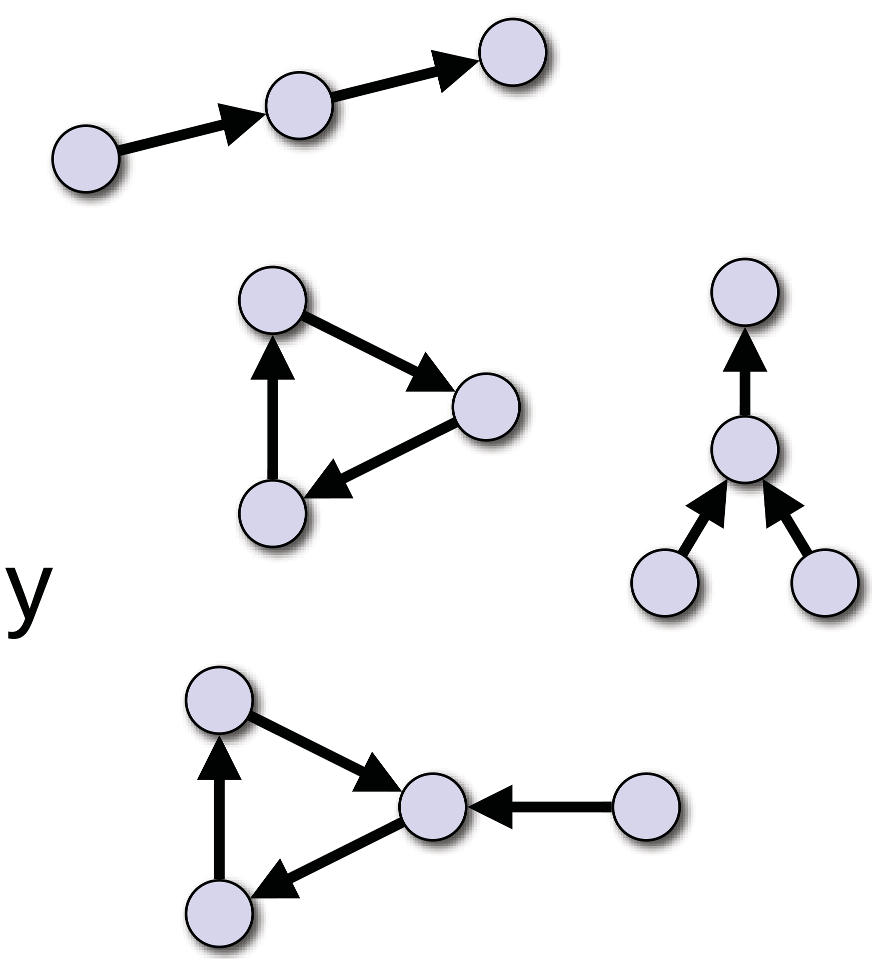
One successor

- Cole–Vishkin technique can be applied in directed graphs in which each node has **at most 1 successor**
    - directed paths
    - rooted trees
    - directed cycles
    - … and in general, **directed pseudoforests**



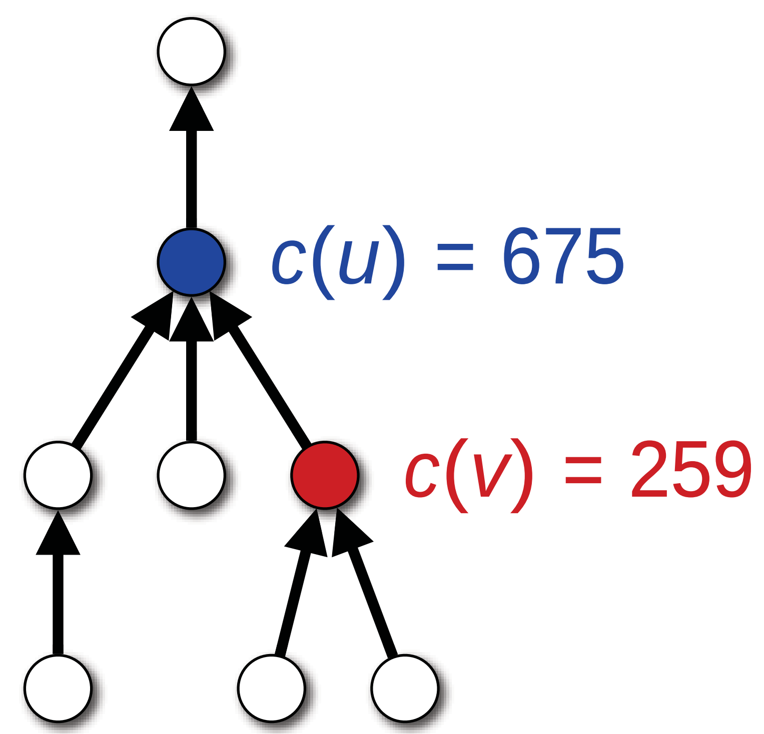
Cole–Vishkin: colour reduction in pseudoforests

- Cole–Vishkin technique can be applied in directed graphs in which each node has **at most 1 successor**
- Reduces k-colouring to $O(\log k)$-colouring in 1 step
- Reduces k-colouring to 6-colouring if applied repeatedly
  - other techniques: 6-colouring to 3-colouring



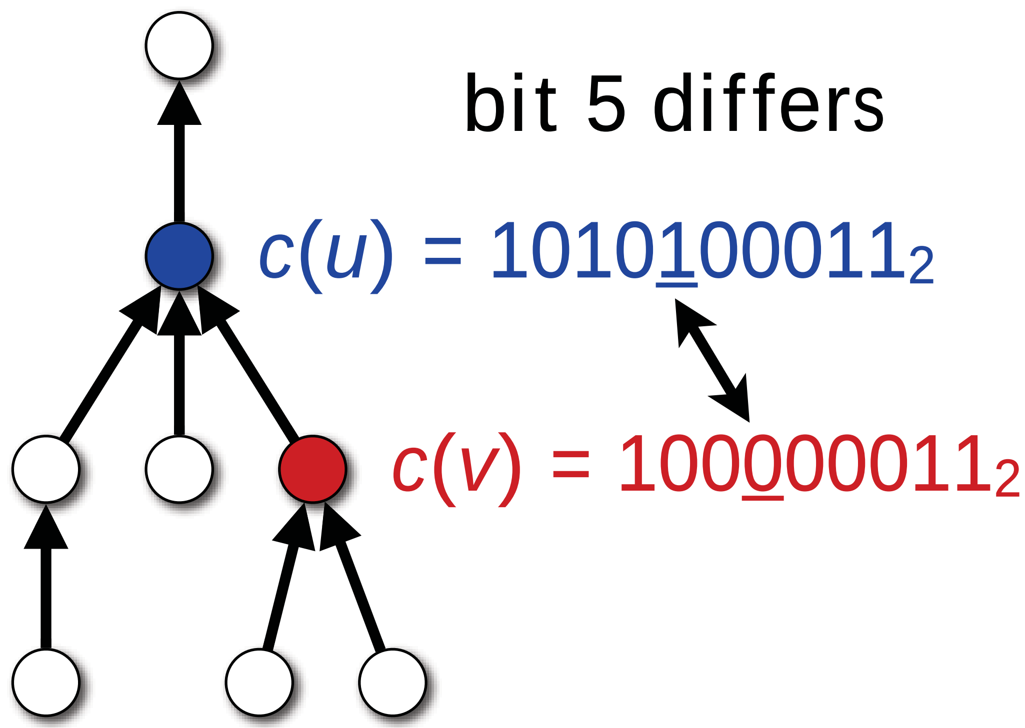
Cole–Vishkin iteration

- Each node v in parallel:
  - receive the colour of the successor u
  - compare your colour $c(v)$ to successor's colour $c(u)$



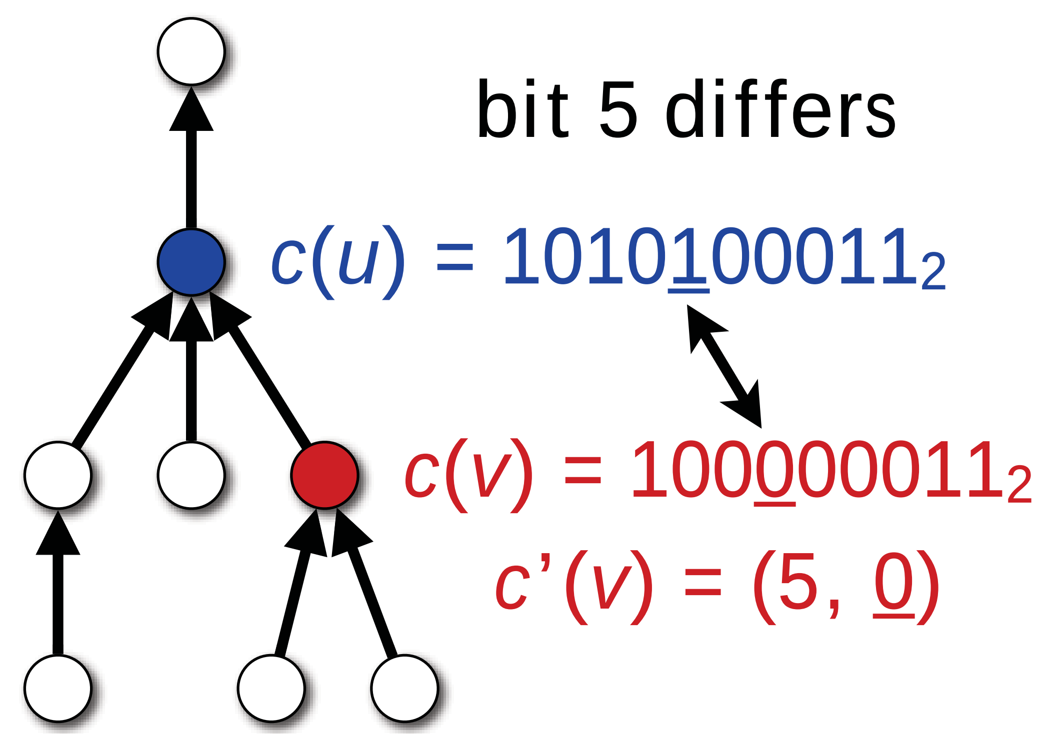
Cole–Vishkin iteration

- Each node *v* in parallel:
  - receive the colour of the successor *u*
  - compare your colour $c(v)$ to successor's colour $c(u)$ — in binary!
  - find the rightmost bit that differs



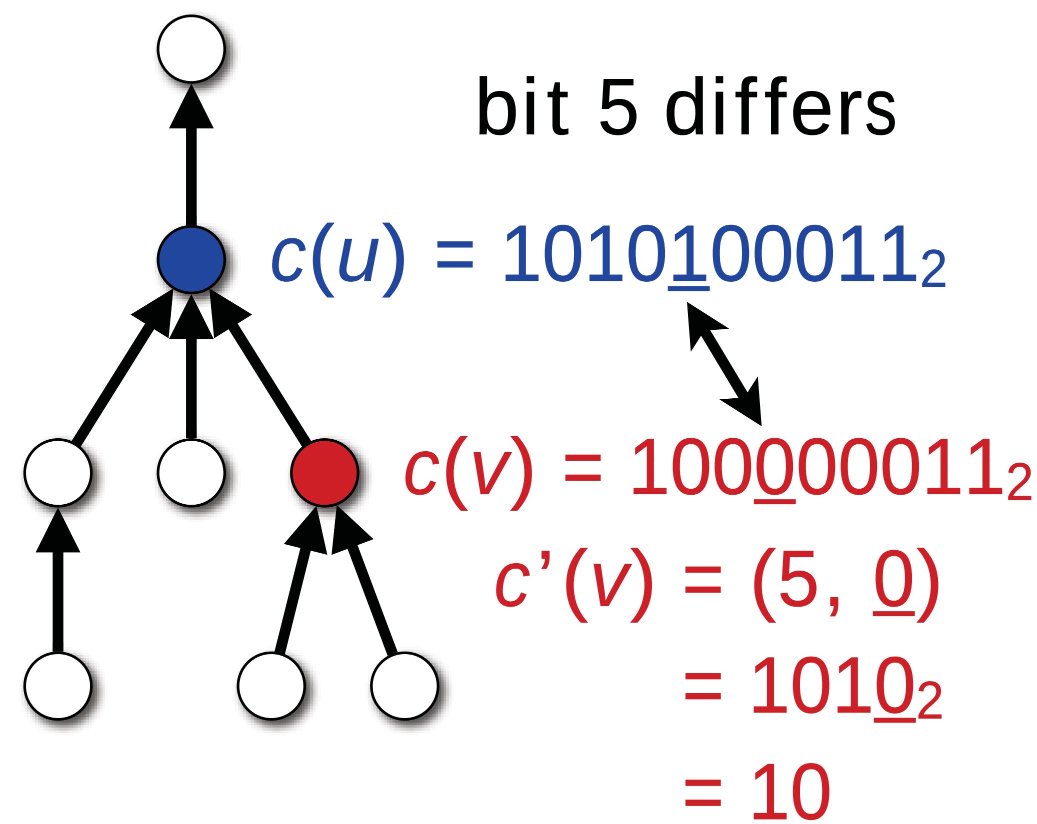
Cole–Vishkin iteration

- Each node *v* in parallel:
  - receive the colour of the successor *u*
  - compare your colour $c(v)$ to successor's colour $c(u)$
  - new colour $c'(v)$ is a pair (index, value):
    - which bit differs
    - value of the bit



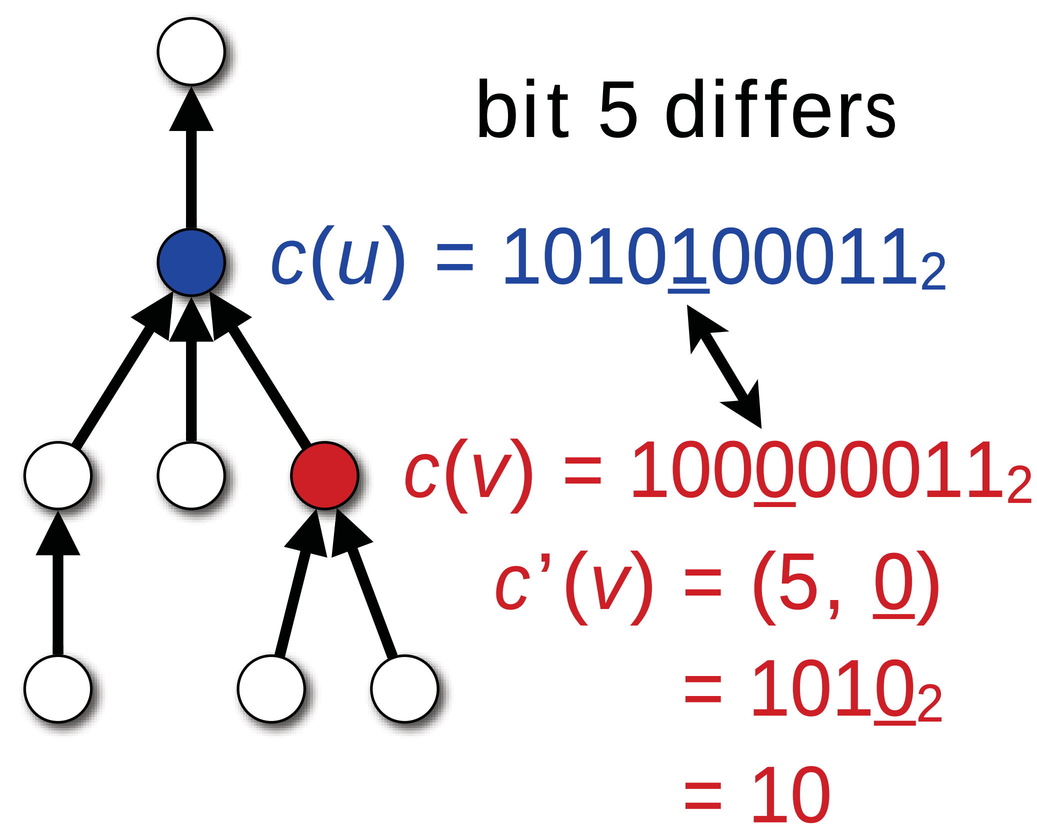
Cole–Vishkin iteration

- Each node *v* in parallel:
  - receive the colour of the successor *u*
  - compare your colour $c(v)$ to successor's colour $c(u)$
  - new colour $c'(v)$ is a pair (index, value)
  - can be encoded in binary or in decimal



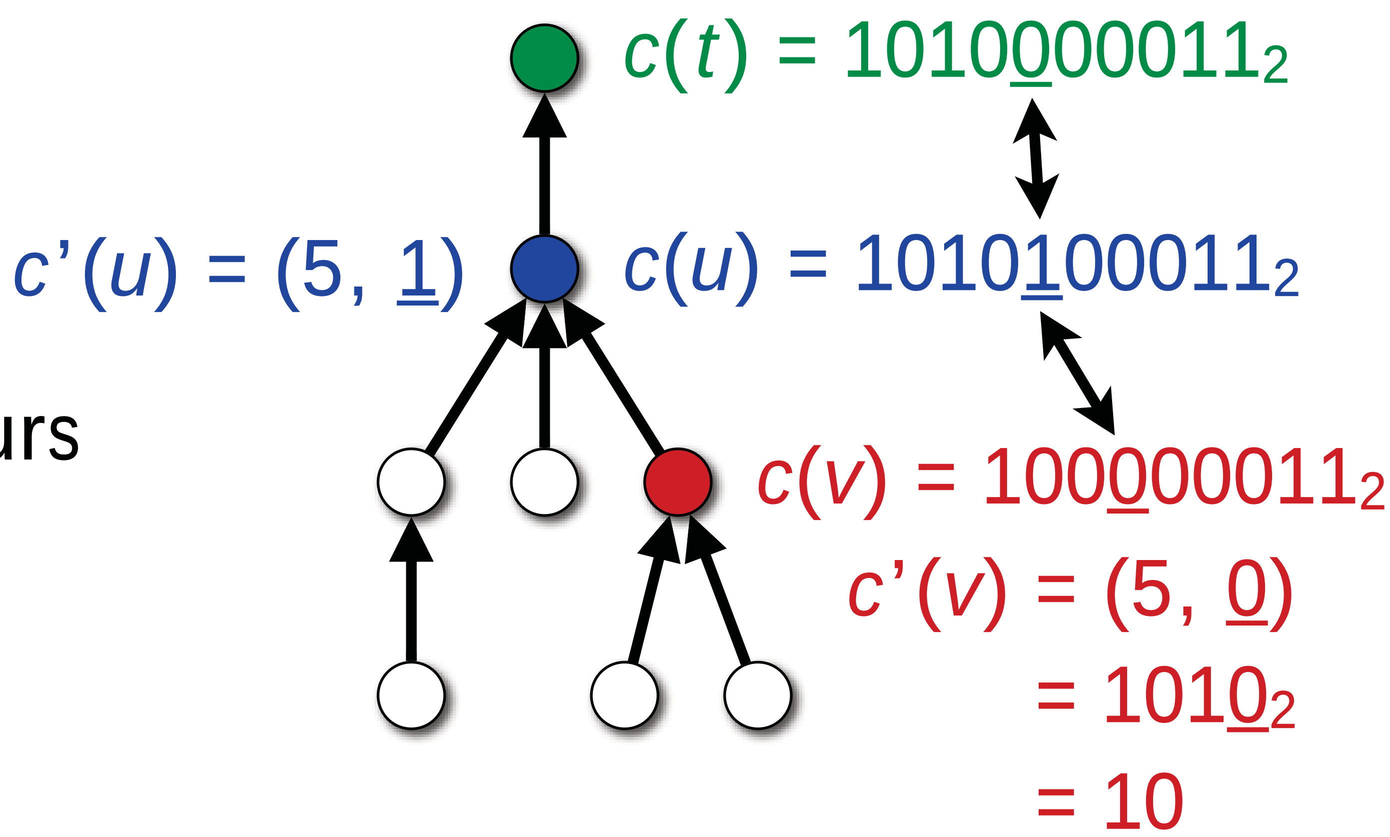
Cole–Vishkin iteration: correctness

- After one iteration, we have much **smaller colours values**
- But do we still have a **proper colouring**?
  - yes — it is enough to show that your successor will choose a different colour



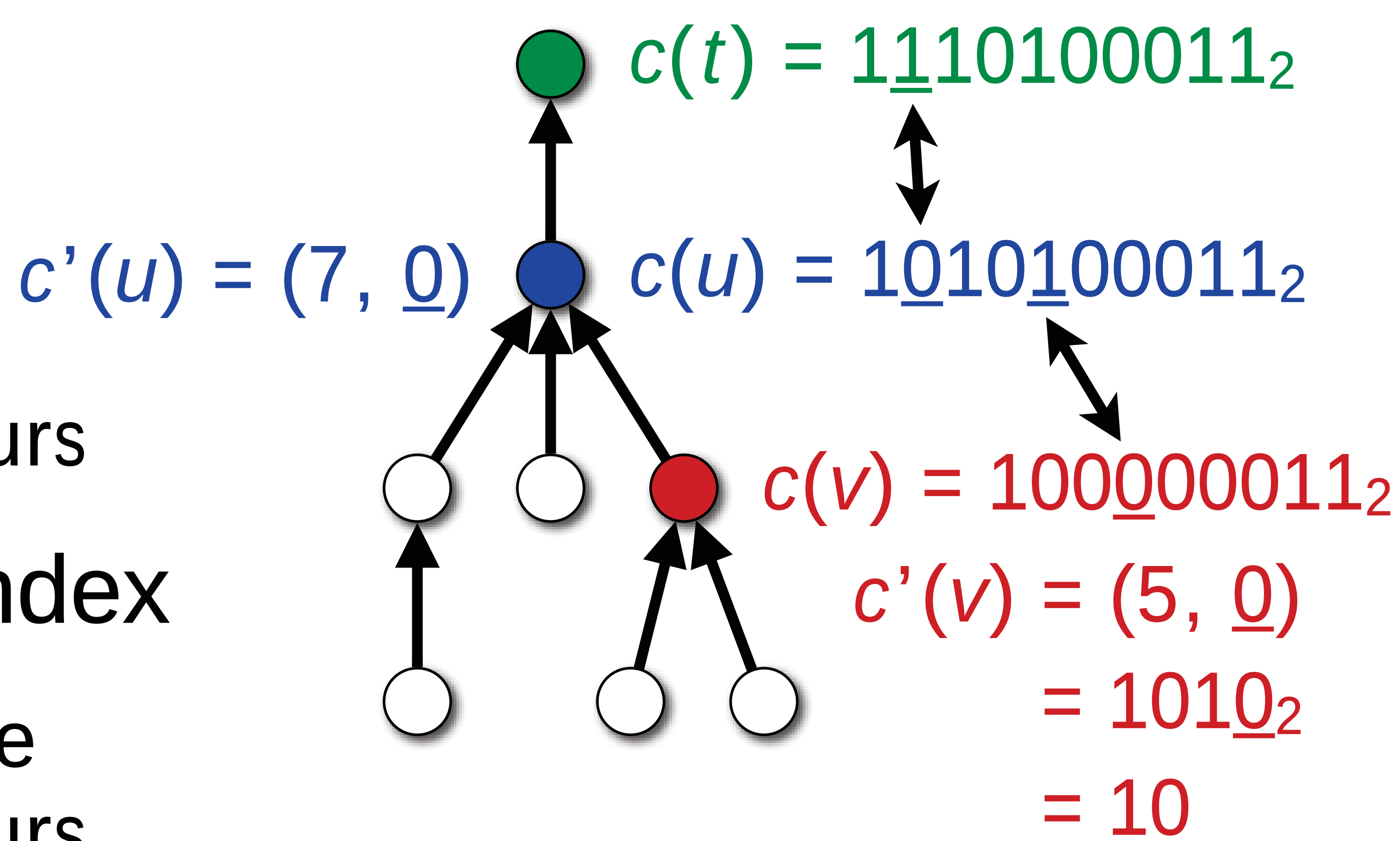
Cole–Vishkin iteration: correctness

- Case 1: successor *u* chooses the same index
    - then *u* chooses a different value!
    - *u* and *v* have different new colours



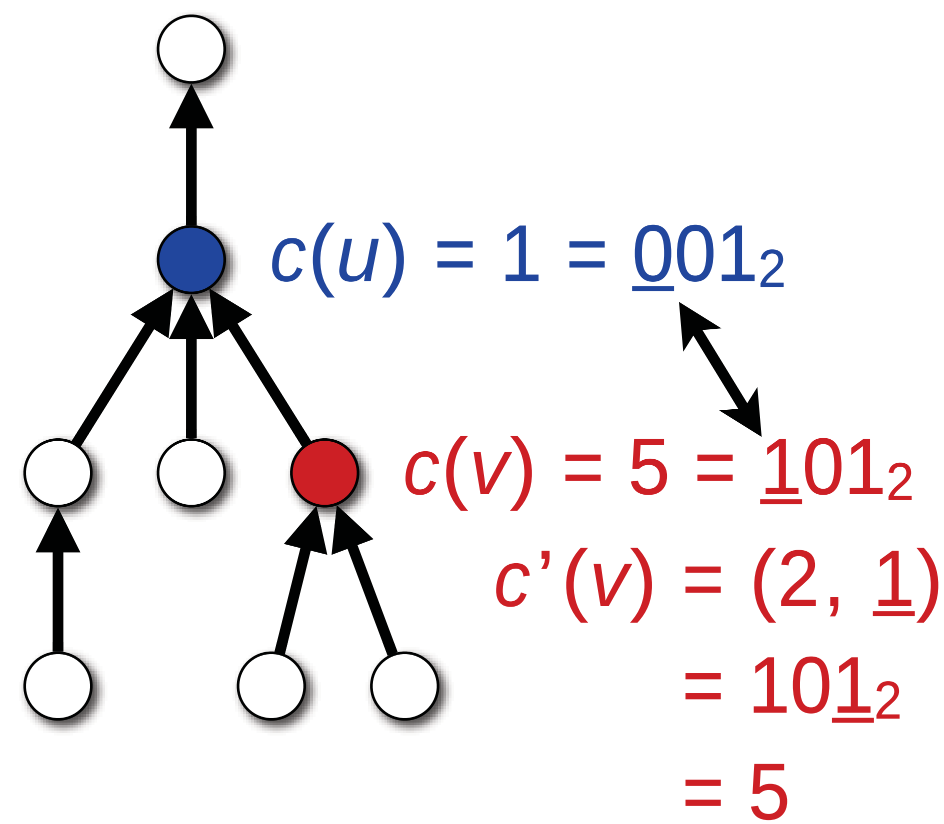
Cole–Vishkin iteration: correctness

- Case 1: successor *u* chooses the same index
  - then *u* chooses a different value!
  - *u* and *v* have different new colours
- Case 2: different index
  - trivial: *u* and *v* have different new colours



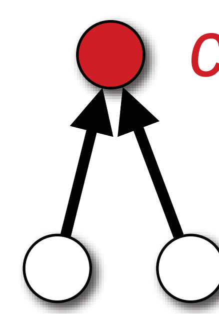
Cole–Vishkin iteration

- Can be used repeatedly until we have k = 6
  - i.e., colours 0, 1, …, 5
  - then we may be stuck and other techniques are needed



Cole–Vishkin iteration

- One special case: what if you don't have a successor?
  - just proceed ***as if*** you had a successor whose colour differs from your colour
  - e.g., pretend that the first bit differs



$$c(v) = 10000001\underline{1}_2$$
$$c'(v) = (0, \underline{1})$$
$$= 0\underline{1}_2$$
$$= 1$$

DDA 2010, lecture 2b: Analysing Cole–Vishkin

- The algorithm is very fast — exactly how fast?
- Let's introduce some notation: $\log^{(i)} x$, $\log^* x$

Logarithms

- Here: all logarithms are to base 2

$$\log x = \log_2 x$$

- Shorthand notation for iterations:

$$\log^{(0)} x = x$$

$$\log^{(1)} x = \log x$$

$$\log^{(2)} x = \log \log x$$

$$\log^{(i)} x = \log^{(i-1)} \log x = \underbrace{\log \log \ldots \log}_{i \text{ times}} x$$

Logarithms: examples

$$\log^{(0)} 1 = 1$$

$$\log^{(1)} 2 = 1$$

$$\log^{(2)} 2^2 = 1$$

$$\log^{(3)} 2^{2^2} = 1$$

$$\log^{(i)} \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{i \text{ times}} = 1$$

$$\log^{(3)} 15 \approx 0.96$$

$$\log^{(3)} 16 = 1$$

$$\log^{(3)} 17 \approx 1.02$$

$$\log^{(5)} 10^{1000} \approx 0.87$$

Iterated logarithm — log*, “log-star”

- log* x = smallest integer i such that $\log^{(i)} x \leq 1$
  - How many times we need to take logarithms until the value is at most 1?

$$\log^* 1 = 0: \quad \log^{(0)} 1 = 1$$

$$\log^* 2 = 1: \quad \log^{(1)} 2 = 1, \quad \log^{(0)} 2 = 2$$

$$\log^* 3 = 2: \quad \log^{(2)} 3 \approx 0.66, \quad \log^{(1)} 3 \approx 1.58$$

$$\log^* 4 = 2: \quad \log^{(2)} 4 = 1, \quad \log^{(1)} 4 = 2$$

$$\log^* 5 = 3: \quad \log^{(3)} 5 \approx 0.28, \quad \log^{(2)} 5 \approx 1.22$$

Iterated logarithm — log*, "log-star"

- log* x = smallest integer i such that $\log^{(i)} x \le 1$
  - How many times we need to take logarithms until the value is at most 1?

$$
\begin{array}{rll}
\log^* 65535 = 4: & \log^{(4)} 65535 < 1.00, & \log^{(3)} 65535 \approx 2.00 \\
\log^* 65536 = 4: & \log^{(4)} 65536 = 1, & \log^{(3)} 65536 = 2 \\
\log^* 65537 = 5: & \log^{(5)} 65537 \approx 0.00, & \log^{(4)} 65537 > 1.00 \\
\log^* 10^{1000} = 5: & \log^{(5)} 10^{1000} \approx 0.87, & \log^{(4)} 10^{1000} \approx 1.83 \\
\log^* 10^{10000} = 5: & \log^{(5)} 10^{10000} \approx 0.98, & \log^{(4)} 10^{10000} \approx 1.97
\end{array}
$$

Cole–Vishkin: one iteration

- One iteration of the Cole–Vishkin algorithm reduces the number of colours:

$$k \text{ colours} \quad \to \quad f(k) = 2\lceil \log k \rceil \text{ colours}$$

- **Proof**: There are $f(k)$ possible (index, value) pairs
  - $\log k$ (rounded up) possible "indexes"
  - 2 possible "values"

Cole–Vishkin: one iteration

- One iteration of the Cole–Vishkin algorithm reduces the number of colours:

$$k \text{ colours} \quad \rightarrow \quad f(k) = 2\lceil \log k \rceil \text{ colours}$$

- **Example**: $k = 100$, $\log k \approx 6.6$, $f(k) = 2 \times 7 = 14$
  - k colours 0, 1, …, 99 can be encoded in 7 bits, therefore "index" is in {0, 1, …, 6}
  - "value" is in {0, 1}

Cole–Vishkin: repeated iterations

- What about repeated iterations?

$$
\begin{aligned}
k \text{ colours} &\rightarrow & f(k) &= 2\lceil \log k \rceil \text{ colours} \\
&\rightarrow & f(f(k)) &= 2\lceil \log 2\lceil \log k \rceil \rceil \text{ colours} \\
&\rightarrow & f(f(f(k))) &= \ldots
\end{aligned}
$$

- Uh-oh, what does that mean in practice?
- **How many iterations until we have 6 colours**?

Cole–Vishkin: repeated iterations

- **Theorem**: Cole–Vishkin reduces the number of colours from k to 6 in at most $\log^* k$ iterations
- **Proof**:
  - Case 1: assume that $\log^* k \leq 2$
  - Then $k \leq 4$ and the claim is trivial: we already have at most 6 colours without any iterations

k colours $\rightarrow$ $f(k) = 2\lceil \log k \rceil$ colours

Cole–Vishkin: repeated iterations

- **Theorem**: Cole–Vishkin reduces the number of colours from k to 6 in at most $\log^* k$ iterations
- **Proof**:
  - Case 2: assume that $\log^* k = 3$
  - Then $k \le 16$, $f(k) \le 8$, $f(f(k)) \le 6$
  - 2 iterations are enough, the claim holds

k colours $\;\rightarrow\;$ $f(k) = 2\lceil \log k \rceil$ colours

Cole–Vishkin: repeated iterations

- **Theorem**: Cole–Vishkin reduces the number of colours from k to 6 in at most $\log^* k$ iterations
- **Proof**:
  - Case 3: assume that $m = \log^* k \geq 4$
  - Let's study the number of colours after 1, 2, …, $m - 3$ iterations…

$$k \text{ colours} \quad \longrightarrow \quad f(k) = 2\lceil \log k \rceil \text{ colours}$$

Cole–Vishkin: repeated iterations

- **Lemma**: If $m = \log^* k \geq 4$ and $i \leq m - 3$, then i iterations reduce the number of colours from k to at most $4 \log^{(i)} k$
- **Proof**: by induction
  - Basis $i = 0$: Trivial, $4 \log^{(0)} k = 4\,k \geq k$
  - Inductive step: Assume that after $i \leq m - 4$ iterations we have at most $4 \log^{(i)} k$ colours. Let's show that after $i + 1$ iterations we have at most $4 \log^{(i+1)} k$ colours…

$$k \text{ colours} \quad \longrightarrow \quad f(k) = 2\lceil \log k \rceil \text{ colours}$$

Cole–Vishkin: repeated iterations

- **Lemma**: If $m = \log^* k \geq 4$ and $i \leq m - 3$, then i iterations reduce the number of colours from k to at most $4 \log^{(i)} k$
  - after $i \leq m - 4$ iterations at most $4 \log^{(i)} k$ colours
  - after $i + 1$ iterations at most $f(4 \log^{(i)} k)$
    $\leq 2(1 + \log(4 \log^{(i)} k))$
    $\leq 2 + 2 \log 4 + 2 \log \log^{(i)} k$
    $< 2 \times 4 + 2 \log^{(i+1)} k$
    $< 4 \log^{(i+1)} k$
    colours

$m = \log^* k$: $\log^{(m-1)} k > 1$,
$\log^{(i+1)} k \geq \log^{(m-3)} k > 4$

k colours $\rightarrow$ $f(k) = 2\lceil \log k \rceil$ colours

Cole–Vishkin: repeated iterations

- **Lemma**: If $m = \log^* k \geq 4$ and $i \leq m - 3$, then i iterations reduce the number of colours from k to at most $4 \log^{(i)} k$
- **Corollary**: After $m - 3$ iterations we have at most $4 \log^{(m-3)} k \leq 4 \times 16 = 64$ colours
- **Corollary**: After m iterations the number of colours is at most $f(f(f(64))) = f(f(12)) = f(8) = 6$

$m = \log^* k$:
$\log^{(m)} k \leq 1$,
$\log^{(m-3)} k \leq 16$

k colours $\rightarrow$ $f(k) = 2\lceil \log k \rceil$ colours

Cole–Vishkin: repeated iterations

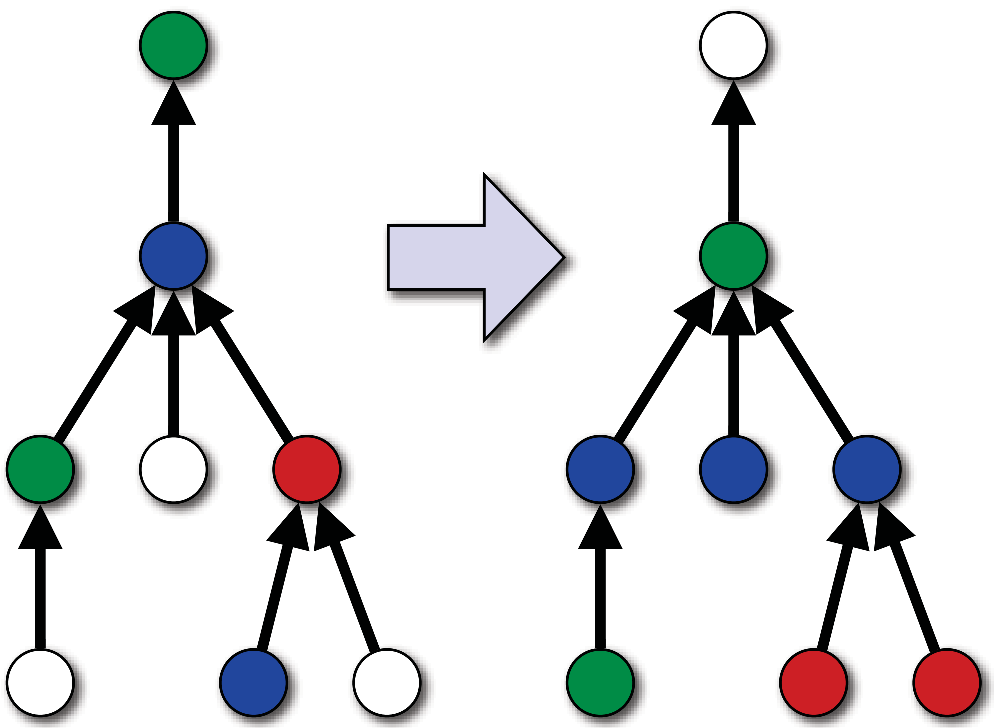
- **Theorem**: Cole–Vishkin reduces the number of colours from k to 6 in at most $\log^* k$ iterations
- Coming up next: how to get from 6 to 3 in at most 3 iterations?

DDA 2010, lecture 2c: Linear-time colour reduction

- Simple algorithm: from k-colouring to $(k - 1)$-colouring in one round
  - in paths, cycles, rooted trees, …
  - slower progress than Cole–Vishkin
  - however, can be used until we have 3 colours

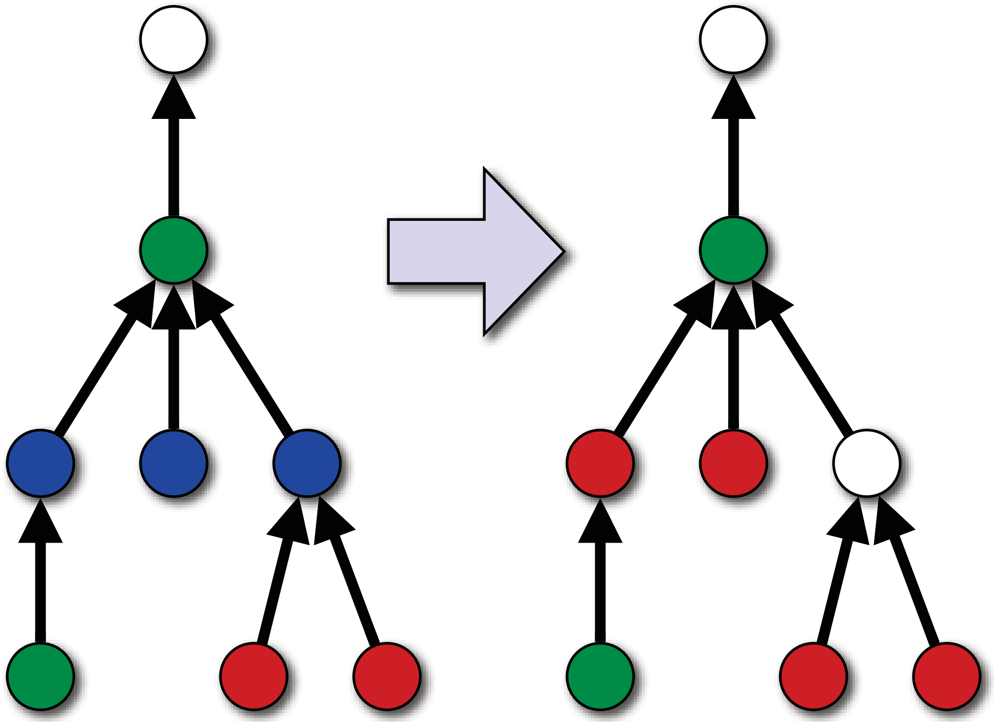
Linear-time colour reduction in pseudoforests

- First "shift" all colours:
  - new colour $c'(v)$ of node v = old colour $c(u)$ of its successor u
  - root: choose another colour
  - siblings have the same colour!



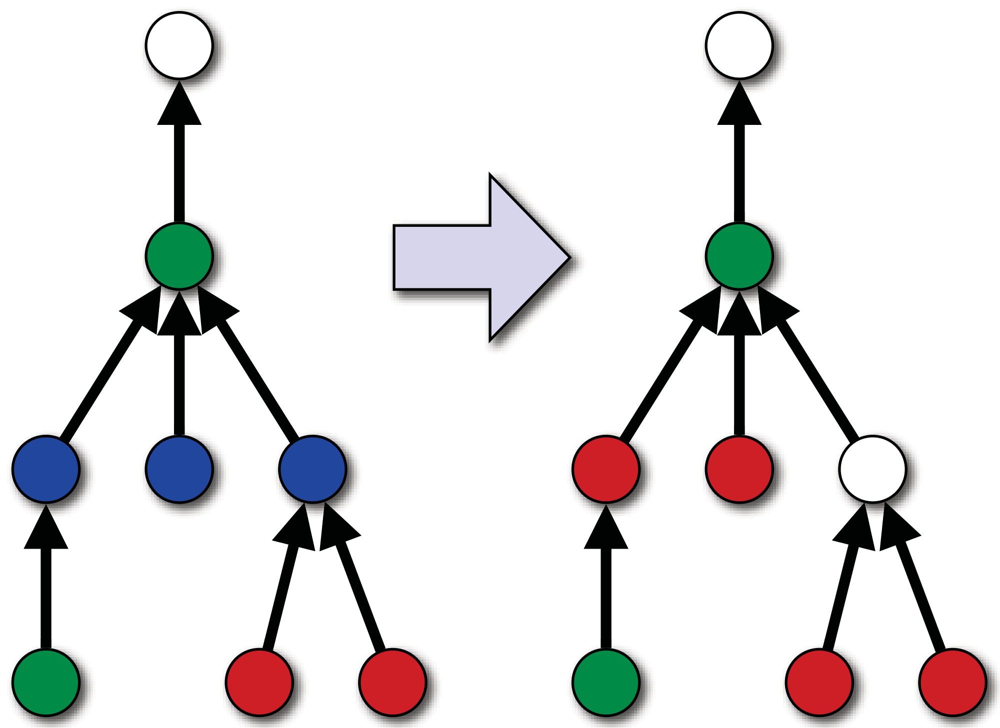
Linear-time colour reduction in pseudoforests

- First “shift” all colours
- Then each node v with colour $k - 1$ chooses a new colour from $\{0, 1, 2\}$
    - always possible: v’s neighbours have at most 2 different colours
    - shifting was needed to achieve this!



Linear-time colour reduction in pseudoforests

- First “shift” all colours
- Then each node v with colour $k - 1$ chooses a new colour from $\{0, 1, 2\}$
- Largest colour $k - 1$ eliminated
- We can repeat until we have a 3-colouring



Linear-time colour reduction in pseudoforests

- Cole–Vishkin:
  - from k to $O(\log k)$ colours in 1 step, until $k = 6$
- Simple algorithm:
  - from k to $k-1$ colours in 1 step, until $k = 3$
- Combine both:
  - from k to 3 colours in at most $3 + \log^* k$ iterations
  - in directed paths, cycles, trees, pseudoforests
  - what can we do in more general graphs?

DDA 2010, lecture 2d: Colouring in general graphs

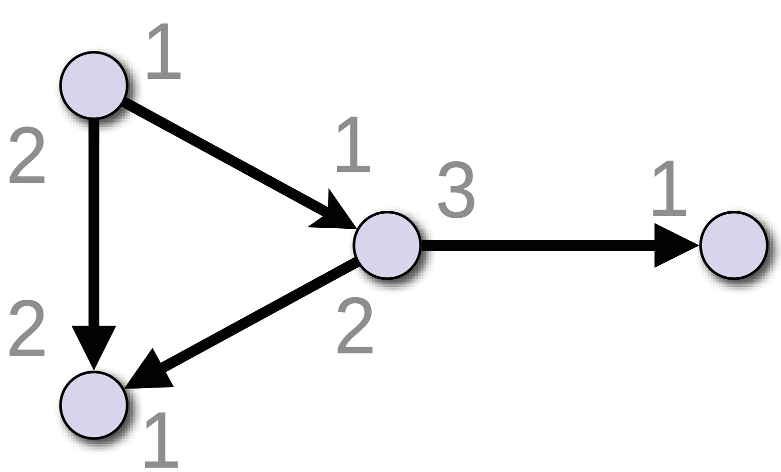
- We know how to colour rooted trees, how does this help in general graphs?
- $(\Delta + 1)$-colouring in $O(\Delta^2 + \log^* n)$ rounds
  - Goldberg, Plotkin & Shannon (1988): "Parallel symmetry-breaking in sparse graphs"
  - Panconesi & Rizzi (2001): "Some simple distributed algorithms for sparse networks"

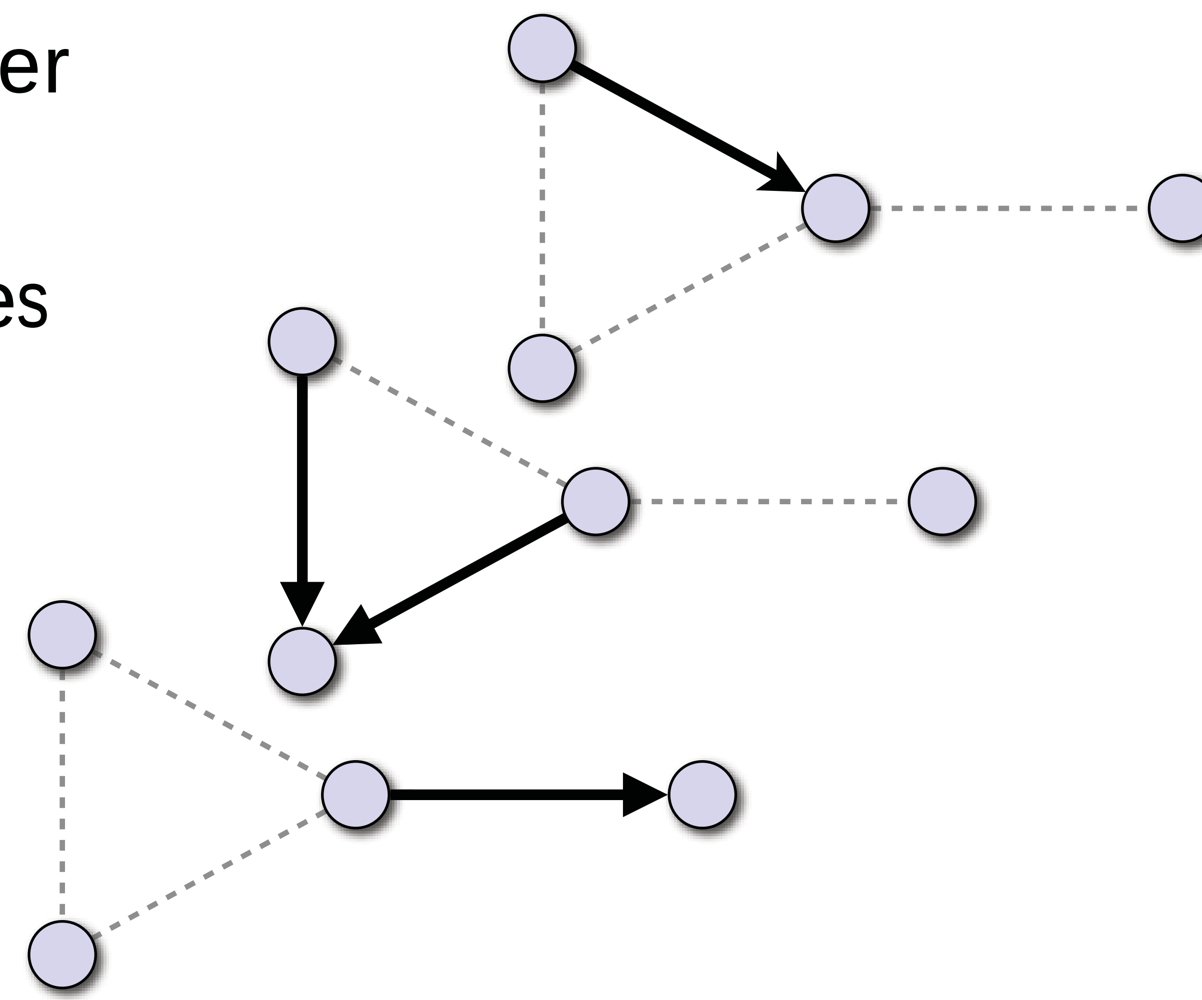
Algorithm for graph colouring

- We will show how to reduce the number of colours from k to $\Delta + 1$ in $O(\Delta^2 + \log^* k)$ rounds
- What if we don't have a k-colouring but only unique identifiers from 1, 2, …, poly(n)?
  - if k = poly(n), then $\log^* k = O(\log^* n)$ — see exercises
  - therefore given unique IDs, we can find a $(\Delta + 1)$-colouring in $O(\Delta^2 + \log^* n)$ rounds

Algorithm for graph colouring

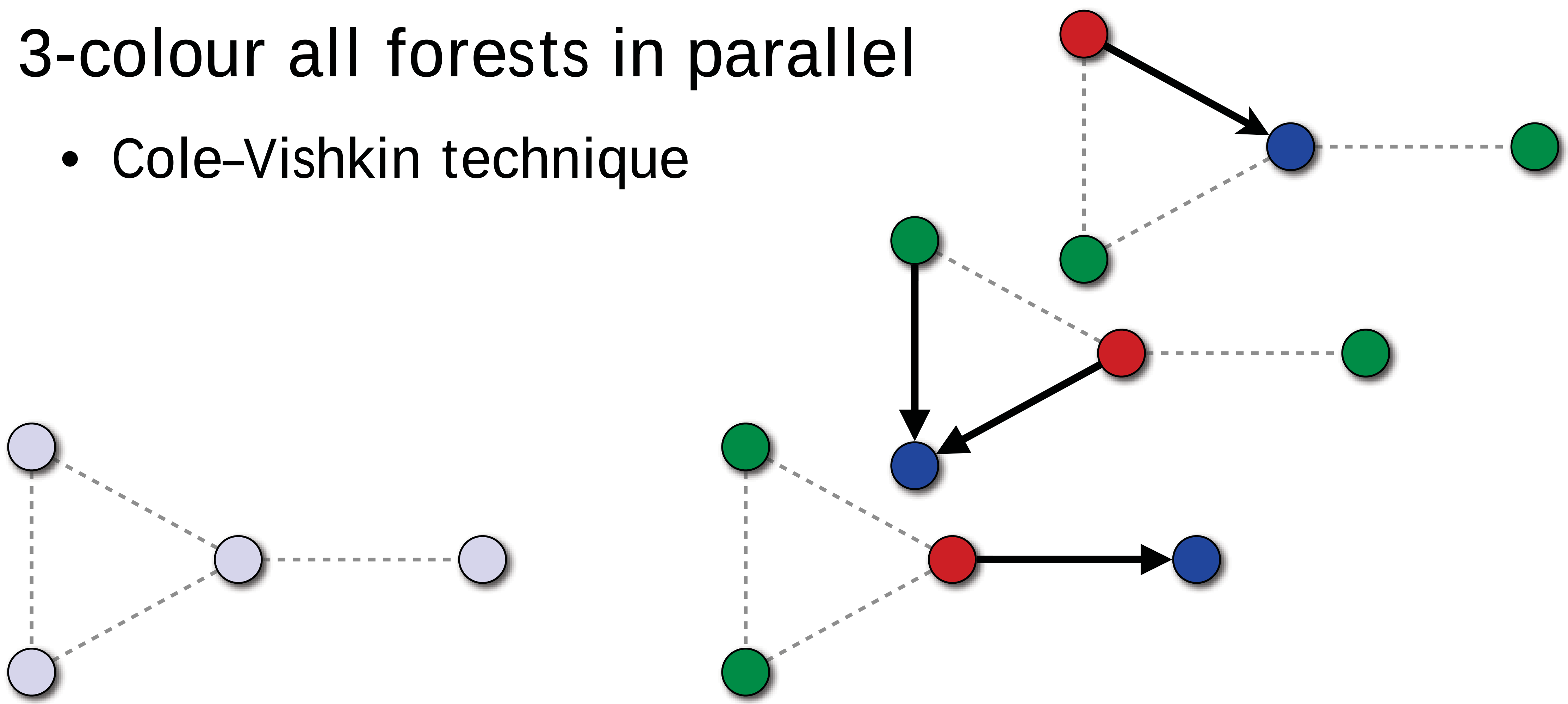
- Partition the graph into Δ directed forests
  - orientation: from smaller to larger colour
  - forest i = outgoing edges from port i





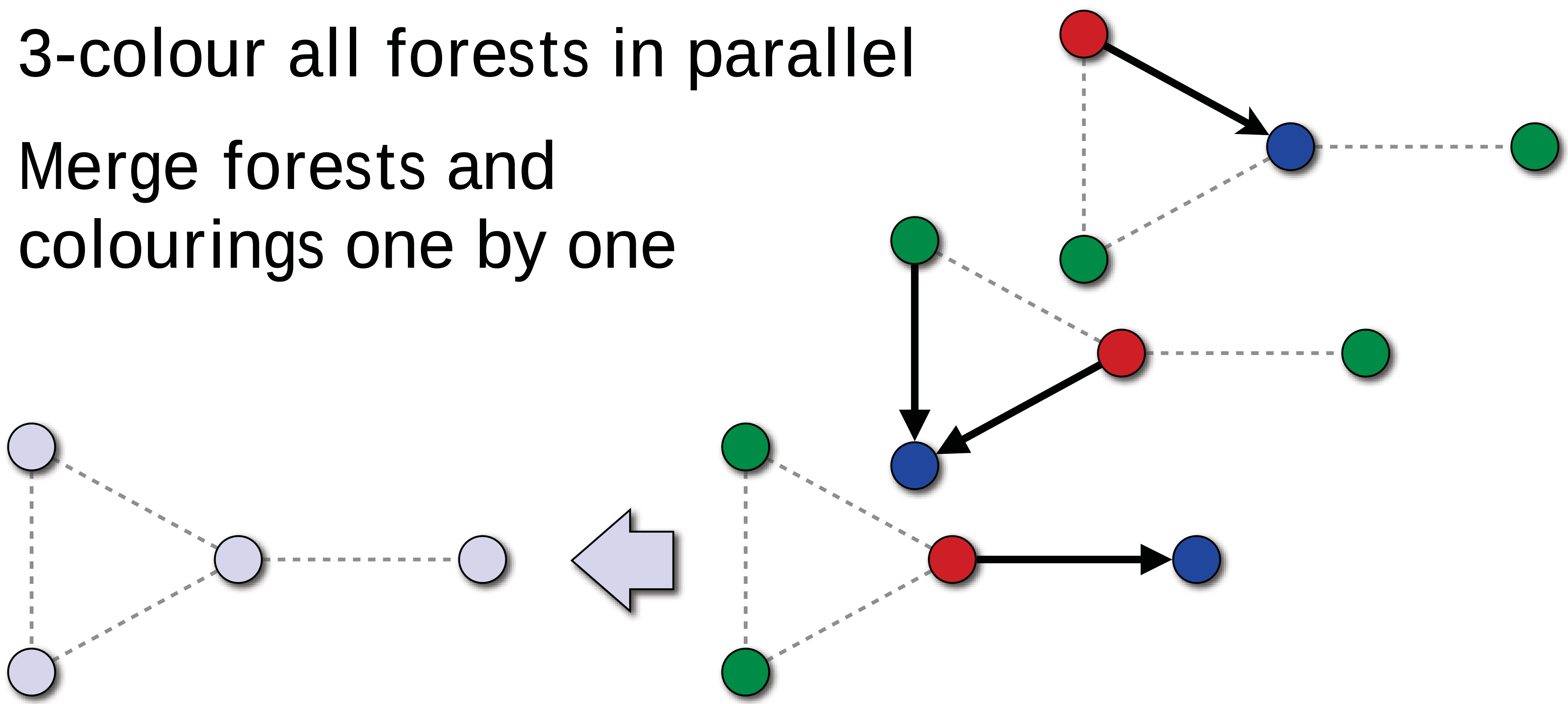
Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
    - Cole–Vishkin technique



Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
- Merge forests and colourings one by one



Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
- M
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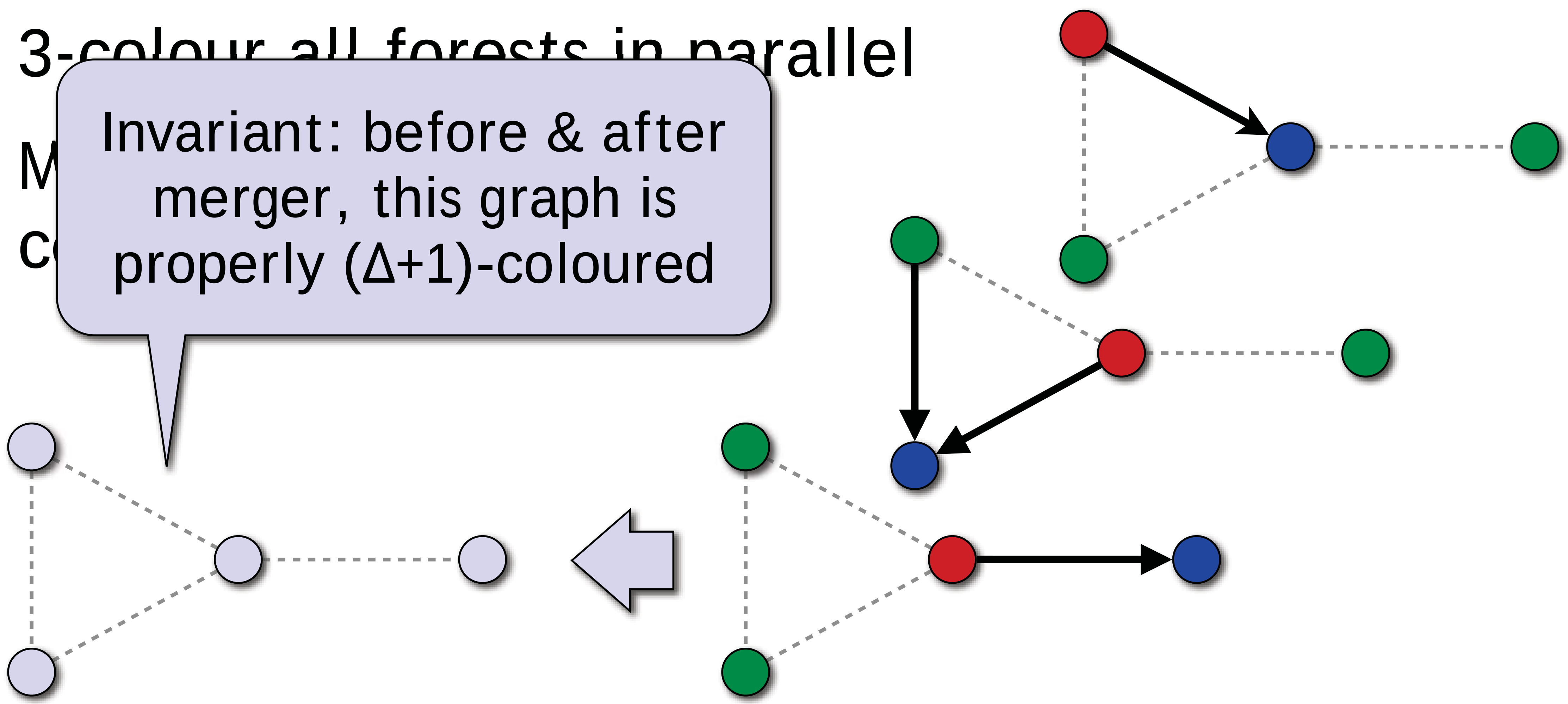
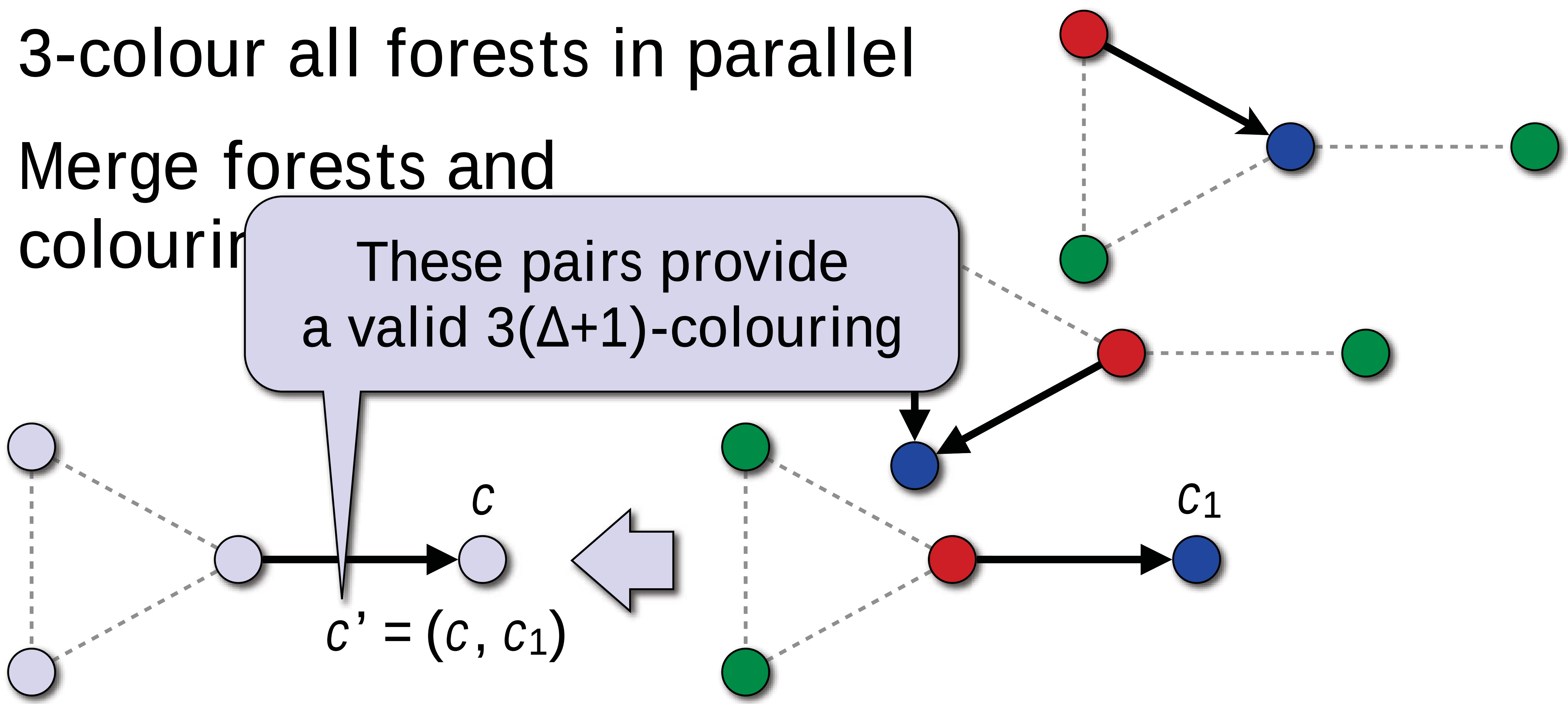


Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
- Merge forests and colourin



Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-co
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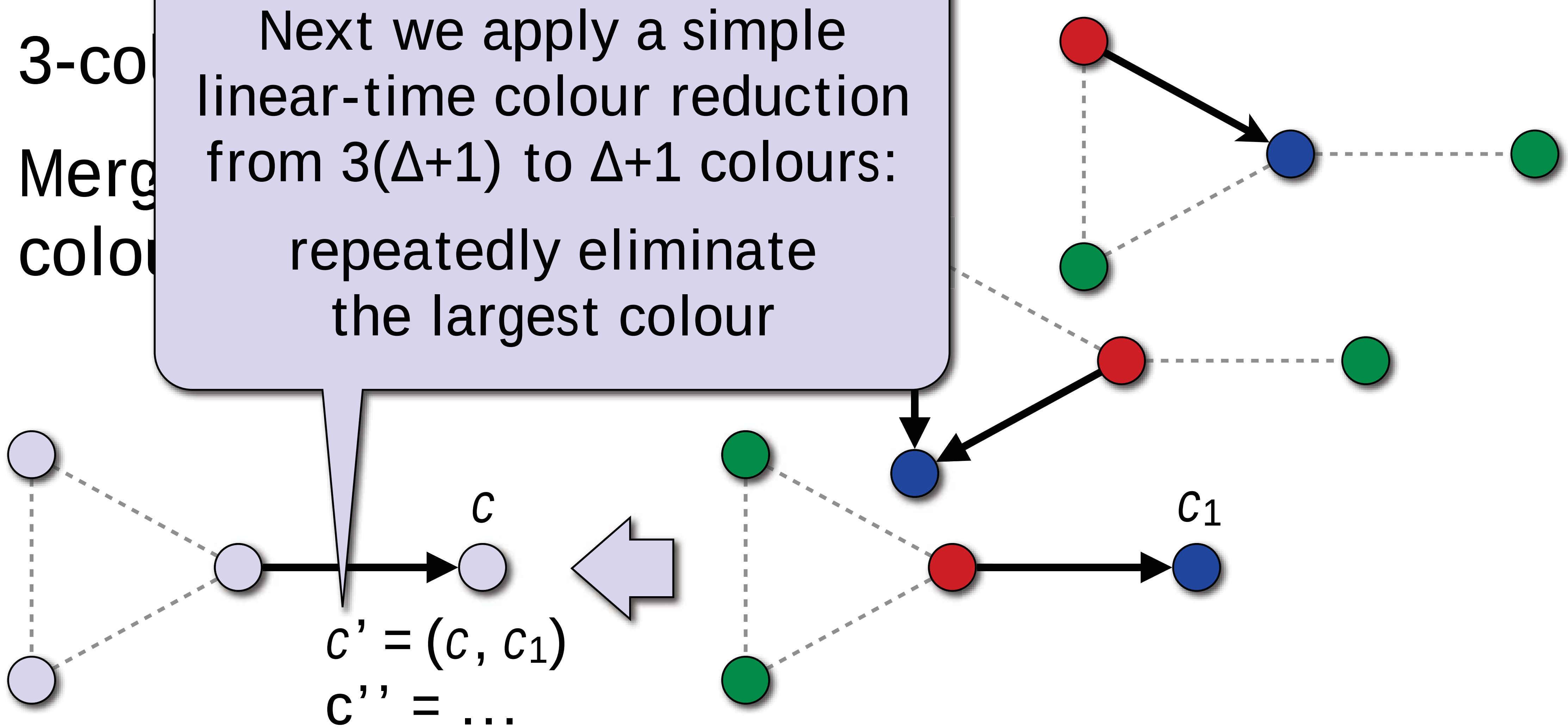
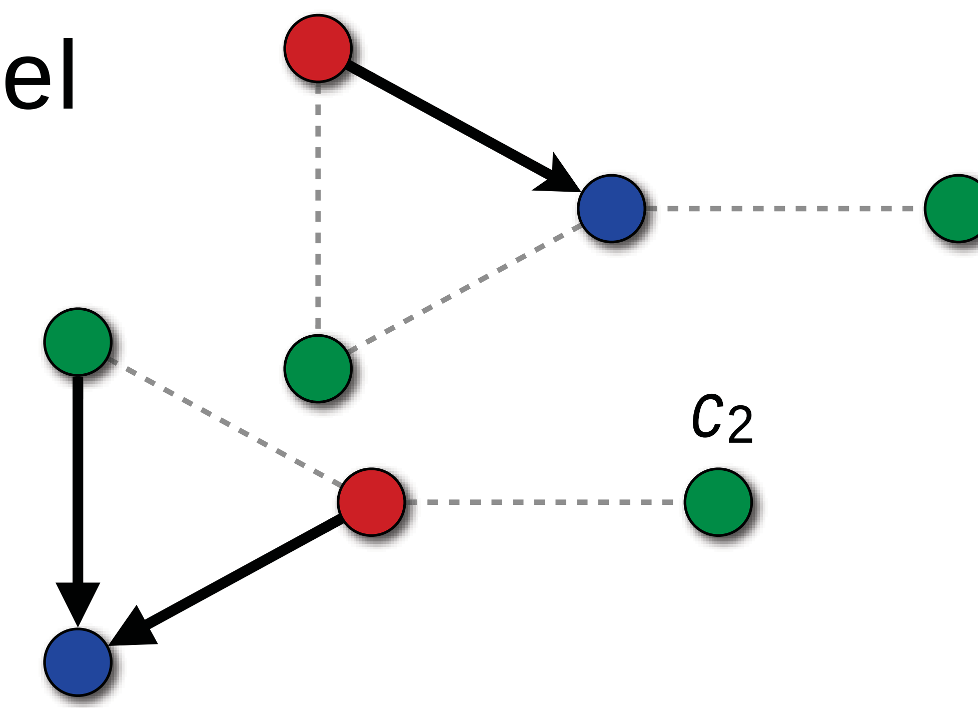
Next we apply a simple linear-time colour reduction from 3(Δ+1) to Δ+1 colours:

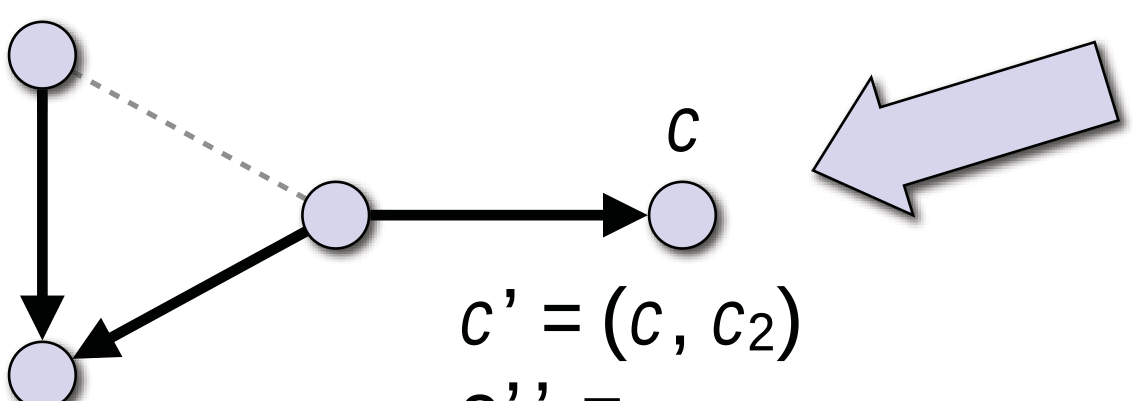
repeatedly eliminate the largest colour



Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
- Merge forests and colourings one by one





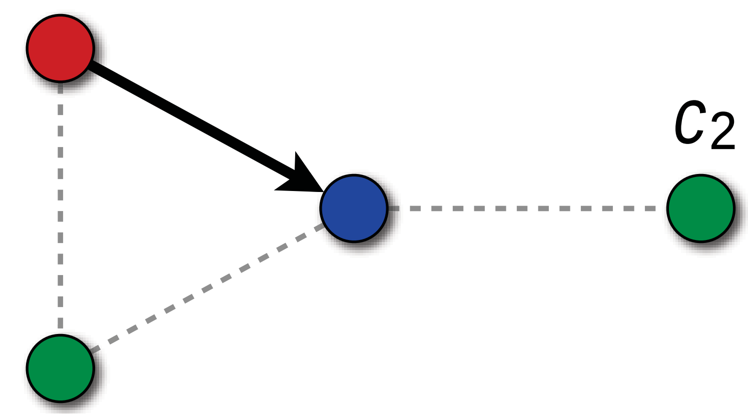
$c' = (c, c_2)$
c'' = …

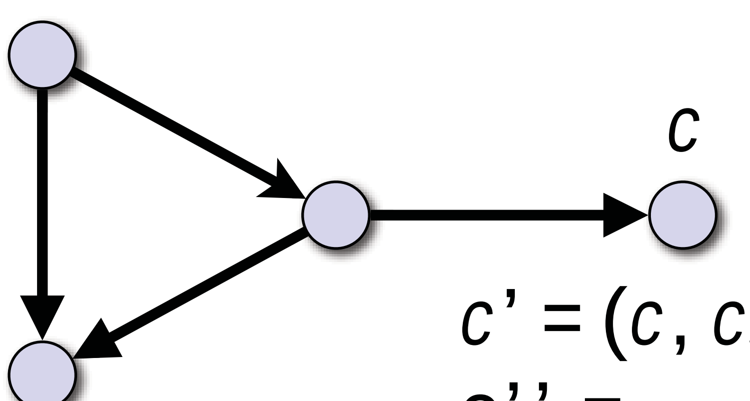
Merge: from Δ+1 to 3(Δ+1)

Reduce: back to Δ+1

Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
- Merge forests and colourings one by one





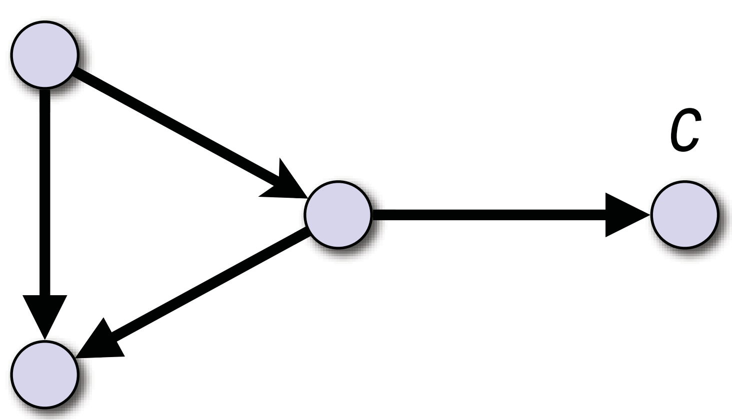
$c' = (c, c_3)$
c'' = …

Merge: from Δ+1 to 3(Δ+1)

Reduce: back to Δ+1

Algorithm for graph colouring

- Partition the graph into Δ directed forests
- 3-colour all forests in parallel
- Merge forests and colourings one by one
  - After Δ steps, we will have a (Δ+1)-colouring of the original graph



Algorithm for graph colouring

- Partition the graph into Δ directed forests
  - $O(1)$ time
- 3-colour all forests in parallel
  - $O(\log^* k)$ time
- Merge forests and colourings one by one
  - Δ steps, each takes $O(\Delta)$ time:
    $O(1)$-time merge + $O(\Delta)$-time colour reduction
- Total running time: $O(\Delta^2 + \log^* k)$

Algorithm for graph colouring

- If we have unique identifiers, we can find a $(\Delta+1)$-colouring in $O(\Delta^2 + \log^* n)$ rounds
  - powerful symmetry-breaking primitive
  - allows us to find a maximal independent set, maximal matching, etc.
  - more recent algorithms: running time $O(\Delta + \log^* n)$
- Could we make it even faster, like $O(\Delta)$? Or is the $O(\log^* n)$ part necessary?
  - we can use Ramsey's theorem to answer this question…